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综 述

一些高振荡积分、高振荡积分方程的高性能计算

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摘要 电磁、声波散射问题的研究涉及一类数学物理问题, 此类问题具有深刻的理论价值和重要的应用背景, 亟待解决. 高振荡微分、积分方程是刻画这些问题的重要的数学模型, 其数值计算存在许多挑战性研究课题. 本文从积分方程解法角度出发, 综述了求解这类高振荡问题的一些最新进展, 特别是针对广义 Fourier 变换、Bessel 变换的高效算法、高振荡核 Volterra 积分方程的数值解法作了详细介绍. 这些数值方法共有特点是振荡频率越高算法精度愈高, 且可望为电磁计算的研究提供一些新的高效算法.

关键词 高振荡问题 高振荡积分 高振荡微分方程 高振荡积分方程 高效算法

MSC (2010) 主题分类 65R20, 65R10, 65D32, 65N21

1 引言

许多高振荡问题的计算来源于电磁、声波散射问题的研究. 电磁、声波散射问题涉及许多数学物理问题, 在无损检测、遥感、雷达、声纳、地矿和海洋探测等领域有着广泛而重要的应用, 其数学模型许多可用 Maxwell 方程、Helmholtz 方程以及具有高振荡核的积分方程加以描述. 以介质的电磁散射研究为例 [1–5], 平面或空间区域时谐电磁波、声波散射问题可用满足适当边界条件的 Helmholtz 方程

$$\Delta u + k^2 u = 0, \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \Omega, \quad d = 2 \text{ 或 } 3 \tag{1.1}$$

加以刻画, 这里 Ω 表示散射区域, k 为波数.

Helmholtz 方程通常的求解方法是微分方程解法 (有限元、有限差分、谱方法) 和积分方程解法. 图 1 是用 Chebyshev 谱方法程序 [6, 第 70 页] 计算满足边界条件 $u=0$ 的对应不同波数的 Helmholtz 方程

$$\Delta u + k^2 u = \mathrm{e}^{-10((y-1)^2+(x-\frac{1}{2})^2)}, \quad -1 < x < 1, \quad -1 < y < 1.$$

从图 1 可以看出, 当 $k \gg 1$, 方程的解具有高频振荡的特点.

对高振荡微分方程发展低成本、高效的数值方法是当前计算数学研究的一个重要课题. 最近 Cohen、Hairer、Lubich 等多位学者 (参见 [7–12]) 研究了高振荡常微分方程的数值解问题. 模 Fourier 变换展开 (modulated Fourier expansions) 是研究这类问题强有力的工具之一 [7–10]. 2002 年, Cox 和

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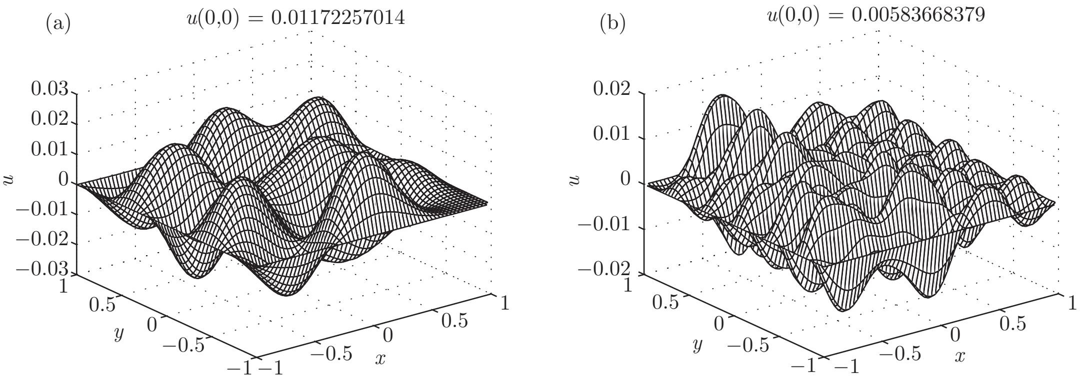

图 1 (a) $k=9$; (b) $k=19$

Matthews[11] 利用谱方法求解某些耗散、扩散高振荡常微分方程的数值解, 并基于 Fourier 变换的高效 Filon 方法提出了指数时间差分 Runge-Kutta 格式; 2009 年 Condon 等 [12] 对简单电路利用广义 Fourier 变换的高效 Filon 型方法给出了高效数值解法 (当电源具有较高频率时, 此系统的解具有高振荡性). 对于复杂电路系统, 其支路电压的求解不仅涉及高振荡微分方程, 还与优化理论的互补问题紧密相关, 满足:

$$\begin{aligned} &x'(t)=Ax(t)+By(t)+f(t),\\ &y(t)\in \mathrm{SOL}(Nx(t)+g(t),M),\\ &x(0)=x_0,\quad t\in[0,T], \end{aligned} \tag{1.2}$$

这里 $A\in\mathbb{R}^{m,m}$, $B\in\mathbb{R}^{m,n}$, $M\in\mathbb{R}^{n,n}$, $N\in\mathbb{R}^{n,m}$, $g:\mathbb{R}\to\mathbb{R}^n$, $f:\mathbb{R}\to\mathbb{R}^m$ 为电源, $\mathrm{SOL}(Nx(t)+g(t),M)$ 表示下列线性互补问题的解集:

$$0\leqslant y(t)\perp Nx(t)+g(t)+My(y)\geqslant 0,$$

其数值求解至今仍是挑战性问题 [13−16].

有限差分、有限元、谱方法求解电磁、声波散射问题近年来已取得许多成果, 如 Engquist、Chen、Bao 等文中的自适应有限元、自适应 PML (perfectly matched layer) 方法、间断 Galerkin 方法等 [17−39]. 微分方程解法的处理难点是针对大波数或高频的离散格式的构造、大规模线性方程组的求解.

积分变换方法也是求解电磁、声波散射问题的有力工具. 对于大波数 Helmholtz 方程 (1.1), Langdon 和 Chandler-Wilde[4] 利用 Hankel 变换给出了等价的积分形式

$$u(\boldsymbol{x})=\int_{T_H}\frac{\partial H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{y}|)}{\partial y_2}\phi(\boldsymbol{y})ds(\boldsymbol{y}),\quad \boldsymbol{x}\in U_H, \tag{1.3}$$

这里 $\phi\in L_\infty(T_H)$ 为密度函数, $H_0^{(1)}$ 表示第一类零阶 Hankel 函数, $U_H=\{(x_1,x_2):x_2>H>0\}$, $T_H=\{(x_1,H):x_1\in\mathbb{R}\}$. 积分变换方法的特点是把多维问题转化为一维问题, 因而数据处理量大为降低, 有其独到的优点. 特别地, 对于二维散射问题, Ashine 和 Huybrechs[1] 指出, Helmholtz 方程的解可用入射波 u^i 和散射波 u^s 表示为

$$u(\boldsymbol{x})=u^i(\boldsymbol{x})+u^s(\boldsymbol{x}),$$

且在边界上满足

$$u^s(\boldsymbol{x})=-u^i(\boldsymbol{x}),\quad \boldsymbol{x}\in\Gamma.$$

因此, 单层位势 (single-layer potential) 的散射波可表示为

$$u^s(\boldsymbol{x}) = (Sq)(\boldsymbol{x}) = \frac{i}{4}\int_\Gamma H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{y}|)q(\boldsymbol{y})ds(\boldsymbol{y}),$$

这里 q 表示单层位势密度函数, 它满足积分方程 [1,5]

$$\frac{i}{4}\int_\Gamma H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{y}|)q(\boldsymbol{y})ds(\boldsymbol{y}) = u^i(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma.$$

基于积分方程解法, Davis 和 Duncan [39] 研究了二维曲面区域中的瞬态声学散射的单层势能积分方程模型

$$\int_\Gamma \frac{u(x', t-|x'-x|)}{|x'-x|}dx' = a(x,t), \qquad (x,t)\in\Gamma\times(0,T).$$

特别地, 若 Γ 为平面区域且当 $t\leqslant 0$ 时 $u\equiv 0$、$a\equiv 0$, 通过连续 Fourier 变换, 上述积分方程也可转化为第一类 Volterra 积分方程 [39]

$$2\pi\int_0^x \widehat{u}(\omega, x-t)J_0(\omega t)dt = \widehat{a}(\omega, x), \quad \omega > 0. \tag{1.4}$$

对于满足非局部边界条件的电磁散射问题, Bao 和 Sun [40] 给出了解的更为复杂的、混合奇异积分方程形式

$$\frac{\partial u}{\partial n} = \frac{ik}{2}\left(\int_0^a \frac{1}{|x-x'|}J_1(k|x-x'|)u(x',0)dx' + i\int_0^a \frac{1}{|x-x'|}Y_1(k|x-x'|)u(x',0)dx'\right).$$

所有上述积分方程有着共同的特征, 当 $\omega\gg 1$ 或 $k\gg 1$, 其积分核 $J_m(\omega x)$、$Y_m(\omega x)$ 均是高振荡的 (见图 2), 其数值解均涉及高振荡问题的高性能计算.

高振荡问题的计算是科学计算领域非常重要、被公认为难的国际热点研究课题, 其研究热点主要集中于 [41]: (1) 高振荡微分方程鲁棒、高性能计算方法研究: 如 Magnus、Neumann 格式积分展开方法以及针对振荡 Hamilton 系统的辛算法 [42−44]、高频 Maxwell 方程、大波数 Helmholtz 方程的数值解, 其应用背景为波动力学中快速振荡 Schrödinger 方程、电路系统、天体力学或分子动力学中的高振荡 Hamilton 系统等; (2) 高振荡积分方程高性能计算方法研究 [45−48], 其应用背景为电磁和声波散射等; (3) 多尺度问题: 许多大型的物理问题涉及的高振荡问题展现出许多非可分规模, 其数值方法求解

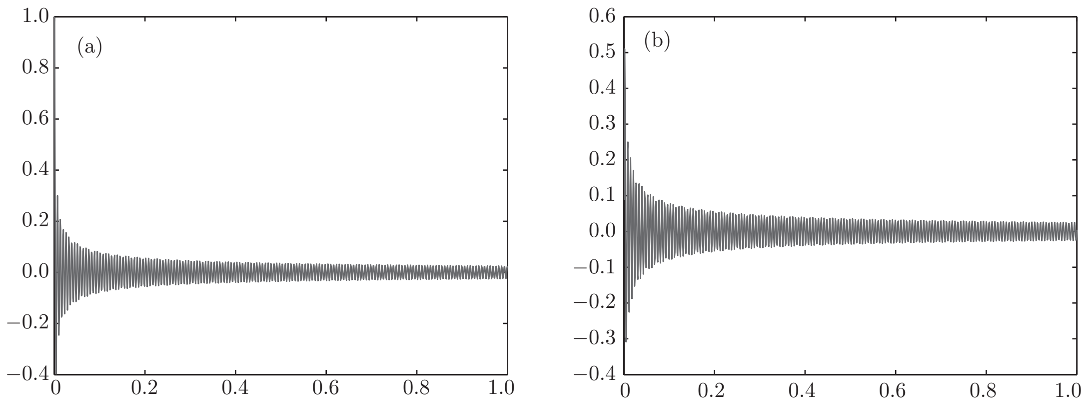

图 2　(a) $J_0(1000x)$; (b) $Y_0(1000x)$

表 1 $\int_{-1}^{1}\cos(10^4x)dx = 2\sin(10^4)/10^4 = -6.112287777765043\cdots\times 10^{-5}$

N	10	100	1000	2000
Q_N^{GL}	1.53709674490791	−0.38401861009781	0.02965639395896	0.05367808387527

难度具有随机性和高度振荡性; (4) 一些高振荡积分变换 (算子) 重要的渐进展开; (5) 高振荡函数高性能数值积分算法.

对于第一类高振荡 Bessel 核的 Volterra 积分方程 (1.4), 2004 年, Davies 和 Duncan [39] 讨论了配置方法求解的稳定性和收敛性; 2009 年, Brunner 等 [46] 研究了不连续 Galerkin 方法. 但是, 无论是配置方法还是不连续 Galerkin 方法均涉及高振荡 Bessel 函数的数值积分. 经典的积分法如 Newton-Cotes 公式、Gauss 方法、Clenshaw-Curtis 公式在计算高振荡积分时完全失效 [45,51,52]. 表 1 显示利用 $N+1$ 点 Gauss-Legendre 法则 Q_N^{GL} 来计算高振荡积分的效果.

因此, 基于传统数值积分算法的配置法、不连续 Galerkin 方法等 [39,46,48–50] 均不适合求解上述高振荡核 Volterra 积分方程. 鉴于此类 Volterra 积分方程具有很强的应用背景, Brunner 等 [45] 把这一类型积分方程数值解作为一个亟待解决的公开问题.

高振荡问题高效数值方法的特点, 正如剑桥大学 Iserles 教授 [51,52] 所言, 恰当的方法可以得到很好的计算结果, 且振荡频率越高算法精度愈高. 本文从积分方程解法角度出发, 针对这类高振荡问题的最新进展、高效算法作综合介绍: 第 2 节详细介绍高振荡函数包括广义 Fourier 变换、Bessel 变换高性能数值积分方法; 第 3 节讨论高振荡核 Volterra 积分方程的一些最新的数值解法; 第 4 节为结束语.

2 高振荡函数高效数值积分算法

高振荡函数高性能数值积分方法包括广义 Fourier 变换、Bessel 变换等是解决高振荡问题的核心基础之一. 传统的数值积分方法是基于对被积函数的多项式插值逼近, 然后对插值多项式进行积分得到逼近结果. 对高频振荡的被积函数, 只有当插值节点数比振荡频率还大时, 才能得到较精确的计算结果, 这样就使得经典的计算方法在计算高振荡积分时计算效率极低或失去了效用.

2.1 广义 Fourier 变换高效数值积分算法

高振荡积分的研究最早可以追溯到上个世纪初, 1928 年, Filon [53] 研究了以下形式的高振荡积分:

$$I[f] = \int_a^b f(x)\sin(\omega x)dx,$$

其主要思想是基于 $f(x)$ 的样条逼近: 设 $x_j = a + \frac{j(b-a)}{2n}$, $j = 0, 1, \ldots, 2n$, $p_k(x) = a_0^{(k)} + a_1^{(k)}x + a_2^{(k)}x^2$ 满足 $p_k(x_{2k+\ell}) = f(x_{2k+\ell})\,(\ell = 0, 1, 2)$, 利用 $p_k(x)$ 近似 $f(x)$ 得到

$$I[f] = \sum_{k=0}^{n-1}\int_{x_{2k}}^{x_{2k+2}} f(x)\sin(\omega x)dx \approx \sum_{k=0}^{n-1}\int_{x_{2k}}^{x_{2k+2}} p_k(x)\sin(\omega x)dx.$$

上式右端积分可以通过分部积分显式求出. 在此基础上, Flinn [54] 和 Luke [55] 进一步发展了 Filon 方法, 研究了在小区间上使用更高次的多项式来逼近函数.

关于广义 Fourier 变换

$$I[f] = \int_a^b f(x)e^{i\omega g(x)}dx, \tag{2.1}$$

当 $g'(x)\neq 0, \forall x\in[a,b]$, Levin [56] 提出了基于匹配算法实现的高效算法 (现称为 Levin 方法), Olver [57] 在 Levin [56] 的基础上引入了 Levin 型方法, Piessens、Poleunis、Sloan 等学者 (参见 [58–62]) 提出了基于 Chebyshev 多项式逼近的、稳定的 Clenshaw-Curtis-Filon 方法. 对于一般的振荡子 $g(x)$, Iserles 和 Nørsett [63,64] 基于分部积分技巧定义了渐进方法、利用 $f(x)$ 的 Hermite 插值逼近提出了 Filon 型方法; 特别地, 若 $f(x)$、$g(x)$ 在复平面解析, Milovanovic [65]、Huybrechs 和 Vandewall [66]、Deaño 和 Huybrechs [67] 利用积分与路径无关, 通过路径选择定义了最速下降方法, 其可基于 Gauss-Laguerre 积分算法实现, 具有更高的关于频率的渐进阶. 积分的最速下降法的思想最早可以追溯到 Riemann 和 Debye, 是研究含大参数积分的渐进展开的一种最重要的工具 [68].

Levin 型方法的优点是不用求矩 $I[x^k]=\int_a^b x^k\mathrm{e}^{i\omega g(x)}dx$, 局限是要求 $g'(x)\neq 0$; 渐进法的局限是对于固定的 ω 不收敛; Filon 型方法的局限是矩 $I[x^k]=\int_a^b x^k\mathrm{e}^{i\omega g(x)}dx$ 必须易于计算. 大多数数值例子显示 Filon 型方法较 Levin 型方法具有更高的计算精度, 因而 Iserles 等 [69] 提出了一个公开问题: 对于复杂振荡子 $g'(\xi)=\cdots=g^{(r)}(\xi)=0$, $g^{(r+1)}(\xi)\neq 0$, 且 $g'(x)\neq 0$ 当 $x\neq\xi$, 是否存在高效算法? Xiang [70] 提出了 moment-free-Filon 型方法, 其思想是在区间 $[a,\xi]$、$[\xi,b]$ 分别利用同胚变换 $t=\sqrt[r+1]{|g(x)-g(\xi)|}$, 把原积分转化为简洁形式 $\int_0^c\widetilde{f}(t)\mathrm{e}^{i\omega t^{r+1}}dt$, 其可用 Filon 型方法快速求解, 从而解决了 Iserles 等 [69] 提出的公开问题, 解决了广义 Fourier 变换的高效计算问题. 这些算法均可推广到高维问题的计算 [71–73]. 这些方法的共同特点是振荡越大计算精度愈高.

Levin 型方法 [56,57,74,75] 设 $g'(x)\neq 0$,$v(x)=\sum_{k=0}^n a_kx^k$ 为 $n=\sum_{k=0}^v m_k-1$ 阶多项式, 在点 $a=c_0<c_1<\cdots<\cdots<c_v=b$ 满足

$$(v'(x)+\mathrm{i}\omega g'(x)v(x))^{(k)}|_{x=c_j}=f^{(k)}(c_j),\quad k=0,1,\dots,m_j-1,\quad m_0,m_v\geqslant s\geqslant 1.$$

Levin 型方法定义为

$$Q_s^L[f]=\int_a^b(v'(x)+\mathrm{i}\omega g'(x)v(x))\mathrm{e}^{\mathrm{i}\omega g(x)}dx=v(b)\mathrm{e}^{\mathrm{i}\omega g(b)}-v(a)\mathrm{e}^{\mathrm{i}\omega g(a)},\tag{2.2}$$

其误差满足

$$I[f]-Q_s^L[f]=O(\omega^{-s-1}).\tag{2.3}$$

Filon 型方法 [63,64] 设 s 为一正整数, $g'(\xi)=g''(\xi)=\cdots=g^{(r)}(\xi)=0$, $g^{(r+1)}(\xi)>0$, $g'(x)\neq 0$ 对 $x\neq\xi$ 成立, $x\in[a,b]$, $v(x)=\sum_{k=0}^n a_kx^k$ 为 $f(x)$ 在点 $a=c_0<c_1<\cdots<c_q=\xi<\cdots<c_v=b$ 的 $n=\sum_{k=0}^v m_k-1$ 阶 Hermite 插值多项式,

$$v(c_j)=f(c_j),\quad\dots,\quad v^{(m_j-1)}(c_j)=f^{(m_j-1)}(c_j),\quad m_0,m_v\geqslant s,\quad m_q\geqslant s(r+1)-1.$$

Filon 型方法定义为

$$Q_s^F[f]\equiv I[v(x)]=\sum_{k=0}^n a_kI[x^k],\quad I[x^k]=\int_a^b x^k\mathrm{e}^{i\omega g(x)}dx,\quad k=0,1,\dots,n,\tag{2.4}$$

其误差满足

$$I[f]-Q_s^F[f]=O(\omega^{-s-1/(r+1)}).\tag{2.5}$$

moment-free-Filon 型方法 [70,76,77] 设 $g'(\xi)=g''(\xi)=\cdots=g^{(r)}(\xi)=0$, $g^{(r+1)}(\xi)>0$, $g'(x)\neq 0$ 对 $\neq\xi$ 成立, $x\in[a,b]$, 则

$$I[f]=\mathrm{e}^{\mathrm{i}\omega g(\xi)}\left(\int_a^\xi+\int_\xi^b\right)f(x)\mathrm{e}^{\mathrm{i}\omega(g(x)-g(\xi))}dx$$

$$= e^{i\omega g(\xi)}\Big[\int_0^{(g(b)-g(\xi))^{\frac{1}{r+1}}}\widetilde{f}_2(t)e^{i\omega t^{r+1}}dt - \int_0^{(g(a)-g(\xi))^{\frac{1}{r+1}}}\widetilde{f}_1(t)e^{i\omega t^{r+1}}dt\Big]. \tag{2.6}$$

上式右端中的积分可基于矩公式

$$v_k(y) = \int_0^y t^k e^{i\omega t^p}dt = \frac{1}{p(-i\omega)^{(k+1)/p}}\left[\Gamma\left(\frac{k+1}{p}\right) - \Gamma\left(\frac{k+1}{p}, -i\omega y^p\right)\right]$$

利用 Filon 型方法高效计算. 这里 $\Gamma(z)$、$\Gamma(z,\alpha)$ 分别表示 Gamma、不完全 Gamma 函数 [78, 第 260 页].

2.2 Bessel 变换高效数值积分算法

对于 Bessel 变换, 1996 年 Levin [79] 提出了匹配算法, 把原积分转化为向量值函数积分来处理: 设 $F(x) = (f_1(x), \dots, f_m(x))^{\mathrm{T}}$ 为非振荡的 m- 向量值函数、$W(\omega, x) = (w_1(\omega, x), \dots, w_m(\omega, x))^{\mathrm{T}}$ 为振荡的、线性无关的 m- 向量值函数,

$$I[F] = \int_a^b \sum_{k=1}^m f_k(x)w_k(\omega, x)dx \equiv \int_a^b F(x)\cdot W(\omega, x)dx.$$

Levin 型方法 [79,80] 若 $W'(\omega, x) = A(\omega, x)W(\omega, x)$ ($A(\omega, x)$ 为 $m\times m$ 矩阵), 令 $n = \sum_{k=0}^v m_k - 1$,

$$P(x) = \left(\sum_{k=0}^n a_k^{(1)}x^k, \dots, \sum_{k=0}^n a_k^{(m)}x^k\right)^{\mathrm{T}}$$

在点 $a = c_0 < c_1 < \cdots < c_v = b$ 满足

$$[P'(x) + A^{\mathrm{T}}(\omega, x)P(x)]^{(k)}|_{x=c_j} = F^{(k)}(c_j), \quad k = 0, 1, \dots, m_j - 1, \quad m_0, m_v \geqslant s \geqslant 1.$$

Levin 型方法为

$$Q_s^L[F] = \int_a^b (P'(x) + A^{\mathrm{T}}(\omega, x)P(x))\cdot W(\omega, x)dx = P(b)\cdot W(\omega, b) - P(a)\cdot W(\omega, a). \tag{2.7}$$

例如, 对于 Bessel 变换 $\int_a^b f(x)J_\mu(\omega x)dx$ 且 $0 \notin [a, b]$, 令

$$W(\omega, x) = (J_{\mu-1}(\omega x), J_\mu(\omega x))^{\mathrm{T}}, \quad F(x) = (0, f(x))^{\mathrm{T}},$$

则

$$W'(\omega, x) = \begin{pmatrix} \frac{\mu-1}{x} & -\omega \\ \omega & -\frac{\mu}{x} \end{pmatrix} W(\omega, x), \quad A(\omega, x) = \begin{pmatrix} \frac{\mu-1}{x} & -\omega \\ \omega & -\frac{\mu}{x} \end{pmatrix}.$$

此时 Levin 型方法的误差为 [80,81]

$$I[f] - Q_s^L[f] = O(\omega^{-s-3/2}). \tag{2.8}$$

Levin 型方法的缺陷是积分区间不能包含零点, 且计算复杂性较高、涉及到向量函数的插值逼近. 为克服其计算复杂性, Evans 与其合作者 [82−84] 提出了一种计算高振荡积分

$$I(f) = \int_a^b f(x)w(\omega, x)dx = \sum_{k=0}^v w_k f(c_k) + E(f) = Q(f) + E(f) \tag{2.9}$$

的广义积分法则.

广义积分法则 设 $\mathcal{L}$ 为微分算子, 满足 $\mathcal{L}w(\omega,x)=0$, $\mathcal{M}$ 为 $\mathcal{L}$ 的自伴算子且 $\mathcal{M}z=f$. 基于 Lagrange 恒等式

$$z\mathcal{L}w(\omega,x)-w(\omega,x)\mathcal{M}z=(Z(w(\omega,x),z(x)))'_x, \tag{2.10}$$

(2.10) 式两边积分得

$$I[f]=\int_a^b f(x)w(\omega,x)dx=\int_a^b w\mathcal{M}zdx=-Z(w(\omega,x),z(x))|_a^b. \tag{2.11}$$

一般地, 求解微分方程 $\mathcal{M}z=f$ 异常困难, 因而选取 $f_k(x)=\mathcal{M}z_k(x)$ 使得 $I[f_k]=Q[f_k]$ $(k=0,1,\dots,v)$ 从而决定权 w_k. 通常选取 $z_k(x)=x^k$ 或者平移 Chebyshev 多项式 $T_k^*(x)$. 当 $w(\omega,x)=\mathrm{e}^{\mathrm{i}\omega g(x)}$、$J_m(\omega g(x))$、$Ai(-\omega g(x))$ 时, 文 [82] 对 $\mathcal{L}$、$\mathcal{M}$ 和 Z 给出了详细的定义.

广义积分法则可推广至更高渐进阶的广义积分法则型方法, 且与 Levin 型方法具有相同的误差阶 [85]. 对 Bessel 变换, 其缺陷也是积分区间不能包含零点.

Filon 型方法 [86,87] 对 $I[f]=\int_a^b f(x)J_m(\omega x)dx$, 设 s 为一正整数, $v(x)=\sum_{k=0}^n a_kx^k$ 为 $f(x)$ 在点 $a=c_0<c_1<\cdots<\cdots<c_v=b$ 的 $n=\sum_{k=0}^v m_k-1$ 阶 Hermite 插值多项式,

$$v(c_k)=f(c_k),\quad \dots,\quad v^{(m_k-1)}(c_k)=f^{(m_k-1)}(c_k),\quad m_0,m_v\geqslant s.$$

Filon 型方法定义为 [87]

$$Q_s^F[f]\equiv I[v(x)]=\sum_{k=0}^n a_kI[k,\mu,\omega],\quad I[k,\mu,\omega]=\int_a^b x^kJ_\mu(\omega x)dx,\quad k=0,1,\dots,n, \tag{2.12}$$

其误差满足

$$I[f]-Q_s^F[f]=\begin{cases}O(\omega^{-s-3/2}), & 0\notin[a,b],\\ O(\omega^{-s-1}), & 0=a<b,\end{cases} \tag{2.13}$$

其中矩 $I[\nu,\mu,\omega]=\int_a^b x^\nu J_\mu(\omega x)dx$ 可表示为 [88−90]:

(i) $0\notin[a,b]$, $\Re(\mu+\nu)>-1$:

$$\begin{aligned}I[\nu,\mu,\omega]=&\frac{b}{\omega^\nu}\{(\nu+\mu-1)J_\mu(\omega b)s^{(2)}_{\nu-1,\mu-1}(\omega b)-J_{\mu-1}(\omega b)s^{(2)}_{\nu,\mu}(\omega b)\}\\&-\frac{a}{\omega^\nu}\{(\nu+\mu-1)J_\mu(ra)s^{(2)}_{\nu-1,\mu-1}(\omega a)-J_{\mu-1}(\omega a)s^{(2)}_{\nu,\mu}(ra)\},\end{aligned} \tag{2.14}$$

这里 $s^{(2)}_{\nu,\mu}(z)$ 为第二类 Lommel 函数 [88,89].

(ii) $a=0$, $\Re(\mu+\nu)>-1$:

$$I[\nu,\mu,\omega]=\frac{2^\mu\Gamma(\frac{\nu+\mu+1}{2})}{\omega^{\nu+1}\Gamma(\frac{\mu-\nu+1}{2})}+\frac{b}{\omega^\nu}[(\nu+\mu-1)J_\mu(\omega b)s^{(2)}_{\nu-1,\mu-1}(\omega b)-J_{\mu-1}(\omega b)s^{(2)}_{\nu,\mu}(\omega b)]. \tag{2.15}$$

注 当 $z\gg1$, $s^{(2)}_{\mu,\nu}(z)$ 有如下展开 [90, 第 361 页]

$$\begin{aligned}s^{(2)}_{\mu,\nu}(z)&=z^{\mu-1}\left[1-\frac{(\mu-1)^2-\nu^2}{z^2}+\frac{\{(\mu-1)^2-\nu^2\}\{(\mu-3)^2-\nu^2\}}{z^4}-\cdots\right]\\&=z^{\mu-1}\left[1-\frac{(\mu-1)^2-\nu^2}{z^2}+\cdots\right.\end{aligned}$$

$$+(-1)^p \frac{\{(\mu-1)^2-\nu^2\}\cdots\{(\mu-2p+1)^2-\nu^2\}}{z^{2p}}\Bigg] + O(z^{\mu-2p-2}). \tag{2.16}$$

因此, $s_{\mu,\nu}^{(2)}(z)$ 通过截断能被快速逼近. 在第 3 节的数值例子中我们取 10 项截断. 特别地, 若 $\mu\pm\nu$ 为正奇数, 则 $s_{\mu,\nu}^{(2)}(z)$ 具有有限表示 [90, 第 447 页].

moment-free-Filon 型方法 [87] 对于一般 Bessel 变换 $I[f]=\int_a^b f(x)J_m(\omega g(x))dx$ 且

$$g(a)=g'(a)=\cdots=g^{(r)}(a)=0,\quad g^{(r+1)}(a)\neq 0,\quad g'(x)\neq 0\,(x\neq a),$$

不妨设 $g^{(r+1)}(a)>0$, 利用同胚变换 $t^{r+1}=g(x)$, 则原积分可转化为

$$I[f]=(r+1)\int_0^{y_0}\widetilde{f}(t)J_\mu(\omega t^{r+1})dt,\quad \widetilde{f}(t)=\frac{f(x)}{g'(x)}g(x)^{r/(r+1)},\quad y_0=\sqrt[r+1]{g(b)},$$

其可利用上述 Filon 型方法高效计算.

moment-free-Filon 型方法解决了 Iserles 等 [63,64] 和 Olver [91] 提出的关于

$$\int_0^1 f(x)J_m(\omega g(x))dx,\qquad \int_0^1 f(x)\mathrm{Ai}(-\omega g(x))dx$$

的高效计算问题.

由于电磁计算中的第一类 Volterra 积分方程往往是病态的 [45,47], 针对这类积分方程, 文 [92] 提出了更为高效、快速的 Clenshaw-Curtis-Filon 型方法.

Clenshaw-Curtis-Filon 型方法 [92] 设 $p_{N+2s}(x)=\sum_{j=0}^{N+2s}a_jT_j^*(x)\,(s\geqslant 0)$ 为 $f(x)$ 在 Clenshaw-Curtis 点 $c_j=\frac{a+b}{2}+\frac{b-a}{2}\cos\left(\frac{j\pi}{N}\right)(j=0,1,\dots,N)$ 的 Hermite 插值多项式 (其可通过快速 FFT 实现),

$$p_{N+2s}^{(k)}(a)=f^{(k)}(a),\quad p_{N+2s}(c_j)=f(c_j),\quad p_{N+2s}^{(k)}(b)=f^{(k)}(b),\quad j=1,\dots,N-1,\quad k=0,\dots,s,$$

$T_j^*(x)$ 表示区间 $[a,b]$ 上平移 Chebyshev 多项式. 对 $I[f]=\int_a^b f(x)J_m(\omega x)dx\ (\Re(m)>-1)$, 定义

$$Q_s^{\mathrm{CCF}}[f]=\sum_{k=0}^{N+2s}a_kM[k,\mu,\omega],\quad M[k,\mu,\omega]=\int_a^b T_k^*(x)J_\mu(\omega x)dx,\quad k=0,1,\dots,N+2s, \tag{2.17}$$

则

$$I[f]-Q_s^{\mathrm{CCF}}[f]=\begin{cases} O\left(\dfrac{\max_{0\leqslant j\leqslant s+2}\{\|f^{(j)}(x)-p_{N+2s}^{(j)}(x)\|_\infty\}}{r^{s+1+3/2}}\right), & 0\notin[a,b],\\ O\left(\dfrac{\max_{0\leqslant j\leqslant s+2}\{\|f^{(j)}(x)-p_{N+2s}^{(j)}(x)\|_\infty\}}{r^{s+2}}\right), & 0=a<b.\end{cases} \tag{2.18}$$

这里 $M[k,\mu,\omega]$ 称为修正矩, 可表示为

$$M[k,\mu,\omega]=M[k,\mu,\omega,b]-M[k,\mu,\omega,a],$$

且

$$M[k,\mu,\omega,y]=\int_0^y T_k^*(x)J_\mu(\omega x)dx=\frac{y}{2}\int_{-1}^1 T_k(t)J_\mu((t+1)\omega y/2)dt,$$

当 $0\leqslant k\leqslant\frac{\omega y}{2}$ 可通过 Piessens 和 Branders [93] 的递归计算:

$$0=M[k+4,\mu,\omega,y]+\frac{16}{\omega_1^2}\left[(k+3)^2-\mu^2-\frac{\omega_1^2}{4}\right]M[k+2,\mu,\omega,y]$$

$$
\begin{aligned}
&+\frac{16}{\omega_1^2}(4\mu^2+2k+4)M[k+1,\mu,\omega,y]-\frac{16}{\omega_1^2}\left[2k^2-6+6\mu^2-\frac{3\omega_1^2}{8}\right]M[k,\mu,\omega,y]\\
&+\frac{16}{\omega_1^2}(4\mu^2-2k+4)M[k-1,\mu,\omega,y]\\
&+\frac{16}{\omega_1^2}\left[(k-3)^2-\mu^2-\frac{\omega_1^2}{4}\right]M[k-2,\mu,\omega,y]+M[k-4,\mu,\omega,y]\quad(\omega_1=\omega y),
\end{aligned}
\tag{2.19}
$$

其中, 修正矩 $M[0,\mu,\omega,y]$、$M[1,\mu,\omega,y]$ $(=M[-1,\mu,\omega,y])$、$M[2,\mu,\omega,y]$ $(=M[-2,\mu,\omega,y])$、$M[3,\mu,\omega,y]$ $(=M[-3,\mu,\omega,y])$ 可用 (2.15) 计算; 当 $k>\frac{\omega_1}{2}$, 修正矩 $M[n,\mu,\omega,y]$ 需用 Oliver 算法 [93–95] 实现. 图 3 显示 Clenshaw-Curtis-Filon 型方法计算 $\int_1^2\cos(x)Y_1(rx)dx$.

注 1　所有这些算法适用于 Bessel 型函数如 Hankel 函数、修正 Bessel 函数、Airy 变换以及高振荡 Hilbert、Hadamard 变换等, 其数值高效性可参见文 [96–107] 中的数值算例.

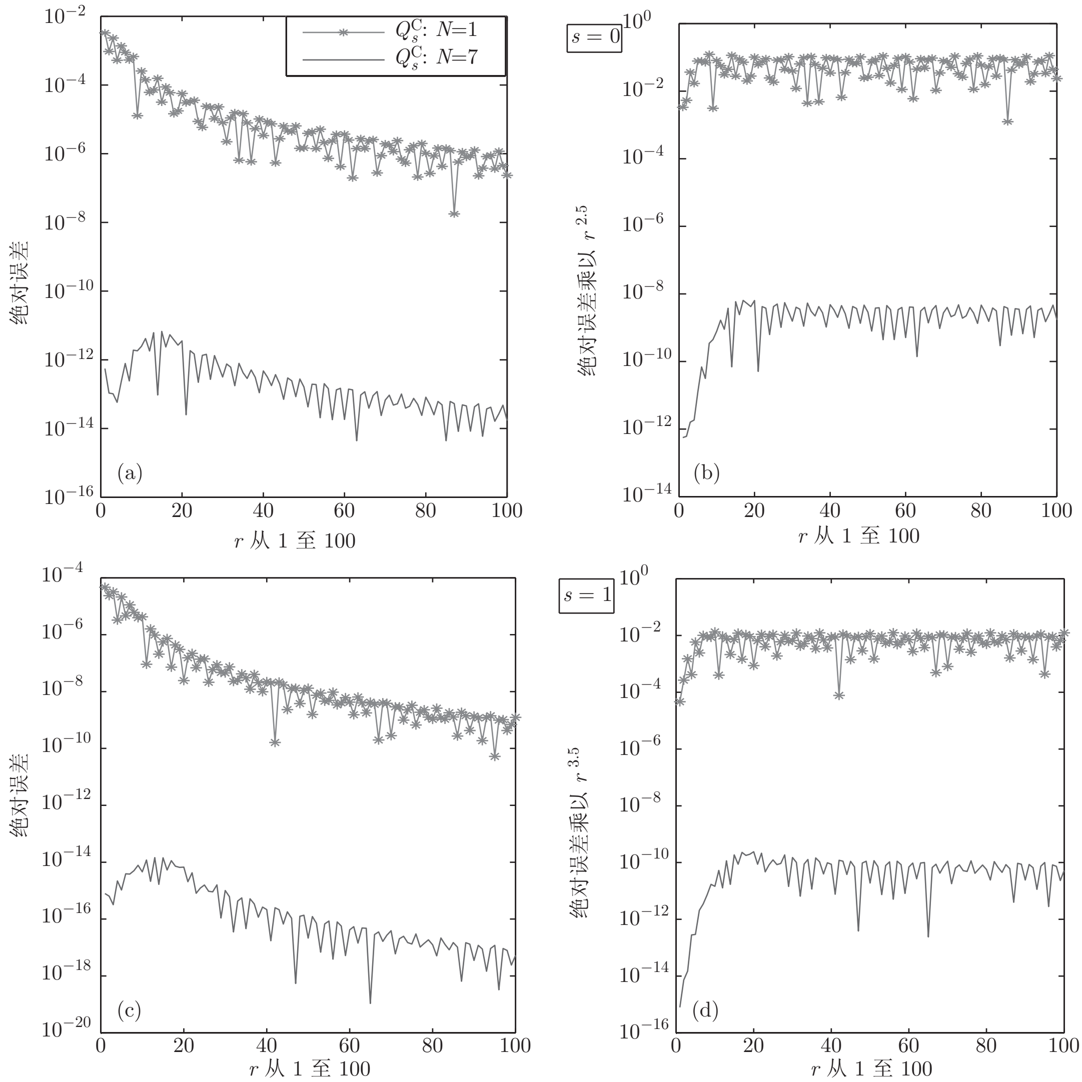

图 3　**绝对误差 ((a),(c)) 以及绝对误差乘以 $r^{s+\frac{5}{2}}$ ((b),(d)): Clenshaw-Curtis-Filon 型方法计算 $\int_1^2\cos(x)Y_1(rx)dx$: $N=1$ 或 7, $s=0$ ((a),(b)), $s=1$ ((c),(d))**

3 高振荡 Bessel 积分核的 Volterra 积分方程高效数值算法

高振荡 Bessel 积分核的 Volterra 积分方程

$$\int_a^x (x-t)^m J_m(\omega(x-t))y(t)dt = g(x), \quad x \in [a,b],$$

近年来引起了许多学者的关注 [39,45−47], 这里 $g(x)$ 是已知函数且满足 $g(a)=0$. 上述方程的特征是当

$$\omega \gg \max\{1, \Re(m)\}$$

时, $J_m(\omega(x-t))$ 呈现高振荡, 这意味着传统的数值解法不适用于求解此类方程. 其求解难度也可从其解的渐进性看出 [92]:

$$y(x) = \begin{cases} O((\omega x)^{1+\Re(m)}), & 若 \quad -\dfrac{1}{2} < n - \Re(m) - 1 < \dfrac{3}{2}, \\ O((\omega x)^{n-\frac{3}{2}}), & 若 \quad \dfrac{3}{2} \leqslant n - \Re(m) - 1, \end{cases}$$

其中, $n = \lfloor 2\Re(m)+1 \rfloor + 1$ ($\lfloor 2\Re(m)+1 \rfloor$ 表示不超过 $2\Re(m)+1$ 的最大整数).

高振荡 Bessel 积分核的 Volterra 积分方程求解方法可分为两类 [47,92,108]: 直接解法和间接解法.

3.1 直接解法

直接解法是基于方程解的解析表达式利用高振荡数值积分方法高效计算. 本小节我们介绍直接解法: Clenshaw-Curtis-Filon 型方法解高振荡 Bessel 积分核的 Volterra 积分方程

$$\int_a^x (x-t)^m J_m(\omega(x-t))y(t)dt = g(x), \quad x \in [a,b], \tag{3.1}$$

和

$$\int_a^x (x-t)^{m/2} J_m(\omega\sqrt{x-t})y(t)dt = g(x), \quad x \in [a,b]. \tag{3.2}$$

关于积分方程 (3.1), 当 $m=0$ 时,

$$\int_0^x J_0(\omega(x-t))y(t)dt = g(x), \quad x \in [0,T],$$

可通过 Laplace 变换求解, 其解可表示为

$$y(x) = g'(x) + \omega^2 \int_0^x g(x-t)J_0(rt)dt - \omega \int_0^x g'(x-t)J_1(rt)dt.$$

这样

$$\int_0^x g(x-t)J_0(\omega t)dt, \quad \int_0^x g'(x-t)J_1(\omega t)dt$$

可利用 Bessel 变换的 Filon 型方法高效求解 [108], 从而解决了 Brunner 等 [45] 提出的一个公开问题.

对于一般情形, 根据 [109] 和 [110] 中的 (1.8.17),

$$\int_a^x (x-t)^m J_m(\omega(x-t))y(t)dt = g(x)$$

的解可表示为

$$y(x)=\left(\frac{2}{\omega}\right)^{n-1}A\left(\frac{d^2}{dx^2}+\omega^2\right)^n\int_a^x (x-t)^{n-m-1}J_{n-m-1}(\omega(x-t))g(t)dt,$$

这里

$$-\frac{1}{2}<\Re(m)<\frac{n-1}{2},\quad n=\lfloor 2\Re(m)+1\rfloor+1,\quad A=\frac{\Gamma(m+1)\Gamma(n-m)}{\Gamma(2m+1)\Gamma(2n-2m-1)}.$$

令

$$\begin{aligned}&g_1(x)=\left(\frac{d^2}{dx^2}+\omega^2\right)g(x),\quad g_{k+1}(x)=\left(\frac{d^2}{dx^2}+\omega^2\right)g_k(x),\\&f_1(x)=(x-a)^{n-m-1}J_{n-m-1}(\omega(x-a))g(a)+\omega(x-a)^{n-m-1}J_{n-m-2}(\omega(x-a))g'(a),\\&f_{k+1}(x)=(x-a)^{n-m-1}J_{n-m-1}(\omega(x-a))g_k(a)+\omega(x-a)^{n-m-1}J_{n-m-2}(\omega(x-a))g_k'(a),\end{aligned}$$

利用 $\frac{d}{dx}x^\nu J_\nu(x)=x^{\nu+1}J_\nu(x)$, 得

$$\begin{aligned}y(x)=A\left(\frac{2}{\omega}\right)^{n-1}&\left\{\int_0^{x-a}t^{n-m-1}J_{n-m-1}(\omega t)\left(\frac{d^2}{dt^2}+\omega^2\right)^n g(x-t)dt\right.\\&\left.+\left(\frac{d^2}{dx^2}+\omega^2\right)^{n-1}f_1(x)+\cdots+\left(\frac{d^2}{dx^2}+\omega^2\right)f_{n-1}(x)\right\}\end{aligned}\tag{3.3}$$

(参见 [92]), 其中 $J_\nu^{(k)}(x)$ 可由以下公式及其递推求得:

$$\begin{pmatrix}J_\nu(\omega x)\\J_{\nu+1}(\omega x)\end{pmatrix}'=\begin{pmatrix}\frac{\nu}{x}&-\omega\\\omega&-\frac{\nu+1}{x}\end{pmatrix}\begin{pmatrix}J_\nu(\omega x)\\J_{\nu+1}(\omega x)\end{pmatrix}.$$

特别地, 若

$$g(a)=g'(a)=\cdots=g^{(n-1)}(a)=0,$$

则

$$y(x)=A\left(\frac{2}{\omega}\right)^{n-1}\int_a^x(x-t)^{n-m-1}J_{n-m-1}(\omega(x-t))\left(\frac{d^2}{dt^2}+\omega^2\right)^n g(t)dt\tag{3.4}$$

(参见 [110]). 因此积分方程 (3.1) 可用 Clenshaw-Curtis-Filon 型方法高效计算.

例 3.1 我们考虑 Clenshaw-Curtis-Filon 型方法解

$$\int_0^x (x-t)^{-\frac{1}{3}}J_{-\frac{1}{3}}(\omega(x-t))y(t)dt=g(x),\quad g(x)=x\mathrm{e}^x.$$

$y(x)$ 可表示为 [92]

$$y(x)=\frac{1}{2}x^{\frac{1}{3}}J_{\frac{1}{3}}(\omega x)+\frac{x^{\frac{4}{3}}}{2}\int_0^1 t^{\frac{1}{3}}(x-xt+2+\omega^2x(1-t))\mathrm{e}^{x-xt}J_{\frac{1}{3}}(\omega xt)dt.$$

对于积分方程 (3.2), 其解可表示为 [92]

$$\begin{aligned}y(x)=\left(\frac{2}{\omega}\right)^{n-2}&\int_0^{x-a}s^{\frac{n-m-2}{2}}I_{n-m-2}(\omega\sqrt{s})g^{(n)}(x-s)dts\\&+(x-a)^{\frac{n-m-2}{2}}I_{n-m-2}(\omega\sqrt{x-a})g^{(n-1)}(a)+\cdots\end{aligned}$$

$$+\frac{d^{n-1}}{dx^{n-1}}(x-a)^{\frac{n-m-2}{2}}I_{n-m-2}(\omega\sqrt{x-a})g(a), \tag{3.5}$$

其中

$$-1<\Re(m)<n-1, \quad n=[\Re(m)+1]+1.$$

特别地, 若

$$g(a)=g'(a)=\cdots=g^{(n-1)}(a)=0, \quad n=[\Re(m)+1]+1,$$

则其解为

$$\begin{aligned} y(x)&=\left(\frac{2}{\omega}\right)^{n-2}\int_a^x (x-t)^{\frac{n-m-2}{2}}I_{n-m-2}(\omega\sqrt{x-t})g^{(n)}(t)dt \\ &=2\left(\frac{2}{\omega}\right)^{n-2}\int_0^{\sqrt{x-a}} u^{n-m-1}I_{n-m-2}(\omega u)g^{(n)}(x-u^2)du. \end{aligned} \tag{3.6}$$

所以积分方程 (3.2) 也可用 Clenshaw-Curtis-Filon 型方法高效计算. 数值结果见表 2, 其中 $c_k=\frac{1}{2}+\frac{1}{2}\cos(k\pi/N)$, $k=0,1,\dots,N$, $N=2$ 及 $s=0,1,2$.

例 3.2 我们考查 Clenshaw-Curtis-Filon 型方法解

$$\int_0^x J_0(\omega\sqrt{x-t})y(t)dt=g(x), \quad g(x)=x\mathrm{e}^x.$$

由 (3.6), 其解可表示为 [92]

$$\begin{aligned} y(x)&=I_0(r\sqrt{x})+\int_0^x I_0(\omega\sqrt{t})(x-t+2)\mathrm{e}^{x-t}dt, \\ &=I_0(\omega\sqrt{x})+2\int_0^{\sqrt{x}} I_0(\omega u)(x-u^2+2)u\mathrm{e}^{x-u^2}du \\ &=I_0(\omega\sqrt{x})+2x\int_0^1 I_0(\omega\sqrt{x}s)(x-xs^2+2)s\mathrm{e}^{x-xs^2}ds. \end{aligned}$$

因为

$$I_\nu(z)=\mathrm{e}^{\pi\nu i/2}J_\nu(z\mathrm{e}^{\pi i/2}),$$

表 2 例 3.1 描述的 Clenshaw-Curtis-Filon 型方法的相对误差

s	$x\backslash\omega$	800	1600	2400
$s=0$	$\frac{1}{4}$	1.79×10^{-5}	1.17×10^{-5}	1.56×10^{-6}
	$\frac{1}{2}$	4.12×10^{-5}	1.53×10^{-5}	2.41×10^{-6}
	1	4.88×10^{-5}	2.57×10^{-5}	1.73×10^{-6}
$s=1$	$\frac{1}{4}$	4.23×10^{-9}	7.01×10^{-10}	2.17×10^{-10}
	$\frac{1}{2}$	1.05×10^{-8}	2.23×10^{-9}	1.73×10^{-9}
	1	3.19×10^{-8}	9.12×10^{-9}	5.73×10^{-9}
$s=2$	$\frac{1}{4}$	2.10×10^{-14}	4.43×10^{-15}	2.41×10^{-16}
	$\frac{1}{2}$	2.50×10^{-13}	2.58×10^{-14}	2.10×10^{-15}
	1	1.31×10^{-12}	1.63×10^{-13}	6.06×10^{-15}

表 3　例 3.2 描述的 Clenshaw-Curtis-Filon 型方法的相对误差

s	$x\omega$	50	100	200
$s=0$	$\frac{1}{4}$	6.53×10^{-6}	2.18×10^{-6}	6.30×10^{-7}
	$\frac{1}{2}$	6.79×10^{-5}	2.11×10^{-5}	5.88×10^{-6}
	1	7.60×10^{-4}	2.25×10^{-4}	6.13×10^{-5}
$s=1$	$\frac{1}{4}$	2.71×10^{-9}	5.81×10^{-10}	9.62×10^{-11}
	$\frac{1}{2}$	9.25×10^{-8}	1.73×10^{-8}	2.66×10^{-9}
	1	3.29×10^{-6}	5.59×10^{-7}	8.17×10^{-8}
$s=2$	$\frac{1}{4}$	6.36×10^{-13}	9.29×10^{-14}	9.17×10^{-15}
	$\frac{1}{2}$	7.32×10^{-11}	8.68×10^{-12}	7.58×10^{-13}
	1	8.45×10^{-9}	8.56×10^{-10}	6.86×10^{-11}

其中 $-\pi<\arg z<\frac{1}{2}\pi$ (见 [78, 方程 9.6.3]), 所以

$$\int_0^1 t^\mu I_0(\omega\sqrt{x}t)dt=\int_0^1 t^\mu J_0(\omega\sqrt{x}it)dt=I[\mu,0,\omega\sqrt{x}i],$$

其可由 Bessel 矩 (2.15) 的公式计算. 数值结果见表 3, 其中 $c_k=\frac{1}{2}+\frac{1}{2}\cos(k\pi/N)$, $k=0,1,\ldots,N$, $N=2$ 及 $s=0,1,2$.

3.2　间接解法

弱奇异高振荡 Bessel 核 Volterra 积分方程

$$y(x)+\int_0^x\frac{J_m(\omega(x-t))}{(x-t)^\alpha}y(t)dt=f(x),\quad x\in[0,1]\tag{3.7}$$

在数学物理问题有广泛的应用 [111,112], 其解在许多情形下无解析表达式, 因而直接方法失去了效用. 方程 (3.7) 具有如下性质 [113]:

(i) 若 $f\in C[0,1]$, $0\leqslant\alpha<1$, 则解 $y(x)$ 关于 $\omega\geqslant0$ 一致有界,

$$\max_{\omega\in[0,+\infty)}\|y(x)\|_\infty<\infty.$$

(ii) 若 $f\in C^1[0,1]$, $0\leqslant\alpha\leqslant m$, 则解 $y\in C^1[0,1]$ 满足

$$y(x)=f(x)+O(\omega^{-1+\alpha}),\quad \omega\gg1,$$

且 $y'(x)$ 关于 $\omega\geqslant0$ 一致有界.

(iii) 若

$$f\in C^q[0,1]\ (q\geqslant1),\quad 0<m<\alpha<1,$$

则解 $y\in C[0,1]\cap C^q(0,1]$, 且 $|y'(x)|\leqslant C_\alpha x^{-\alpha}$ $(x\in(0,1])$, 其中 C_α 为一与 ω 无关常数.

Filon 方法 $\boldsymbol{Q_N^F}$ [113]　设 $0=t_0<t_1<t_2<\cdots<t_N=1$,

$$L[y(0),y(t_j)]=y(0)+\frac{y(t_j)-y(0)}{t_j}t$$

表示 $y(0), y(t_j)$ 的线性插值. 由 $y(0)=f(0)$ 得到 $j=1,2,\dots,N$,

$$y(t_j)+\int_0^{t_j}\frac{J_m(\omega(t_j-t))}{(t_j-t)^\alpha}L[y(0),y(t_j)]dt=f(t_j)-\int_0^{t_j}\frac{J_m(\omega(t_j-t))}{(t_j-t)^\alpha}\{y(t)-L[y(0),y(t_j)]\}dt.$$

Filon 方法定义为

$$y_j=\frac{t_jf(t_j)-f(0)I[1-\alpha,m,\omega,t_j]}{t_j+t_jI[-\alpha,m,\omega,t_j]-I[1-\alpha,m,\omega,t_j]},\quad j=1,2,\dots,N. \tag{3.8}$$

Filon 方法的误差: 设 $f\in C^1[0,1]$, $0\leqslant\alpha<1$, 则

$$y_j-y(t_j)=O\left(\frac{1}{\omega^{1-\alpha}}\right). \tag{3.9}$$

若 $f(0)=0$ 且 $f\in C^2[0,1]$, 则

$$y_j-y(t_j)=O\left(\frac{1}{\omega^{2-\alpha}}\right). \tag{3.10}$$

分段常数或线性连续匹配法 [113] 设

$$I_\Delta=\{t_j:\ j=0,1,\dots,N\},$$

$\widehat{y}(x)$ 为 $y(x)$ 的近似,

$$\widehat{y}(x)|_{(t_j,t_{j+1}]}\quad\begin{cases}\text{为一常数},\ j=0,1,\dots,N-1,\\ \text{为线性函数},\ j=0,1,\dots,N-1\end{cases}$$

满足

$$\widehat{y}(t_j)+\int_0^{t_j}\frac{J_m(\omega t)}{t^\alpha}\widehat{y}(t_j-t)dt=f(t_j),\quad j=1,2,\dots,N.$$

分段常数匹配法 $Q_N^{L,0}$ 定义为

$$\widehat{y}_1=f(0),\quad \widehat{y}_j=\frac{f(t_j)-\sum_{i=1}^{j-1}\widehat{y}_i(I[-\alpha,m,\omega,t_{j-i+1}]-I[-\alpha,m,\omega,t_{j-i}])}{1+I[-\alpha,m,\omega,t_1]},\quad j=2,\dots,N, \tag{3.11}$$

其数值算法具有较好的稳定性.

线性连续匹配法 $Q_N^{L,1}$ 定义为

$$\begin{aligned}\widehat{y}_1&=\frac{f(t_1)t_1-f(0)I[1-\alpha,m,\omega,t_1]}{t_1+t_1I[-\alpha,m,\omega,t_1]-I[1-\alpha,m,\omega,t_1]},\\ \widehat{y}_j&=\frac{f(t_1)t_1-\widehat{y}_{j-1}I[1-\alpha,m,\omega,t_1]-Q_1-Q_2}{t_1+t_1I[-\alpha,m,\omega,t_1]-I[1-\alpha,m,\omega,t_1]},\quad j=2,\dots,N,\end{aligned} \tag{3.12}$$

其中

$$\begin{aligned}Q_1&=\sum_{k=1}^{j-1}\frac{t_1(t_{k+1}\widehat{y}_{j-k}-t_k\widehat{y}_{j-k-1})}{t_{k+1}-t_k}(I[-\alpha,m,\omega,t_{k+1}]-I[-\alpha,m,\omega,t_k]),\\ Q_2&=\sum_{k=1}^{j-1}\frac{t_1(\widehat{y}_{j-k-1}-\widehat{y}_{j-k})}{t_{k+1}-t_k}(I[1-\alpha,m,\omega,t_{k+1}]-I[1-\alpha,m,\omega,t_k]).\end{aligned}$$

表 4–5 显示 Filon 方法、匹配法求解第二类高振荡核 Volterra 积分方程的高效性.

表 4 Filon 方法、匹配法求解 $y(x)+\int_0^x \frac{J_0(\omega t)}{\sqrt{t}} y(x-t)dt=\sin(x)$ $(\omega=10^4)$

x	0.1	0.5	1
Q_{10}^F	0.09778806153725	0.46960145889132	0.82422771691863
$Q_{10}^{L,0}$	0.09778746085725	0.46960046983603	0.82422691381786
$Q_{100}^{L,0}$	0.09778852007547	0.46960196841630	0.82422791394085
$Q_{10}^{L,1}$	0.09778806153725	0.46960143190203	0.82422759760585
$Q_{100}^{L,1}$	0.09778806023321	0.46960142272695	0.82422758074811

表 5 Filon 方法、匹配法求解 $y(x)+\int_0^x \frac{J_0(\omega t)}{\sqrt{t}} y(x-t)dt=\sin(x)$ $(\omega=10^8)$

x	0.1	0.5	1
Q_{10}^F	0.09981253487447	0.47932525900365	0.84129497754997
$Q_{10}^{L,0}$	0.09981253487324	0.47932525900275	0.84129497754926
$Q_{100}^{L,0}$	0.09981253487543	0.47932525900435	0.84129497755031
$Q_{10}^{L,1}$	0.09981253487447	0.47932525900362	0.84129497754985
$Q_{100}^{L,1}$	0.09981253487447	0.47932525900361	0.84129497754983

分段常数或线性连续匹配法具有以下的误差分析: 设

$$f \in C^1[0,1], \quad 0 \leqslant \alpha < 1,$$

$\{t_j\}_{j=1}^N$ 为一致分布, $h=1/N$. 分段常数或线性连续匹配法误差为

$$\max_{1\leqslant j\leqslant N} |\widehat{y}_j - y(t_j)| = O\left(\frac{h^{1-\alpha}}{\omega^{1-\alpha}}\right).$$

注 2 上述间接方法可直接推广到弱奇异高振荡 Bessel 核 Fredholm 积分方程的计算, 也很容易应用于弱奇异高振荡 Bessel 核第一类 Volterra 方程 [114] (见表 6–7)、弱奇异高振荡三角函数核的 Volterra 和 Fredholm 积分方程的求解.

表 6 Filon 方法、匹配法求解 $\int_0^x \frac{J_0(\omega t)}{\sqrt{t}} y(x-t)dt=\sin(x)$ $(\omega=10^4)$

x	0.1	0.5	1
Q_{10}^F	4.77301777319465	22.91705064883280	40.22228925024157
$Q_{10}^{L,0}$	4.77158713415073	22.91469534218337	40.22037690297926
$Q_{100}^{L,0}$	4.77410835173220	22.91826457983211	40.22275820099042
$Q_{10}^{L,1}$	4.77301777319465	22.91698632215610	40.22200508730764
$Q_{100}^{L,1}$	4.77301442196750	22.91696447036630	40.22196494232428

表 7　Filon 方法、匹配法求解 $\int_0^x \frac{J_0(\omega t)}{\sqrt{t}} y(x-t)dt = \sin(x)$ $(\omega = 10^8)$

x	0.1	0.5	1
Q_{10}^{F}	0.47719255862206e+3	2.29160037728700e+3	4.02213705066743e+3
$Q_{10}^{L,0}$	0.47719253063212e+3	2.29160035676694e+3	4.02213703440968e+3
$Q_{100}^{L,0}$	0.47719258060041e+3	2.29160039331785e+3	4.02213705821737e+3
$Q_{10}^{L,1}$	0.47719255862206e+3	2.29160037664509e+3	4.02213704782770e+3
$Q_{100}^{L,1}$	0.47719255859097e+3	2.29160037642550e+3	4.02213704742428e+3

4　结束语

一些数学物理问题具有深远的物理意义和重大的应用价值, 例如众所周知的量子物理、天体物理中的 Schrödinger 方程、Boltzmann 方程、电磁、声波散射中的 Maxwell 方程、Helmholtz 方程、Weierstrass 积分方程等. 数学物理问题的研究背景在许多情形下具有高频 (或大波数)、呈现高振荡性, 传统的数值方法存在求解难度, 因此高振荡问题的理论与计算被公认为是科学计算领域最具挑战性研究领域之一. 本文从积分方程解法角度出发, 探讨了高振荡问题的理论与计算核心基础之一的广义 Fourier 变换、Bessel 变换的高效算法、高振荡积分方程求解的最新进展.

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参考文献

1　Asheim A, Huybrechs D. Local solutions to high frequency $2D$ scattering problems. J Comput Phys, 2009, 229: 5357–5372
2　Cakoni F, Colton D. Qualitative Methods in Inverse Scattering Theory. Berlin: Springer, 2006
3　Colton D, Kress R. Integral equation methods in scattering theory. New York: Wiley, 1983
4　Langdon S, Chandler-Wilde S N. A wavenumber independent boundary element method for an acoustic scattering problem. SIAM J Numer Anal, 2006, 43: 2450–2477
5　Nédélec J C. Acoustic and Electromagnetic Equations. Applied Mathematical Sciences, vol. 144. Berlin: Springer, 2001
6　Trefethen L N. Spectral Methods in MATLAB. Philadelphia: SIAM, 2000
7　Condon M, Deaño A, Iserles A. On second order differential equations with highly oscillatory forcing terms. Proc Royal Soc A, 2010, 466: 1809–1828
8　Cohen D, Hairer E, Lubich C. Modulated Fourier expansions of highly oscillatory differential equations. Found Comput Math, 2003, 3: 327–345
9　Engquist B, Ying L. Fast directional algorithms for the Helmholtz kernel. J Comput Appl Math, 2010, 234: 1851–1859
10　Candes E, Demanet L, Ying L. Fast computation of Fourier integral operators. SIAM J Sci Comput, 2007, 29: 2464–2493
11　Cox S M, Matthews P C. Exponential time differencing for stiff systems. J Comput Phys, 2002, 176: 430–455
12　Condon M, Deano A, Iserles A, et al. On numerical methods for highly oscillatory problems in circuit simulation. Compel, 2009, 28: 1607–1618
13　Camlibel M K, Heemels W P M H, Schumacher J M. Consistency of a time-stepping method for a class of piecewise-linear networks. IEEE Trans Circuits Syst I Fund Theo Appl, 2009, 49: 349–357
14　Han L, Tiwari A, Camlibel M K, et al. Convergence of time-stepping schemes for passive and extended linear complementarity systems. SIAM J Numer Anal, 2009, 47: 3768–3796
15　Chen X, Xiang S. Implicit solution function of P_0 and Z matrix complementarity constraints. Math Program Ser A, 2011, 128: 1–18

16 Chen X, Xiang S. Newton iterations in implicit time-stepping scheme for differential linear complementarity systems. Math Program Ser A, 2012, in press
17 Chen Z, Wu H. An adaptive finite element method with perfectly matched absorbing layers for the wave scattering by periodic structures. SIAM J Numer Anal, 2003, 41: 799–826
18 Chen Z, Liu X. An adaptive perfectly matched layer technique for time-harmonic scattering problems. SIAM J Numer Anal, 2005, 43: 645–671
19 Chen Z, Wang L, Zheng W. An adaptive multilevel method for time-harmonic Maxwell equations with singularities. SIAM J Sci Comput, 2007, 29: 118–138
20 Chen J, Chen Z. An adaptive perfectly matched layer technique for 3-D time-harmonic electromagnetic scattering problems. Math Comput, 2008, 77: 673–698
21 Chen Z, Xiao Y, Zhang L. The adaptive immersed interface finite element method for elliptic and Maxwell interface problems. J Comput Phys, 2009, 228: 5000–5019
22 Chen Z, Guo B, Xiao Y. An hp addaptive uniaxial perfectly matched layer method for Helmholtz scattering problems. Comm Comput Phys, 2009, 5: 546–564
23 Chen Z, Zheng W. Convergence of the uniaxial perfectly matched layer method for time-harmonic scattering problems in two-layered media. SIAM J Numer Anal, 2010, 48: 2158–2185
24 Bao G, Wu H. Convergence analysis of the PML problems for time-harmonic Maxwell's equations. SIAM J Numer Anal, 2005, 43:2121–2143
25 Bao G, Li P, Wu H. An adaptive edge element method with perfectly matched absorbing layers for wave scattering by biperiodic structures. Math Comput, 2010, 79: 1–34
26 Candes E, Demanet L, Ying L. A fast butterfly algorithm for the computation of Fourier integral operators. SIAM J Multi Model Simul, 2009, 7: 1727–1750
27 Demanet L, Ying L. Scattering in flatland: efficient representations via wave atoms. Found Comput Math, 2010, 10: 569–613
28 Demanet L, Ying L. Wave atoms and time upscaling of wave equations. Numer Math, 2009, 113: 1–71
29 Engquist B, Ying L. A fast directional algorithm for high frequency acoustic scattering in two dimensions. Comm Math Sci, 2009, 7: 327–345
30 Engquist B, Ying L. Sweeping preconditioner for the Helmholtz equation: Moving perfectly matched layers. Multi Model Simul, 2011, 9: 686–710
31 Engquist B, Ying L. Sweeping preconditioner for the Helmholtz equation: Hierarchical matrix representation. Comm Pure Appl Math, 2011, 64: 697–735
32 Feng X and Wu H. hp-discontinuous Galerkin methods for the Helmholtz equation with large wave number. Math Comput, 2011, 80: 1997–2024
33 Feng X, Wu H. Discontinuous Galerkin methods for the Helmholtzequation with large wave number. SIAM J Numer Anal, 2009, 47: 2872–2896
34 Feng X, Wu H. A posteriori error estimates and an adaptive finite element method for the Allen-Cahn equation and the mean curvature flow. J Sci Comput, 2005, 24: 121–146
35 Li P, Wu H, Zheng W. Electromagnetic scattering by unbounded rough surfaces. SIAM J Math Anal, 2011, 43: 1205–1231
36 Qian J, Ying L. Fast multiscale Gaussian wavepacket transforms and multiscale Gaussian beams for the wave equation. SIAM Multiscale Model Simul, 2010, 8: 1803–1837
37 Tsuji P, Ying L. A fast directional algorithm for high-frequency electromagnetic scattering. J Comput Phys, 2011, 230: 5471–5487
38 Shen J, Wang L. Analysis of a spectral-Galerkin approximation to the Helmholtz equation in exterior domains. SIAM J Numer Anal, 2007, 45: 1954–1978
39 Davies P J, Duncan D B. Stability and convergence of collocation schemes for retarded potential integral equations. SIAM J Numer Anal, 2004, 42: 1167–1188
40 Bao G, Sun W W. A fast algorithm for the electromagnetic scattering from a large cavity. SIAM J Sci Comput, 2005, 27: 553–574
41 http://www.newton.ac.uk/programmes/HOP/index.html
42 Iserles A. Think globally, act locally: Solving highly-oscillatory ordinary differential equations. Appl Numer Math, 2002, 43: 145–160
43 Iserles A. On the method of Neumann series for highly oscillatory equations. BIT, 2004, 44: 473–488
44 Hairer E, Lubich C. Long-time energy conservation of numerical methods for oscillatory differential equations. SIAM

J Numer Anal, 2001, 38: 414–441
45 Brunner H, Iserles A, Nørsett P. Open problems in the computational solution of Volterra functional equations with highly oscillatory kernel. Isaac Newton Institute: HOP 2007: Effective Computational Methods for Highly Oscillatory Solutions, 2007
46 Brunner H, Davies P J, Duncan D B. Discontinuous Galerkin approximations for Volterra integral equations of the first kind. IMA J Numer Anal, 2009, 29: 856–881
47 Brunner H. On the numerical solution of first-kind Volterra integral equations with highly oscillatory kernels. Isaac Newton Institute: HOP 2009: Highly Oscillatory Problems: to Applications, 2009
48 Linz P. Product integration methods for Volterra integral equations of the first kind. BIT, 1971, 11: 413–421
49 de Hoog F, Weiss R. On the solution of Volterra integral equations of the first kind. Numer Math, 1973, 21: 22–32
50 Baker C T H. A perspective on the numerical treatment of Volterra equations. J Comput Appl Math, 2000, 125: 217–249
51 Iserles A. On the numerical quadrature of highly-oscillating integrals I: Fourier transforms. IMA J Numer Anal, 2004, 24: 365–391
52 Iserles A. On the numerical quadrature of highly-oscillating integrals II: Irregular oscillators. IMA J Numer Anal, 2005, 25: 25–44
53 Filon L N G. On a quadrature formula for trigonometric integrals. Proc Royal Soc Edinburgh, 1928, 49: 38–47
54 Flinn E A. A modification of Filon's method of numerical integration. J ACM, 1960, 9: 181–184
55 Luke Y K. On the computation of oscillatory integrals. Proc Cambridge Philos Soc, 1954, 50: 269–277
56 Levin D. Procedures for computing one-and-two dimensional integrals of functions with rapid irregular oscillations. Math Comput, 1982, 38: 531–538
57 Olver S. Moments-free numerical integration of highly oscillatory functions. IMA J Numer Anal, 2006, 26: 213–227
58 Piessens R, Poleunis F. A numerical method for the integration of oscillatory functions. BIT, 1971, 11: 317–327
59 Sloan I H, Smith W E. Product-integration with the Clenshaw-Curtis and related points. Numer Math, 1978, 30: 415–428
60 Sloan I H, Smith W E. Product integration with the Clenshaw-Curtis points: implementation and error estimates. Numer Numer Math, 1980, 34: 387–401
61 Xiang S, Chen X, Wang H. Error bounds for approximation in Chebyshev points. Numer Math, 2010, 116: 463–491
62 Domínguez V, Graham I G, Smyshlyaev V P. Stability and error estimates for Filon-Clenshaw-Curtis rules for highly-oscillatory integrals. IMA J Numer Anal, 2011, 31: 1253–1280
63 Iserles A, Nørsett S P. On quadrature methods for highly oscillatory integrals and their implementation. BIT, 2004, 44: 755–772
64 Iserles A, Norsett S P. Efficient quadrature of highly-oscillatory integrals using derivatives. Proc Roy Soc A, 2005, 461: 1383–1399
65 Milovanovic G V. Numerical calculation of integrals involving oscillatory and singular kernels and some applications of quadratures. Comput Math Appl, 1998, 36: 19–39
66 Huybrechs D, Vandewalle S. On the evaluation of highly oscillatory integrals by analytic continuation. SIAM J Numer Anal, 2006, 44: 1026–1048
67 Deaño A, Huybrechs D. Complex Gaussian quadrature of oscillatory integrals. Numer Math, 2009, 112: 197–219
68 Wong R. Asymptotic Approximations of Integrals. New York: Academic Press, 1989
69 Iserles A, Norsett S P, Olver S. Highly oscillatory quadrature: The story so far. In: Bermudez de Castro A, ed. Proceedings of ENumath, Santiago de Compostela 2006. Berlin: Springer, 2006, 97–118
70 Xiang S. Efficient Filon-type methods for $\int_a^b f(x)\mathrm{e}^{i\omega g(x)}dx$. Numer Math, 2007, 105: 633–658
71 Iserles A, Nørsett S P. Quadrature methods for multivariate highly oscillatory integrals using derivatives. Math Comput, 2006, 75: 1233–1258
72 Olver S. On the quadrature of multivariate highly oscillatory integrals over non-polytope domains. Numer Math, 2006, 103: 643–665
73 Huybrechs D, Vandewalle S. The construction of cubature rules for multivariate highly oscillatory integrals. Math Comput, 2007, 76: 1955–1980
74 Olver S. Shifted GMRES for oscillatory integrals. Numer Math, 2010, 114: 607–628
75 Li J B, Wang X S, Wang T. A universal solution to one-dimensional oscillatory integrals. Sci China Ser F Inf Sci, 2008, 51: 1614–1622
76 Xiang S. On the Filon and Levin methods for highly oscillatory integral $\int_a^b f(x)\mathrm{e}^{i\omega g(x)}dx$. J Comput Appl Math, 2007, 208: 434–439

77 Xiang S. Efficient quadrature for highly oscillatory integrals involving critical points. J Comput Appl Math, 2007, 206: 688–698
78 Abramowitz M, Stegun I A. Handbook of Mathematical Functions. Washington: National Bureau of Standards, 1964
79 Levin D. Fast integration of rapidly oscillatory functions. J Comput Appl Math, 1996, 67: 95–101
80 Xiang S, Gui W, Mo P. Numerical quadrature for Bessel transformation. Appl Numer Math, 2008, 58: 1247–1261
81 Xiang S. Numerical analysis of a fast integration method for highly oscillatory functions. BIT, 2007, 47: 469–482
82 Chung K C, Evans G A, Webster J R. A method to generate generalized quadrature rules for oscillatory integrals. Appl Numer Math, 2000, 34: 85–93
83 Evans G A, Webster J R. A high order, progressive method for the evaluation of irregular oscillatory integrals. Appl Numer Math, 1997, 23: 205–218
84 Evans G A, Chung K C. Some theoretical aspects of generalised quadrature methods. J Complexity, 2003, 19: 272–285
85 Xiang S, Gui W. On generalized quadrature rules for fast oscillatory integrals. Appl Math Comp, 2008, 197: 60–75
86 Xiang S. On quadrature of Bessel transformations. J Comput Appl Math, 2005, 177: 231–139
87 Xiang S, Wang H. Fast integration of highly oscillatory integrals with exotic oscillators. Math Comput, 2010, 79: 829–844
88 Gradshteyn I S, Ryzhik I M. Table of Integrals, Series, and Products. New York: Academic Press, 1994
89 Luke Y L. The special functions and their approximations, vol. 2. New York-London: Academic Press, 1969
90 Watson G N. A Treatise on the Theory of Bessel Functions. New York: Cambridge University Press, 1952
91 Olver S. Numerical approximation of vector-valued highly oscillatory integrals. BIT, 2007, 47: 637–655
92 Xiang S, Cho Y, Wang H, et al. Clenshaw-Curtis-Filon-type method for highly oscillatory Bessel transforms and applications. IMA J Numer Anal, 2011, 31: 1281–1314
93 Piessens R, Branders M. Modified Clenshaw-Curtis method for the computation of Bessel function integrals. BIT, 1983, 23: 370–381
94 Piessens R. Automatic computation of Bessel function integrals. Comput Phys Commun, 1982, 25: 289–295
95 Oliver J. The numerical solution of linear recurrence relations. Numer Math, 1968, 11: 349–360
96 Wu J M, Sun W W. The superconvergence of Newton-Cotes rules for the Hadamard finite-part integral on an interval. Numer Math, 2008, 109: 143–165
97 Okecha G E. Quadrature formulae for Cauchy principal value integrals of oscillatory kinds. Math Comput, 1987, 49: 259–268
98 Capobianco M R, Criscuolo G. On quadrature for Cauchy principal value integrals of oscillatory functions. J Comput Appl Math, 2003, 156: 471–486
99 Wang H, Xiang S. Uniform approximations to Cauchy principal value integrals of oscillatory functions. Appl Math Comp, 2009, 215: 1886–1894
100 Hascelik A I. On numerical computation of integrals with integrands of the form $f(x)\sin(w/x^r)$ on $[0,1]$. J Comput Appl Math, 2009, 223: 399–408
101 Wang H, Xiang S. On the evaluation of Cauchy principal value integrals of oscillatory functions. J Comput Appl Math, 2010, 234: 95–100
102 Xiang S, Wang H. On the Levin iteration method for oscillatory integrals. J Comput Appl Math, 2008, 217: 38–45
103 Chen R, Xiang S, Zhou Y. A paprameter method for computing highly oscillatory integrals. Comp Math Appl, 2009, 58: 1830–1837
104 Chen R, Xiang S. Note on the homotopy perturbation method for multivariate vector-value oscillatory integrals. Appl Math Comp, 2009, 215: 78–84
105 Kang H, Xiang S. On the calculation of highly oscillatory integrals with an algebraic singularity. Appl Math Comp, 2010, 216: 3890–3897
106 Mo H, Xiang S. On the asymptotic order of Filon-type methods for highly oscillatory integrals with an algebraic singularity. Appl Math Comp, 2011, 217: 9105–9110
107 Kang H, Xiang S. Efficient integration for a class of highly oscillatory integrals. Appl Math Comp, 2011, 218: 3553–3564
108 Wang H, Xiang S. Asymptotic expansion and Filon-type methods for a Volterra integral equation with a highly oscillatory kernel. IMA J Numer Anal, 2011, 31: 469–490
109 Samko S G, Kilbas A A, Marichev O I. Fractional Integrals and Derivatives: Theory and Applications. New York: Gordon & Breach Sci Publ, 1993
110 Polyanin A D, Manzhirov A V. Handbook of Integral Equations. London: CRC Press, 1998
111 Aleksandrov V M, Kovalenko E V. Mathematical method in the displacement problem. Inzh Zh Mekh Tverd Tela,

1984, 2: 77–89

112 Kiryakova V, Al-Saqabi B. Explicit solutions to hyper-Bessel integral equations of second kind. Comp Math Appl, 1999, 37: 75–86

113 Xiang S, Brunner H. Efficient Methods for Volterra Integral Equations with Highly Oscillatory Bessel Kernels. Technical Report, Central South University, 2011

114 Xiang S, He K. Discontinuous Galerkin method for weakly singular Volterra integral equations with highly oscillatory Bessel kernels. Technical Report, Central South University, 2011

Efficient methods for some highly oscillatory integrals and integral equations

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Abstract A number of mathematical and physical problems occur in the areas of electromagnetics and acoustic scattering simulations, which are of important theoretical values and have wide applications. High oscillation plays a crucial role and represents formidable mathematical and computational challenge. Highly oscillatory differential equations and integral equations are two fundamental models for these problems, whose computations are difficult and of many challenging problems. From the view of reformulation by means of integral equations, this paper gives a survey on the new developments on highly oscillatory problems, particularly, the details on generalized Fourier transforms, Bessel transforms and Volterra integral equations with highly oscillatory kernels. These methods share that the higher the frequency the more accurate of the numerical solution, which provides a new way of solving these kinds of equations.

Keywords **highly oscillatory problem, highly oscillatory integral, highly oscillatory differential equation, highly oscillatory integral equation, efficient method**

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