SOME FIXED POINT THEOREMS FOR SET-CONTRACTION OPERATORS

SUN JINGXIAN (孙经先)
(*Department of Mathematics, Shandong University, Jinan*)
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In this paper we study the computation of the topological degree and give some fixed point theorems for strict-set-contraction operators. These theorems extend and improve a series of results in [1], [2] and [4].

Theorem 1. *Let X be an infinite dimensional Banach space, set $B_r=\{x|x\in X, \|x\|<r\}$, and $A:\bar{B}_r\to X$ a k-set-contraction operator, $k<1$. Suppose that*

(i) *there exists $\delta>0$ such that $\|Ax\|\geqslant(k+\delta)\|x\|$ for $x\in\partial B_r$;*

(ii) *$Ax\neq\mu x$ for $x\in\partial B_r$ and $0<\mu\leqslant 1$.*

Then $\deg(I-A, B_r, 0)=0$.

Remark 1. The following simple example shows that condition (i) of Theorem 1 is optimal. Let l^2 denote the Banach space of all real-number sequence $x=(x_1, x_2,\cdots)$ with norm $\|x\|=\left(\sum_{n=1}^{\infty}x_n^2\right)^{\frac{1}{2}}$. Let $A:l^2\to l^2$ be the mapping defined by

$$Ax=\left(0,\frac{1}{2}x_1,\frac{1}{2}x_2,\cdots\right)\text{ for } x=(x_1,x_2,\cdots).$$

Then A is a $\frac{1}{2}$-set-contraction mapping. Obviously,

$$\|Ax\|\geqslant\frac{1}{2}\|x\|$$

for $x\in\partial B_1=\{x|x\in l^2,\|x\|=1\}$ and $Ax\neq\mu x$ for $x\in\partial B_1, 0<\mu\leqslant 1$. If $\deg(I-A,B_1,0)=0$, then we can easily prove that there exist $x_0\in\partial B_1, \lambda_0>0$ such that $Ax_0=\lambda_0x_0$. On the other hand, $Ax\neq\lambda x$ for any $x\in\partial B_1$, we have arrived at a contradiction. Hence $\deg(I-A,B_1,0)\neq 0$.

Theorem 2. *Let X be an infinite dimensional Banach space, $B_r=\{x|x\in X, \|x\|<r\}$, $B_R=\{x|x\in X,\|x\|<R\}, R>r>0$. Let $A:\bar{B}_R\to X$ be a k-set-contraction operator, $k<1$. Suppose that one of the following conditions is satisfied:*

(i) *$\|Ax\|\geqslant\|x\|$ for $x\in\partial B_R$ and $\|Ax\|\leqslant\|x\|$ for $x\in\partial B_r$;*

(ii) *$\|Ax\|\leqslant\|x\|$ for $x\in\partial B_R$ and $\|Ax\|\geqslant\|x\|$ for $x\in\partial B_r$.*

Then A has a fixed point on $\bar{B}_R\setminus B_r$ at least.

From Theorem 1 we can obtain the following corollaries:

Corollary 1. *Let X be an infinite dimensional Banach space, $A:\bar{B}_R \to X$ a strict-set-contraction operator. Suppose that A maps ∂B_R into ∂B_R. Then A has a fixed point on ∂B_R at least.*

Corollary 2. *Let X be an infinite dimensional Hilbert space, $A:\bar{B}_R \to X$ a strict-set-contraction operator. Suppose that $(Ax, x) = (x, x)$ for $x \in \partial B_R$. Then A has a fixed point on ∂B_R at least.*

Corollary 3. *Let X be an infinite dimensional Banach space, $A:\bar{B}_R \to X$ a k-set-contraction operator (we do not suppose $k \leqslant 1$). Suppose that*

$$\inf_{x \in \partial B_R} \|Ax\| = c > 0, \quad c > kR.$$

Then (i) A has at least a positive eigenvalue λ, corresponding to eigenvector $x_\lambda \in \partial B_R$; (ii) A has at least a negative eigenvalue μ, corresponding to eigenvector $x_\mu \in \partial B_R$.

Remark 2. Corollary 3 is a generalization of the Birkhoff-Kellogg theorem[3].

Using Theorem 2 we can prove the following corollary.

Corollary 4. *Let X be an infinite dimensional Banach space, $A:\bar{B}_R \to X$ a condensing operator. Suppose that there exists $\delta > 0$ such that one of the following conditions is satisfied:*

(i) $\|Ax\| \geqslant (1 + \delta)\|x\|$ *for* $x \in \partial B_R$ *and* $\|Ax\| \leqslant \|x\|$ *for* $x \in \partial B_r$;

(ii) $\|Ax\| \leqslant \|x\|$ *for* $x \in \partial B_R$ *and* $\|Ax\| \geqslant (1 + \delta)\|x\|$ *for* $x \in \partial B_r$.

Then A has a fixed point on $\bar{B}_R \backslash B_r$ at least.

Theorem 3. *Let X be a Banach space, P a cone in X, $B_r = \{x | x \in X, \ \|x\| < r\}$. Let $A:\bar{B}_r \cap P \to P$ be a k-set-contraction operator, $k < 1$. Suppose that*

(i) *there exists $\delta > 0$ such that $\|Ax\| \geqslant (k + \delta)\|x\|$ for $x \in \partial B_r \cap P$;*

(ii) $Ax \neq \mu x$ *for* $x \in \partial B_r \cap P$ *and* $0 < \mu \leqslant 1$.

Then $i(A, B_r \cap P, P) = 0$, where $i(A, B_r \cap P, P)$ is the fixed point index of A over $B_r \cap P$ with respect to P.

Remark 3. If A is a completely continuous operator, then Theorem 3 has been proved in [4].

Theorem 4. *Let X be a Banach space, P a cone in X, $B_r = \{x \in X | \|x\| < r\}$, $B_R = \{x | x \in X, \|x\| < R\}, R > r > 0$. Let $A:\bar{B}_R \cap P \to P$ be a strict-set-contraction operator. Suppose that one of the following conditions is satisfied:*

(i) $\|Ax\| \geqslant \|x\|$ *for* $x \in \partial B_R \cap P$ *and* $\|Ax\| \leqslant \|x\|$ *for* $x \in \partial B_r \cap P$;

(ii) $\|Ax\| \leqslant \|x\|$ *for* $x \in \partial B_R \cap P$ *and* $\|Ax\| \geqslant \|x\|$ *for* $x \in \partial B_r \cap P$.

Then A has a fixed point on $(\bar{B}_R \backslash B_r) \cap P$ at least.

Remark 4. Theorem 4 is the generalization of Theorems 1.2 and 1.3 in [5] (see also Corollaries 3.10 and 3.11 in [6]).

Remark 5. Corollaries 1—4 in this paper can also be extended to the case that A is a cone mapping.

We can also prove the following theorem:

Theorem 5. *Let X be a Banach space, P a normal cone, Ω a bounded open set in X, $\theta\in\Omega$. Let $A:\bar{\Omega}\cap P\to P$ be a k-set-contraction operator. Suppose that the following conditions are satisfied:*

(i) *$kN<1$, where N is a normal constant of P, i. e. if $\theta\leqslant x\leqslant y$, then $\|x\|\leqslant N\|y\|$;*

(ii) *there exists $\delta>0$ such that $\|Ax\|\geqslant(Nk+\delta)\|x\|$ for any $x\in\partial\Omega\cap P$;*

(iii) *$Ax\neq\mu x$ for $x\in\partial\Omega\cap P$ and $0<\mu\leqslant 1$.*

Then $i(A,\Omega\cap P,P)=0$.

Remark 6. As applications of Théorems 3 and 5, we can extend Theorems 2.1 and 2.4 of Chapter V in [7] and the main results in [5] and [6] to the case that A is a set-contraction operator.

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