SCIENCE CHINA
Mathematics

• REVIEWS •

March 2010 Vol. 53 No. 3: 513–522
doi: 10.1007/s11425-010-0045-3

Value distribution theory and the research of Yang Lo

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Received November 19, 2009; accepted January 14, 2010

Abstract Value distribution theory is concerned with the position and frequency of solutions of the equation $f(z) = a$. Here f may be entire, i.e. an everywhere convergent power series or meromorphic, i.e. the ratio of two such series, or a function in some other domains, such as an angle or a disk.

Yang Lo's significant contributions to this area will be highlighted. Some of his important contributions to normal families will also be described.

Keywords meromorphic function, value distribution, deficient value, angular distribution

MSC(2000): 30D10, 30D20

Citation: Hayman W K, Wu S J. Value distribution theory and the research of Yang Lo. Sci China Math, 2010, 53(3): 513–522, doi: 10.1007/s11425-010-0045-3

0 Introduction

We consider functions of a complex variable $z = x + iy$ in the open plane $\mathbb{C}$ and in particular 4 classes of such functions

a) the class P of polynomials

$$f(z) = a_0 + a_1 z + \cdots + a_d z^d, \tag{1}$$

where the a_j are constants and d is the degree of $f(z)$;

b) the class R of rational functions

$$f(z) = p_1/p_2, \tag{2}$$

where p_1 and p_2 are polynomials;

c) the class E of entire functions

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \tag{3}$$

i.e. everywhere convergent power series;

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d) the class M of meromorphic functions

$$f(z) = f_1/f_2, \tag{4}$$

where f_1 and f_2 are entire.

Value distribution theory is concerned with the distribution of the roots of the equation

$$f(z) = a, \tag{5}$$

when f belongs to one of these classes or the corresponding functions in other domains. A good account of the subject is given in Yang's monograph [85].

If P is a polynomial of degree d, then it follows from the Fundamental Theorem of Algebra that for all values of a, the equation (5) has exactly d roots, counting multiplicity. We say that (5) has a root of multiplicity q at $z = z_0$ if

$$f(z_0) = a, \quad f'(z_0) = 0, \quad \ldots, \quad f^{(q-1)}(z_0) = 0, \quad f^{(q)}(z_0) \neq 0.$$

If $q > 1$, then f' has a zero of multiplicity $q - 1$ at z_0. Suppose now that f is a polynomial of degree d. Then f' has exactly $d - 1$ zeros. Thus there can be at most $d - 1$ values a for which the equation (5) has a multiple root. For all other values of a the equation (5) has precisely d distinct simple roots. If the equation $f(z) = a_\nu$ has a root of multiplicity $q_\mu(a_\nu)$ at $z_{\mu,\nu}$, then f' has a zero of multiplicity $q_\mu(a_\nu) - 1$ at $z_{\mu,\nu}$. Thus

$$\sum_{\mu,\nu}(q_\mu(a_\nu) - 1) = d - 1. \tag{6}$$

Suppose that for some a the equation (5) has only multiple roots. Then we say that a is totally branched. A polynomial can have one totally branched value (e.g. $f(z) = z^d$) but no more. For if a and b are totally branched, then (6) shows that for at least one such value, say a,

$$\sum_\mu q_\mu(a) \leqslant 2\sum_\mu(q_\mu(a) - 1) \leqslant d - 1,$$

which contradicts $\sum_\mu q_\mu(a) = d$. A rather more complicated argument shows that a rational function can have at most 3 totally branched values. For instance $f(z) = (z + 1/z)^2$ is totally branched at the values 0, 4 and ∞.

When we try to extend the theory to entire or meromorphic functions, the situation becomes more complicated. The function

$$f(z) = \mathrm{e}^z$$

never assumes the value 0. However this is the worst case. Picard [16] showed that every transcendental entire function, i.e. not a polynomial, assumes every finite value with at most one exception infinitely often. Picard was justly proud of this famous theorem and there is a story that for some time afterwards every Paris PhD thesis had to begin with the words "it follows from the famous theorem of Picard$\cdots$."

For polynomials of positive degree d all values are assumed equally often, counting multiplicity (CM), i.e. precisely d times. In order to generalize this result to entire functions we have to introduce a concept of size for such functions. This was accomplished by Hadamard [6] and Borel [1]. They used the Maximum Modulus

$$M(r, f) = \sup_{|z|=r} |f(z)|, \quad 0 \leqslant r < \infty. \tag{7}$$

They also defined the order of f by comparing $M(r,\ f)$ with e^{r^ρ}, i.e.,

$$\rho = \rho(f) = \lim_{r\to\infty} \sup \frac{\log\log M(r, f)}{\log r}. \tag{8}$$

We also introduce the counting function $n(t,a)$ for the number of roots of (5) in $|z| \leqslant t$ (CM). It turns out to be useful to define also the logarithmic integral

$$N(r,a) = n(0,a)\log r + \int_0^r \frac{n(t,a)-n(0,a)}{t}dt. \tag{9}$$

This simplifies to

$$N(r,a) = \int_0^r \frac{n(t,a)}{t}dt, \quad \text{if } a \neq f(0).$$

If we want to count distinct roots of $f(z) = a$, i.e., ignoring multiplicity (IM), we write $\overline{n}$, $\overline{N}$ instead of n, N in (9). The order of the number of roots is then defined analogously to (8) by

$$\rho(a,f) = \lim_{r\to\infty}\sup \frac{\log n(r,a)}{\log r} = \lim_{r\to\infty}\sup \frac{\log N(r,a)}{\log r}, \tag{10}$$

with a similar definition for the order $\overline{\rho}(a)$ of the number of distinct roots. We then have the following results:

For all values of a,

$$\overline{\rho}(a) \leqslant \rho(a) \leqslant \rho(f). \tag{11}$$

Equality holds for all values of a except for at most one so-called Borel exceptional value. We can only have $\rho(a) < \rho(f)$ for some a if $\rho(f)$ is a positive integer or ∞. For instance if

$$f(z) = \mathrm{e}^{z^p} + a, \quad \text{or } \mathrm{e}^{\mathrm{e}^z} + a,$$

then $f(z) \neq a$ and f has order p or ∞ respectively.

Julia also showed that every entire function has a Julia direction $\arg z = \theta_0$. This means that f assumes every value with at most one exception infinitely often in every angle

$$\theta_0 - \varepsilon < \arg z < \theta_0 + \varepsilon, \quad \text{when } \varepsilon > 0.$$

The theory of Hadamard and Borel, beautiful as it is, has some disadvantages.

(i) It has nothing to say about meromorphic functions.

(ii) It has only rather weak information about functions of infinite order.

(iii) The notion of a Borel exceptional value is quite strong. Can we compare the growth of $n(r,a)$ or $\overline{n}(r,a)$ for different values of a more precisely?

These problems were largely overcome by Nevanlinna [13] in what has been described by Hermann Weyl as one of the finest achievement of 20th century Mathematics.

We consider from now on meromorphic functions (see (4)). These are functions f such that at every point f or $1/f$ is holomorphic. If $1/f = 0$ at some point z_0 we say that $f = \infty$, or f has a pole at z_0. The counting functions $n, \overline{n}, N$ and $\overline{N}$ can then be extended to the value $a = \infty$, if f is meromorphic. Nevanlinna's starting point was a result, which the author Jensen [9], with no false modesty, described as "a new and important theorem in function theory". It is the formula

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(r\mathrm{e}^{i\theta})|d\theta + N(r,\infty) - N(r,0), \tag{12}$$

in the case that $f(0) \neq 0, \infty$.

If we write, for $x \geqslant 0$, $\log^+ x = \max(\log x, 0)$, so that

$$\log x = \log^+ x - \log^+ \frac{1}{x},$$

we obtain from (12)

$$m(r,\infty) + N(r,\infty) = m(r,0) + N(r,0) + \log|f(0)|, \tag{13}$$

where

$$m(r,\infty,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta, \tag{14}$$

and, for finite a,

$$m(r,a,f)=m\bigg(r,\infty,\frac{1}{f-a}\bigg).$$

Nevanlinna now noted that if we replace f by $f-a$, where a is a finite constant, then $N(r,\infty)$ is unaltered and $m(r,\infty)$ is only altered by a bounded amount.

Applying (13) to $f-a$ we obtain for every a

$$m(r,\infty)+N(r,\infty)=m(r,a)+N(r,a)+O(1),\qquad \text{as } r\to\infty. \tag{15}$$

Nevanlinna defined the characteristic function $T(r,f)$ as the left-hand side of (15), namely

$$T(r,f)=m(r,\infty,f)+N(r,\infty,f). \tag{16}$$

Now (15) is Nevanlinna's first fundamental theorem (FFT).

We can now define the order of the meromorphic function f by

$$\rho(f)=\lim_{r\to\infty}\sup\frac{\log T(r,f)}{\log r}. \tag{17}$$

For entire functions f we have the inequality

$$T(r,\,f)\leqslant\log M(r,\,f)\leqslant\frac{R+r}{R-r}T(R,\,f),\quad 0\leqslant r<R<\infty,$$

and taking e.g. $R=2r$ we see that the order defined by (17) coincides with that given by (8) in this case. We cannot use $M(r,f)$ to defined the order for meromorphic functions, since $M(r,f)=\infty$, whenever f has a pole on $|z|=r$.

The FFT shows that a meromorphic function f has the same "affinity" for every complex value a, including ∞. If $N(r,a)$ is smaller, $m(r,a)$ must be bigger. Nevanlinna's second fundamental theorem (SFT) shows that it is usually m, that is the smaller term. We define

$$\delta(a)=\lim_{r\to\infty}\inf\frac{m(r,a)}{T(r,\,f)}=1-\lim_{r\to\infty}\sup\frac{N(r,a)}{T(r,\,f)}. \tag{18}$$

We can also take into account multiplicities by defining

$$\theta(a)=\lim_{r\to\infty}\inf\frac{N(r,a)-\overline{N}(r,a)}{T(r,\,f)}. \tag{19}$$

Then

$$\delta(a)+\theta(a)\leqslant\Theta(a)=1-\lim_{r\to\infty}\sup\frac{\overline{N}(r,a)}{T(r,\,f)}. \tag{20}$$

If $\delta(a)>0$, then a is called a deficient value of f and $\delta(a)$ is the deficiency. Now SFT states that

$$\sum_a(\delta(a)+\theta(a))\leqslant\sum_a\Theta(a)\leqslant 2. \tag{21}$$

Drasin [3] has shown that all $\delta(a)$ and $\theta(a)$ allowed by SFT can actually occur. We can have $\theta(0)=\theta(\infty)=1$ for a function of any order ρ, $0\leqslant\rho\leqslant\infty$. However $\sum\delta(a)<2$, unless ρ is a positive integer or ∞.

If f assumes the value a only with multiplicity at least p, we have

$$\overline{N}(r,a)\leqslant\frac{1}{p}N(r,a)\leqslant\frac{1}{p}T(r,\,f)+O(1),$$

so that

$$\Theta(a) \geqslant 1 - \frac{1}{p}.$$

If $f \neq a$, then $\Theta(a) = \delta(a) = 1$. Thus f can omit at most 2 (finite or infinite) values. Also f in M can have at most 4 totally branched values, while an entire f can have at most 2 totally branched values. Thus if f is in one of our classes P, E, R and M, f can have at most respectively 1, 2, 3 and 4 totally branched values. The examples z^2, $\cos z$, $(z - 1/z)^2$, which is totally branched at 0, ∞ and -4, and Weierstrass's function $\wp(z)$, which is totally branched at e_1, e_2, e_3 and ∞, show that these estimates are all sharp.

1 Angular distributions of meromorphic functions

We are now ready to describe some of the work of Yang Lo. In some cases collaborating with Zhang Guanghou they obtained significant results in several aspects of the value distribution theory of meromorphic functions. In this section we discuss their contributions to the theory of angular distributions of meromorphic functions (ADMF).

What are the main subjects in ADMF? Generally speaking, they are the singular directions. First let us define the Borel exceptional value of a meromorphic function in some unbounded domains. Let $f(z)$ be a meromorphic function of finite positive order λ and let Ω be an unbounded domain in the complex plane. We say that $a \in \bar{\mathcal{C}}$ is a Borel exceptional value of f in Ω if the following inequality holds:

$$\lim_{r\to\infty} \sup \frac{\log n(D_r \cap \Omega, f = a)}{\log r} < \lambda,$$

where $D_r = \{z; |z| \leqslant r\}$.

In order to explain the notation of singular directions of a meromorphic function, we need some more preparations. A property of meromorphic functions is called a Picard type property, if whenever a meromorphic function $f(z)$ fails to have the property, then $f(z)$ must be a constant function. For example, the property that a meromorphic function has at most two Picard values is a typical Picard type property.

To give another example, we note that the first author studied the value distributions of meromorphic functions with their derivatives [7]. As a consequence of the main result in [7] we have a Picard type property as follows. If a meromorphic function $f(z)$ in the complex plane has a finite Picard value, then $f'(z)$ cannot have a non-zero Picard value.

Corresponding to each Picard type property, there is a Borel type property. For example, the property that a meromorphic function of finite positive order has at most two Borel exceptional values in the plane is a Borel type property. Corresponding to Hayman's result, the following Borel type property is true. If $f(z)$ has a finite Borel exceptional value in the plane then $f'(z)$ cannot have any non-zero finite Borel exceptional value.

We can now give an informal definition of the singular direction of a meromorphic function. A ray from the origin is a singular direction of a meromorphic function $f(z)$, if some property that for all non-constant meromorphic functions holding in the whole plane, also holds in any angular domain containing the ray.

We know that, corresponding to Picard's theorem, any transcendental meromorphic function $f(z)$ which satisfies the condition that

$$\lim_{r\to\infty} \sup \frac{T(r,f)}{(\log r)^2} = \infty$$

must have a Julia direction. Another basic result in ADMF is due to Valiron who showed that for any meromorphic function of finite positive order there exists at least one Borel direction (refer to [85]).

Corresponding to Hayman's result, a natural problem is, is there a Hayman direction for every transcendental meromorphic function? The existence of the Hayman direction of Picard type is due to Yang [45] and the existence of the Hayman direction of Borel type is due to Yang and Zhang [51].

Yang Lo also proved some existence theorems of other singular directions which involve the multiple values as well as the derivatives of the meromorphic functions [37].

As we pointed out above, one of the basic results in angular distributions of meromorphic functions is the existence of Borel directions. Let $f(z)$ be a meromorphic function of finite positive order. It is easy to see that the set $\{e^{i\vartheta}; \arg z = \vartheta$ is a Borel direction of $f\}$ forms a closed set in the unit circle. Yang Lo and Zhang Guanghou proved another basic result in the theory of ADMF.

Theorem 1 [30]. *Let E be a closed set in the unit circle and let a positive λ be given. Then there exists a meromorphic function f of order λ such that*

$$E = \{\mathrm{e}^{i\vartheta}; \arg z = \vartheta \textit{ is a Borel direction of } f\}.$$

For entire functions, the distributions of Borel directions must have some restrictions and the existence problem was solved by Drasin and Weitsman [4]. The restriction for the distribution of Borel directions of an entire function follows from a result of Cartwright which says that for an entire function of finite order $\lambda > \frac{1}{2}$, there must exist at least two Borel directions $\arg z = \vartheta_1; \vartheta_2$ such that $0 < \vartheta_1 - \vartheta_2 \leqslant \frac{\pi}{\lambda}$ (refer to [85]). Yang and Zhang [24] generalized the result of Cartwright to the case of meromorphic functions with at least one deficient value (finite or infinite).

An old problem of Valiron asks whether f and f' must have a common Borel direction. Milloux [10] proved this for entire functions and Yang [35] proved that every Borel direction of f' is also a Borel direction of f, if ∞ is a Borel exceptional value of f. Zhang [18] proved that every Borel direction of f is also a Borel direction of f', if f has a finite Borel exceptional value.

Now we introduce another [28, 29] striking and surprising result of Yang Lo and Zhang Guanghou. As we know, the deficient value is one of the most important concept in the modular distribution of meromorphic functions and the Borel direction is the most important one in ADMF. The Yang-Zhang Theorem below establishes a close relation between the number of deficient values and that of the Borel directions.

Theorem 2 [28, 29]. *If f is a meromorphic function of finite positive order and p, q are respectively the number of finite deficient values and the number of Borel directions of f, then $p \leqslant q$. If f is entire, $p \leqslant q/2$. Both these inequalities are sharp.*

On the face of it Borel directions seem to have nothing to do with deficient values, which makes this theorem all the more striking. Tragically Zhang Guanghou died in 1987 at the relatively young age of 50.

Finally in this section we would like to mention the joint work of Yang Lo and the first author on functions in an angle [36, 44]. They discussed a conjecture that Littlewood made in about 1930 and that Mary Cartwright discovered in a note book of his. Littlewood's hypothesis need to be strengthened, but they showed that if f has order ρ on a sufficiently dense set of arcs in an angle S and if f has zeros of order ρ in an interior angle S' then f assumes all values a with order at least ρ in S.

2 The deficient values of meromorphic functions

In this section we mainly discuss some of the contributions of Yang Lo to the modular distributions of meromorphic functions. In [19], Yang Lo introduced an extension of the deficiency relation involving multiplicities. This has been called by Gol'dberg [5] the Yang Lo relation and is sharp. Yang Lo also proved the inequality

$$\sum_{a\in\mathbb{C}} \delta(a, f^{(k)}) \leqslant \frac{2k+2}{2k+1},$$

sharpening an earlier inequality of Mues [12].

Yang [67] and working with Wang [70], solving some problems of Drasin, also proved the relation

$$\sum_{a\in\mathbb{C}} \delta(a, f) + \sum_{b\in\bar{\mathbb{C}}} \delta(b, f^{(k)}) \leqslant 3, \tag{22}$$

and obtained all cases of equality.

Suppose that $a(z)$ is a function small compared with f, in the sense that

$$T(r, a) = o\{T(r, f)\}, \quad \text{as } r \to \infty.$$

Then Yang [39] investigated to what extent small functions $a(z)$ can replace constants a in the SFT (21).

3 Normal families

A family of functions $\mathcal{F}$ is called normal, following Montel [11], if from every sequence $\{f_n\}$ of functions in $\mathcal{F}$ we can select a locally uniformly convergent subsequence $\{f_{n_p}\}$. There is a heuristic principle called Bloch's princliple which can be roughly stated as follows. If for a family $\mathcal{F}$ of functions in the open plane $\mathbb{C}$ certain properties imply that $\mathcal{F}$ consists only of constants, then the corresponding properties for the class $\mathcal{F}(D)$ of functions in a subdomain D of $\mathbb{C}$ imply that the functions in $\mathcal{F}(D)$ form a normal family. Yang and Zhang [23] proved that this principle applies to the family of meromorphic functions f, such that the zeros of f have order at least m and the zeros of $f^{(k)} - 1$ have order at least n, where

$$\frac{k+1}{m} + \frac{1}{n} < 1.$$

Taking $k = 1, n = \infty$ and writing $F = f^{m+1}/(m+1)$, we see that if f is holomorphic and $f'f^m \neq 1$ the above results apply to F if $m \geqslant 2$. So the corresponding family in the unit disk is normal. This answers a problem (see [8, Problem 5.12]) originally posed by Xiong as Drasin points out [2, p. 252] at least for $m \geqslant 2$. The case $m = 1$ was latter settled by Oshkin [14]. A proof of Bloch's principle in many cases has been given by Zalcman [17] and extended by Pang and Zalcman [15].

Acknowledgements The second author is partially supported by National Natural Science Foundation of China (Grant Nos. 10671004, 10831004) and the Doctoral Education Program Foundation of China (Grant No. 20060001003). The authors are most grateful for the help of several people in the preparation of the paper. Our fellow lecturers David Drasin and Lawrence Zalcman cleared up a number of priority questions for us.

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