

Left-symmetric algebra structures on the twisted Heisenberg-Virasoro algebra

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Abstract The compatible left-symmetric algebra structures on the twisted Heisenberg-Virasoro algebra with some natural grading conditions are completely determined. The results of the earlier work on left-symmetric algebra structures on the Virasoro algebra play an essential role in determining these compatible structures. As a corollary, any such left-symmetric algebra contains an infinite-dimensional nontrivial subalgebra that is also a submodule of the regular module.

Keywords twisted Heisenberg-Virasoro algebra, left-symmetric algebra, Virasoro algebra

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1 Introduction

The notion of left-symmetric algebras was originally introduced by Vinberg, which are also called Vinberg algebras [19], Koszul algebras or quasi-associative algebras [13]. Cayley firstly introduced a kind of left-symmetric algebras (LSAs, for short), called rooted tree algebras in [5]. LSAs are closely related to vector fields, rooted tree algebras, vertex algebras, operad theory, convex homogeneous cones, affine manifolds, Lie groups, Lie algebras and so on. For more information on origins and the applications of LSAs see the survey paper [4] by Burde. Recently much attention has been paid to such objects and many related papers appeared (i.e., [2, 4, 7, 10–12, 17]). We know that LSAs and RSAs (the opposite algebras of LSAs, called right-symmetric algebras) are examples of Lie-admissible algebras, i.e., the commutator defines a Lie bracket. An important project related to LSAs is to determine all the compatible left-symmetric algebra structures on a Lie algebra. The left-symmetric algebra structures on the Virasoro algebra were investigated in [12] and [14], whose super cases were determined in [11]. Recently, the authors classified all the compatible left-symmetric algebra structures on the W -algebra $W(2, 2)$ in [6].

In the present paper, we classify all the left-symmetric algebra structures on the twisted Heisenberg-Virasoro algebra with similar natural grading conditions. The twisted Heisenberg-Virasoro algebra \mathcal{H} ,

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introduced by Arbarello et al. [1] more than twenty years ago, is an infinite-dimensional Lie algebra with a \mathbb{C} -basis $\{L_n, I_n, c_1, c_2, c_3 \mid n \in \mathbb{Z}\}$ and the following non-vanishing brackets:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c_1, \\ [L_m, I_n] &= -nI_{m+n} + (m^2-m)\delta_{m+n,0}c_2, \\ [I_m, I_n] &= n\delta_{m+n,0}c_3. \end{aligned} \quad (1.1)$$

The twisted Heisenberg-Virasoro algebra is the central extension of the Lie algebra $\{f(t)\frac{d}{dt} + g(t) \mid f, g \in \mathbb{C}[t, t^{-1}]\}$ of differential operators of order at most one [1] and contains the infinite-dimensional Heisenberg algebra (the subalgebra spanned by $\{c_3, I_n \mid n \in \mathbb{Z}\}$) and the Virasoro algebra (the subalgebra spanned by $\{c_1, L_n \mid n \in \mathbb{Z}\}$, denoted by \mathcal{V}) as its subalgebras, which both play important roles in mathematical physics. Until now the structures and representations of the twisted Heisenberg-Virasoro have been investigated in many papers, among which Arbarello et al. [1] studied the irreducible highest weight representations for \mathcal{H} , Billig [3] described the structure of the irreducible highest weight modules for the twisted Heisenberg-Virasoro Lie algebra at level zero, Shen and Jiang [18] determined its derivation algebra and automorphism group, Lu and Zhao [16] gave a complete classification of irreducible Harish-Chandra modules, Liu and Jiang [15] classified all indecomposable Harish-Chandra modules of the intermediate series and Fu et al. [9] exhausted all the possible nontrivial superextensions of this type algebra.

Throughout this paper, \mathbb{C} and \mathbb{Z} denote the field of complex numbers and the ring of integers, respectively. Unless specified otherwise, all the Lie algebras and left-symmetric algebras are defined over \mathbb{C} and we assume that $\epsilon \in \mathbb{C}$ possesses the following properties:

$$\operatorname{Re} \epsilon > 0, \quad \epsilon^{-1} \notin \mathbb{Z} \quad \text{or} \quad \operatorname{Re} \epsilon = 0, \quad \operatorname{Im} \epsilon > 0.$$

2 Preliminaries and main results

In this section, we introduce some definitions and notation for left-symmetric algebras, and follow with some results about the left-symmetric algebra structures on the Virasoro algebra.

Definition 2.1. Let \mathcal{A} be a vector space over a field \mathbb{F} equipped with a bilinear product $(x, y) \mapsto xy$. \mathcal{A} is called a *left-symmetric algebra* if for any $x, y, z \in \mathcal{A}$, the associator

$$(x, y, z) = (xy)z - x(yz)$$

is symmetric in x, y , that is,

$$(x, y, z) = (y, x, z), \quad \text{or equivalently} \quad (xy)z - x(yz) = (yx)z - y(xz).$$

Modules over left-symmetric algebras can also be defined [8]. In particular, for any left-symmetric algebra \mathcal{A} , its underlying vector space can be endowed with the natural \mathcal{A} -module structure: $(a, m) \mapsto am$, $(m, a) \mapsto ma$, $a, m \in \mathcal{A}$.

Left-symmetric algebras are Lie-admissible algebras (see [17]).

Proposition 2.2. Let \mathcal{A} be a left-symmetric algebra. For any $x \in \mathcal{A}$, denote by L_x the left multiplication operator (i.e., $L_x(y) = xy$, for all $y \in \mathcal{A}$).

(1) The commutator

$$[x, y] = xy - yx, \quad \forall x, y \in \mathcal{A},$$

defines a Lie algebra $\mathcal{G}(\mathcal{A})$, which is called the sub-adjacent Lie algebra of \mathcal{A} and \mathcal{A} is called a compatible left-symmetric algebra structure on the Lie algebra $\mathcal{G}(\mathcal{A})$.

(2) Let $L : \mathcal{G}(\mathcal{A}) \rightarrow \mathfrak{gl}(\mathcal{A})$ with $x \mapsto L_x$. Then (L, \mathcal{A}) gives a representation of the Lie algebra $\mathcal{G}(\mathcal{A})$, that is,

$$[L_x, L_y] = L_{[x, y]}, \quad \forall x, y \in \mathcal{A}.$$

Let $\rho : \mathcal{G} \rightarrow \mathfrak{gl}(V)$ be a representation of any Lie algebra \mathcal{G} . A 1-cocycle $q : \mathcal{G} \rightarrow V$ is a linear map on vector space associated with ρ (denoted by (ρ, q)) satisfying

$$q[x, y] = \rho(x)q(y) - \rho(y)q(x), \quad \forall x, y \in \mathcal{G}.$$

Let \mathcal{A} be a left-symmetric algebra and $\rho : \mathcal{G}(\mathcal{A}) \rightarrow \mathfrak{gl}(V)$ be a representation of its sub-adjacent Lie algebra. If g is a homomorphism of the representations from \mathcal{A} to V , then g is a 1-cocycle of $\mathcal{G}(\mathcal{A})$ associated with ρ . There is not always a compatible left-symmetric algebra structure on any Lie algebra \mathcal{G} . A sufficient and necessary condition for a Lie algebra with a compatible left-symmetric algebra structure is given as follows.

Proposition 2.3. *Let \mathcal{G} be a Lie algebra. Then there is a compatible left-symmetric algebra structure on \mathcal{G} if and only if there exists a bijective 1-cocycle of \mathcal{G} .*

In fact, let (ρ, q) be a bijective 1-cocycle of \mathcal{G} , then

$$x * y = q^{-1}\rho(x)q(y), \quad \forall x, y \in \mathcal{G},$$

defines a left-symmetric algebra structure on \mathcal{G} . Conversely, for a left-symmetric algebra \mathcal{A} , the identity transformation id is a 1-cocycle of $\mathcal{G}(\mathcal{A})$ associated with the regular representation L (see [10, 17]).

To avoid confusion, for a Lie algebra \mathcal{G} we denote a compatible left-symmetric algebra on \mathcal{G} by $\mathcal{A}(\mathcal{G})$. A compatible left-symmetric algebra structure on the Virasoro algebra $\mathcal{V} = \bigoplus_{n \in \mathbb{Z}} L_n \oplus \mathbb{C}c_1$ is said to have the natural grading condition if the multiplication of $\mathcal{A}(\mathcal{V})$ satisfies

$$c_1 c_1 = c_1 L_m = L_m c_1 = 0, \quad L_m L_n = f(m, n)L_{m+n} + \omega(m, n)c_1, \quad (2.1)$$

for two complex-valued functions $f(m, n), \omega(m, n)$ on $\mathbb{Z} \times \mathbb{Z}$. The condition (2.1) is said to be natural because it means that $\mathcal{A}(\mathcal{V})$ is still \mathbb{Z} -graded and c_1 is also a central extension given by $\omega(m, n)$. Such left-symmetric algebra structures were classified in [12].

Theorem 2.4. *Any left-symmetric algebra structure on the Virasoro algebra \mathcal{V} satisfying (2.1) is isomorphic to one of the left-symmetric algebras given by the multiplication*

$$L_m L_n = \frac{-n(1 + \epsilon n)}{1 + \epsilon(m + n)} L_{m+n} + \frac{c_1}{24} (m^3 - m + (\epsilon - \epsilon^{-1})m^2) \delta_{m+n, 0}.$$

The Lie algebra \mathcal{H} contains a Heisenberg subalgebra and a Virasoro subalgebra, and is also \mathbb{Z} -graded: $\mathcal{H} = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_m$, where $\mathcal{H}_m = \mathbb{C}L_m \oplus \mathbb{C}I_m \oplus \delta_{m,0}(\mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}c_3)$ for all $m \in \mathbb{Z}$. Then it is natural to suppose that the compatible left-symmetric algebra structures satisfy similar conditions. Hence we can safely make the following assumptions on $\mathcal{A}(\mathcal{H})$:

$$\begin{aligned} L_m L_n &= f(m, n)L_{m+n} + \omega(L_m, L_n)c_1, \\ L_m I_n &= g(m, n)I_{m+n} + \omega(L_m, I_n)c_2, \\ I_m L_n &= h(m, n)I_{m+n} + \omega(I_m, L_n)c_2, \\ I_m I_n &= a(m, n)L_{m+n} + b(m, n)I_{m+n} + \omega(I_m, I_n)c_3, \\ c_i c_j &= c_i L_m = L_m c_i = c_i I_n = I_n c_i = 0, \end{aligned} \quad (2.2)$$

where $f(m, n), g(m, n), h(m, n), \omega(\cdot, \cdot)$ are all complex-valued functions.

The main result of this paper can be formulated as the following theorem.

Theorem 2.5. *Any left-symmetric algebra structure on the twisted Heisenberg-Virasoro algebra \mathcal{H} satisfying relation (2.2) is isomorphic to one of the left-symmetric algebras determined by the following*

functions:

$$\begin{aligned}
 f(m, n) &= \frac{-n(1 + \epsilon n)}{1 + \epsilon(m + n)}, \quad g(m, n) = -n(1 + (1 - \epsilon n)\alpha\delta_{m+n,0}), \\
 h(m, n) &= n(1 + \epsilon n)\alpha\delta_{m+n,0}, \quad a(m, n) = b(m, n) = 0, \\
 \omega(L_m, L_n) &= \frac{1}{24}(m^3 - m + (\epsilon - \epsilon^{-1})m^2)\delta_{m+n,0}, \\
 \omega(L_m, I_n) &= (m^2 - m + (\epsilon m^2 + m)\beta)\delta_{m+n,0}, \\
 \omega(I_m, L_n) &= n(1 + \epsilon n)\beta\delta_{m+n,0}, \quad \omega(I_m, I_n) = \frac{n}{2}\delta_{m+n,0}
 \end{aligned} \tag{2.3}$$

for any $m, n \in \mathbb{Z}$ and some constants $\alpha, \beta \in \mathbb{C}$.

By the main theorem, the following corollary is obvious.

Corollary 2.6. Let $\mathcal{A}(\mathcal{H})$ be a compatible left-symmetric algebra on \mathcal{H} satisfying relation (2.2). Denote $\mathcal{A}(\mathcal{V}) = \text{span}_{\mathbb{C}}\{L_n, c_1 \mid n \in \mathbb{Z}\}$, $\mathcal{A}_2 = \text{span}_{\mathbb{C}}\{I_n, c_2 \mid n \in \mathbb{Z}\}$, $\mathcal{A}_3 = \text{span}_{\mathbb{C}}\{I_n, c_3 \mid n \in \mathbb{Z}\}$ and $\mathcal{A}_{23} = \mathcal{A}_2 + \mathcal{A}_3$. Then all $\mathcal{A}(\mathcal{V})$, \mathcal{A}_3 and \mathcal{A}_{23} are infinite-dimensional subalgebras of $\mathcal{A}(\mathcal{H})$. Furthermore, \mathcal{A}_{23} is both an $\mathcal{A}(\mathcal{H})$ -module and an $\mathcal{A}(\mathcal{V})$ -module. However, the $\mathcal{A}(\mathcal{V})$ -module \mathcal{A}_2 is not always an $\mathcal{A}(\mathcal{H})$ -submodule.

3 Proof of Theorem 2.5

We will divide the proof of main theorem into two steps. The first step is to determine the left-symmetric algebra structures on the centerless Heisenberg-Virasoro algebra, denoted by \mathcal{H} , which is defined by the relations (1.1) with $c_i = 0$. Then we obtain the central extensions of the left-symmetric algebra on \mathcal{H} , i.e., the left-symmetric algebra structures on \mathcal{H} .

3.1 The centerless case

The grading conditions of the left-symmetric algebra structures on \mathcal{H} are those given in the relations (2.2) with $c_i = 0$, which give the following lemma.

Lemma 3.1. A bilinear product defined by (2.2) with $c_i = 0$ gives a compatible left-symmetric algebra structure on \mathcal{H} if and only if

$$\begin{aligned}
 f(m, n) - f(n, m) &= m - n, \quad g(m, n) - h(n, m) = -n, \\
 a(m, n) &= a(n, m), \quad b(m, n) = b(n, m), \\
 f(n, k)f(m, n + k) - f(m, k)f(n, m + k) &= (m - n)f(m + n, k), \\
 g(n, k)g(m, n + k) - g(m, k)g(n, m + k) &= (m - n)g(m + n, k), \\
 h(n, k)g(m, n + k) - f(m, k)h(n, m + k) &= -nh(m + n, k), \\
 h(n, k)a(m, n + k) - h(m, k)a(n, m + k) &= 0, \\
 h(n, k)b(m, n + k) - h(m, k)b(n, m + k) &= 0, \\
 a(n, k)f(m, n + k) - g(m, k)a(n, m + k) &= -na(m + n, k), \\
 b(n, k)g(m, n + k) - g(m, k)b(n, m + k) &= -nb(m + n, k), \\
 b(n, k)a(m, n + k) - b(m, k)a(n, m + k) &= 0, \\
 a(n, k)h(m, n + k) + b(n, k)b(m, n + k) &= a(m, k)h(n, m + k) + b(m, k)b(n, m + k)
 \end{aligned} \tag{3.1}$$

hold for all $m, n, k \in \mathbb{Z}$.

By Theorem 2.4, in order to obtain a compatible left-symmetric algebra structure on \mathcal{H} with the grading condition (2.2), one can suppose

$$f(m, n) = \frac{-n(1 + \epsilon n)}{1 + \epsilon(m + n)}. \tag{3.2}$$

Furthermore, we have the following technical lemma that determines all the compatible left-symmetric algebra structures on \mathcal{H} with $f(m, n)$ defined by (3.2).

Lemma 3.2. *Fixing some $\alpha, \epsilon, a, b \in \mathbb{C}$ and the corresponding $f(m, n)$ defined by (3.2), there is only the following solution simultaneously satisfying all equations in (3.1):*

$$\begin{aligned} h(m, n) &= n(1 + \epsilon n)\alpha\delta_{m+n,0}, & g(m, n) &= -n(1 + (1 - \epsilon n)\alpha\delta_{m+n,0}), \\ a(m, n) &= \frac{a\delta_{\alpha,0}}{1 + \epsilon(m+n)}, & b(m, n) &= b\delta_{\alpha,0}. \end{aligned} \quad (3.3)$$

Hence these functions define a compatible left-symmetric algebra structure on \mathcal{H} .

Proof. It is easy to check that the complex-valued functions $f(m, n)$, $g(m, n)$, $h(m, n)$, $a(m, n)$ and $b(m, n)$ given in (3.3) simultaneously satisfy (3.1).

Conversely, for any $m, n \in \mathbb{Z}$, introduce the notation:

$$G(m, n) = \frac{1 + \epsilon(m+n)}{1 + \epsilon n}g(m, n), \quad H(n, m) = \frac{1 + \epsilon(m+n)}{1 + \epsilon m}h(n, m),$$

then (3.1) implies the following relations:

$$(1 + \epsilon n)G(m, n) - (1 + \epsilon m)H(n, m) = -n(1 + \epsilon(m+n)), \quad (3.4)$$

$$G(n, k)G(m, n+k) - G(m, k)G(n, m+k) = (m-n)G(m+n, k), \quad (3.5)$$

$$H(n, k)G(m, n+k) + kH(n, m+k) = -nH(m+n, k). \quad (3.6)$$

Taking $m = 0$ in (3.4) and in (3.6), we obtain $H(n, k)H(n+k, 0) = 0$. Immediately, we have

$$H(n, 0) = 0, \quad G(0, n) = -n, \quad \forall n \in \mathbb{Z}.$$

Noticing that (3.6) implies

$$H(0, k)G(m, k) + kH(0, m+k) = 0$$

for all $m, k \in \mathbb{Z}$, thus, if there is some $k \neq 0$ such that $H(0, k) = 0$, then $H(0, m) = 0$ for all $m \in \mathbb{Z}$. According to the above discussion, the proof of this lemma shall be fall into two cases.

Case 1. $H(0, l) = 0$ for all $l \in \mathbb{Z}$.

In this case, taking $n = -k = m$ in (3.6), we obtain $H(-2k, k) = 0$ for all $k \in \mathbb{Z}$. Observing the sequence

$$H(-2k, k) \xrightarrow{n=-2k=2m, (3.6)} H(-3k, k) \xrightarrow{n=-3k=3m, (3.6)} \cdots \xrightarrow{n=-ak=am, (3.6)} H(-(a+1)k, k) \cdots,$$

one can deduce

$$H(-am, m) = 0, \quad G(m, -am) = am \frac{1 + \epsilon(m-am)}{1 - \epsilon am}, \quad \forall a, m \in \mathbb{Z}, a \geq 2.$$

Similarly, one has

$$H(0, k) \xrightarrow{n=k=-m, (3.6)} H(k, k) \xrightarrow{n=2k=-2m, (3.6)} \cdots \xrightarrow{n=ak=-am, (3.6)} H(ak, k) \cdots,$$

which implies $H(am, m) = 0$ for all $a \in \mathbb{Z} \setminus \{-1\}$. By these vanishing cases and (3.6), for $m+n \neq 0$, we have

$$H(m, n) = 0, \quad \forall m, n \in \mathbb{Z}, \quad m+n \neq 0.$$

Now, taking $m+n+k=0$ in (3.6), we have

$$kH(n, -n) = -nH(-k, k), \quad \forall n, k \in \mathbb{Z}.$$

If we denote $H(-1, 1)$ by α , then we have $H(-k, k) = k\alpha$, $\forall k \in \mathbb{Z}$.

Case 2. $G(-k, k) = 0$, for all $k \in \mathbb{Z}$.

Firstly, taking $m + n + k = 0$ in (3.6), we obtain

$$kH(n, -n) = -nH(-k, k), \quad \forall n, k \in \mathbb{Z},$$

which implies

$$\frac{kn}{1 - \epsilon n} = \frac{kn}{1 + \epsilon k}, \quad \forall n, k \in \mathbb{Z}$$

by using (3.4). But this is impossible.

According to (3.1), we have obtained

$$f(m, n) = \frac{-n(1 + \epsilon n)}{1 + \epsilon(m + n)}, \quad h(m, n) = n(1 + \epsilon n)\alpha\delta_{m+n,0}, \quad g(m, n) = -n(1 + (1 - \epsilon n)\alpha\delta_{m+n,0}).$$

The left thing we have to do is to calculate $a(m, n)$ and $b(m, n)$. If $\alpha = 0$, then one has $g(m, n) = -n$ and $h(m, n) = 0$. Similarly as before, setting $A(m, n) = (1 + \epsilon(m + n))a(m, n)$, we have the relation

$$(n + k)A(n, k) = kA(n, m + k) + nA(m + n, k). \quad (3.7)$$

Taking $k = 0$ in the above relation, we obtain

$$A(m, 0) = A(0, n) = A(0, 0), \quad \forall m, n \in \mathbb{Z}.$$

Considering the case $n = k$ in (3.7) and using $A(m, n) = A(n, m)$, one has $A(m, n) = a$, which implies

$$a(m, n) = \frac{a}{1 + \epsilon(m + n)}$$

for some constant $a \in \mathbb{C}$ and for all $m, n \in \mathbb{Z}$. Similarly, we have $b(m, n) = b$ for some constant $b \in \mathbb{C}$.

If $\alpha \neq 0$, we have $h(n, -n)a(m, 0) = 0$ by taking $n = -k \neq m$ in the sixth relation in (3.1). This implies $a(m, 0) = 0$. Thus $g(m, k)a(n, m + k) = -ka(n, m + k)$. By the similar discussion to that of the $\alpha = 0$ case, we get

$$a(m, n) = b(m, n) = 0, \quad \forall m, n \in \mathbb{Z}.$$

The proof of Lemma 3.2 is now complete. □

3.2 Left-symmetric algebra structures on the twisted Heisenberg-Virasoro algebra

We give our classification of left-symmetric algebra structures on the twisted Heisenberg-Virasoro algebra by determining the central extensions of the left-symmetric algebra structures on \mathcal{H} . First, one has the following lemma:

Lemma 3.3. *A multiplication defined by relations in (2.2) gives a compatible left-symmetric algebra structure on \mathcal{H} if and only if (3.1) and the following identities hold for all $m, n, k \in \mathbb{Z}$,*

$$\begin{aligned} \omega(L_m, L_n) - \omega(L_n, L_m) &= \frac{m^3 - m}{12} \delta_{m+n,0}, \\ \omega(L_m, I_n) - \omega(I_n, L_m) &= (m^2 - m) \delta_{m+n,0}, \\ \omega(I_m, I_n) - \omega(I_n, I_m) &= n \delta_{m+n,0}, \\ f(n, k) \omega(L_m, L_{n+k}) - f(m, k) \omega(L_n, L_{m+k}) &= (m - n) \omega(L_{m+n}, L_k), \\ g(n, k) \omega(L_m, I_{n+k}) - g(m, k) \omega(L_n, I_{m+k}) &= (m - n) \omega(L_{m+n}, I_k), \\ h(n, k) \omega(L_m, I_{n+k}) - f(m, k) \omega(I_n, L_{m+k}) &= -n \omega(I_{m+n}, L_k), \\ h(n, k) \omega(I_m, I_{n+k}) &= h(m, k) \omega(I_n, I_{m+k}), \\ g(m, k) \omega(I_n, I_{m+k}) - n \omega(I_{m+n}, I_k) &= 0, \\ a(n, k) \omega(L_m, L_{n+k}) &= b(n, k) \omega(L_m, I_{n+k}) = 0, \\ a(n, k) \omega(I_m, L_{n+k}) &= a(m, k) \omega(I_n, L_{m+k}), \\ b(n, k) \omega(I_m, I_{n+k}) &= b(m, k) \omega(I_n, I_{m+k}). \end{aligned} \quad (3.8)$$

According to Theorem 2.4 and Lemma 3.2, one can suppose

$$f(m, n) = \frac{-n(1 + \epsilon n)}{1 + \epsilon(m + n)}, \quad \omega(L_m, L_n) = \frac{1}{24}(m^3 - m + (\epsilon - \epsilon^{-1})m^2)\delta_{m+n,0}, \quad (3.9)$$

$$h(m, n) = n(1 + \epsilon n)\alpha\delta_{m+n,0}, \quad (3.10)$$

$$g(m, n) = -n(1 + (1 - \epsilon n)\alpha\delta_{m+n,0}), \quad b(m, n) = b\delta_{\alpha,0}, \quad (3.11)$$

for some constants $b, \alpha \in \mathbb{C}$ and it is clear that (3.9) implies that $a(m, n) = 0$ for all $m, n \in \mathbb{Z}$.

Proof of Theorem 2.5. It is easy to check that the complex-valued functions $f(m, n)$, $g(m, n)$, $h(m, n)$, $a(m, n)$, $b(m, n)$ and $\omega(\cdot, \cdot)$ given in (3.3) simultaneously satisfy (3.1) and (3.8). Set $\varphi(m, n) = \omega(L_m, I_n)$ and $\psi(m, n) = \omega(I_m, L_n)$. Then we have the relations

$$\varphi(m, n) - \psi(n, m) = (m^2 - m)\delta_{m+n,0}, \quad (3.12)$$

$$g(n, k)\varphi(m, n + k) - g(m, k)\varphi(n, m + k) = (m - n)\varphi(m + n, k), \quad (3.13)$$

$$h(n, k)\varphi(m, n + k) - f(m, k)\psi(n, m + k) = -n\psi(m + n, k), \quad (3.14)$$

which together give

$$\psi(m, 0) = \varphi(m, 0) = \psi(0, m) = \varphi(0, m) = 0, \quad \forall m \in \mathbb{Z}.$$

Then (3.13) and (3.14) become

$$k\varphi(n, m + k) - k\varphi(m, n + k) = (m - n)\varphi(m + n, k), \quad (3.15)$$

and

$$k\frac{\psi(n, m + k)}{1 + \epsilon(m + k)} = -n\frac{\psi(m + n, k)}{1 + \epsilon k}. \quad (3.16)$$

Setting $m = 0$ in the above equations, one can suppose

$$\varphi(m, n) = \delta_{m+n,0}\varphi(m), \quad \psi(m, n) = (1 + \epsilon n)\delta_{m+n,0}\psi(m).$$

Now taking $m + n + k = 0$ in (3.15) and (3.16), we obtain

$$(m - n)\varphi(m + n) = (m + n)(\varphi(m) - \varphi(n)), \quad (m + n)\psi(n) = n\psi(m + n).$$

By induction on m, n , one can deduce

$$\varphi(m) = \frac{m^2 - m}{2}\varphi(2) - (m^2 - 2m)\varphi(1) \quad \text{and} \quad \psi(m) = -m\psi(-1).$$

Plugging them into (3.12), we obtain

$$\frac{m^2 - m}{2}\varphi(2) - (m^2 - 2m)\varphi(1) - m(1 + \epsilon m)\psi(-1) = m^2 - m,$$

which implies

$$\varphi(2) = 2(1 + \psi(-1) + 2\epsilon\psi(-1)), \quad \varphi(1) = (1 + \epsilon)\psi(-1). \quad (3.17)$$

Denoting $\psi(-1)$ by β , we get

$$\begin{aligned} \omega(L_m, I_n) &= \varphi(m)\delta_{m+n,0} = (m^2 - m + (\epsilon m^2 + m)\beta)\delta_{m+n,0}, \\ \omega(I_m, L_n) &= (1 + \epsilon n)\psi(m)\delta_{m+n,0} = n(1 + \epsilon n)\beta\delta_{m+n,0}. \end{aligned}$$

Furthermore, recalling the ninth relation in (3.8), one has $b(m, n) = 0$, $\forall m, n \in \mathbb{Z}$.

Finally, we aim to determining $\omega(I_m, I_n)$. First, we have $(n + k)\omega(I_n, I_k) = 0$ by taking $m = 0$ in the eighth relation in (3.8). Then we can suppose $\omega(I_m, I_n) = \theta(m)\delta_{m+n,0}$. Taking $k = 0$ in (3.8), one has $\theta(0) = 0$. Then we can rewrite the relation in (3.8) as (taking $m + n + k = 0$)

$$(m + n)\theta(n) = n\theta(m + n).$$

Therefore, there exists some $\theta \in \mathbb{C}$ such that $\theta(n) = n\theta$. Taking it back to the third relation of (3.8), we obtain $\theta = -\frac{1}{2}$. Thus we obtain

$$\omega(I_m, I_n) = \theta(m)\delta_{m+n,0} = \frac{n}{2}\delta_{m+n,0}.$$

Then the main theorem follows. □

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