ON THE DIOPHANTINE EQUATION $\dfrac{ax^m-1}{abx-1}=by^2$

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Let $a, b \in \mathbb{N}$. In this note we study the solutions of the Diophantine equation

$$\frac{ax^m-1}{abx-1}=by^2, \qquad 2\nmid m>1 \tag{1}$$

with elementary method of Pell's equation[1], and the results we get is generalization of the studies of Ljunggren [2] ($a=b=1$) and Sun Qi et al. [3] ($b=1$).

Theorem 1. *If a, $b\in N$, then the Diophantine equation (1) only has solutions in positive integers $x=y=1$ (when $a>1$, $b=1$) and $m=4s+1$, $x=3$, $y=3^{2s}+2$ (when $a=\frac{1}{4}(3^{2s-1}+1)$, $b=1$), where s is a positive integer.*

For the Diophantine equation

$$\frac{ax^m+1}{abx+1}=by^2,\ 2\nmid m>1, \tag{2}$$

we have

Theorem 2. *If a, $b\in\mathbb{N}$, then the Diophantine* Eq. (2) *only has solutions in positive integers $x=y=1$ (when $b=1$) and $m=4s+3$, $x=3$, $y=3^{2s+1}-2$ (when $a=\frac{1}{4}(3^{2s}-1)$, $b=1$), where s is a positive integer.*

From Theorems 1 and 2, we have

Corollary. *If $a\in\mathbb{N}$, then the Diophantine equations*

$$ax^m=axy^{2n-2}+y^n-1,\ 2\nmid m>1,\ n>1,$$

$$ax^m=axy^{2n-2}-y^n+1,\ 2\nmid m>1,\ n>1$$

only have the solution $x=y=1$, respectively.

Proof of Theorem 1. Suppose (m, x, y) is any positive integer solution of (1). Then (1) gives

$$ax^m - b(abx-1)y^2 = 1,$$

and so

$$(ax^m + b(abx-1)y^2)^2 - 4abx(abx-1)(x^{\frac{m-1}{2}}y)^2 = 1. \qquad (2')$$

Since $\varepsilon = 2abx - 1 + \sqrt{4abx(abx-1)}$ is the fundamental solution of Pell's equation $u^2 - 4abx(abx-1)v^2 = 1$ [1], thus (2') gives

$$ax^m + b(abx-1)y^2 = \frac{\varepsilon^n + \bar{\varepsilon}^n}{2} = 2ax^m - 1, \qquad (3)$$

$$x^{\frac{m-1}{2}}y = \frac{\varepsilon^n - \bar{\varepsilon}^n}{2\sqrt{4abx(abx-1)}}, \qquad (4)$$

where $\bar{\varepsilon} = 2abx - 1 - \sqrt{4abx(abx-1)}$, $n \in \mathbb{N}$.

First, we suppose $2|n$. Put $n = 2l$, $l \in \mathbb{N}$. Then by (3), we have

$$2ax^m - 1 = \frac{\varepsilon^{2l} + \bar{\varepsilon}^{2l}}{2} = 2\left(\frac{\varepsilon^l + \bar{\varepsilon}^l}{2}\right)^2 - 1,$$

and so $ax = x_1^2$ $(x_1 \in \mathbb{N})$, $\frac{\varepsilon^l + \bar{\varepsilon}^l}{2} = x^{\frac{m-1}{2}} \cdot x_1$. From (4), we have

$$x^{\frac{m-1}{2}}y = 2 \cdot \frac{\varepsilon^l + \bar{\varepsilon}^l}{2} \cdot \frac{\varepsilon^l - \bar{\varepsilon}^l}{2\sqrt{}} = 2x^{\frac{m-1}{2}} \cdot x_1 \cdot \frac{\varepsilon^l - \bar{\varepsilon}^l}{2\sqrt{}}, \qquad (5)$$

where $\sqrt{} = \sqrt{4abx(abx-1)}$. Clearly, (5) gives $x_1 | y$, and so $x_1 = 1$ by (1). Thus $ax = x_1^2 = 1$, $y = 0$ which is impossible.

Next, we suppose $2 \nmid n$. From (3) and (4), we have $x = y = 1$ when $b = 1$, $a > 1$ if $n = 1$. If $n > 1$, and let p be any prime divisor of n, then $n = pl$, $2 \nmid l \in \mathbb{N}$. Put $\varepsilon^l = a_l + b_l\sqrt{}$, $\bar{\varepsilon}^l = a_l - b_l\sqrt{}$. Then (3) and (4) give respectively

$$2ax^m - 1 = a_l^p + \binom{p}{2}a_l^{p-2}(b_l\sqrt{})^2 + \cdots + \binom{p}{p-1}a_l(b_l\sqrt{})^{p-1}, \qquad (6)$$

$$x^{\frac{m-1}{2}}y = b_l\left[\binom{p}{1}a_l^{p-1} + \binom{p}{3}a_l^{p-3}(b_l\sqrt{})^2 + \cdots + \binom{p}{p}(b_l\sqrt{})^{p-1}\right]. \qquad (7)$$

Since $a_l + b_l\sqrt{} = \varepsilon^l \equiv (-1)^l + l(-1)^{l-1}\sqrt{} \pmod{x}$, thus by $2 \nmid l$ we have

$$a_l \equiv -1 \pmod{x}, \quad b_l \equiv l \pmod{x}. \qquad (8)$$

If $p \nmid x$, by $\sqrt{}^{\,2} \equiv 0 \pmod{x}$ and (8), we have

$$\left(x, \binom{p}{1} a_l^{p-1} + \binom{p}{3} a_l^{p-3}\left(b_l\sqrt{\ }\ \right)^2 + \cdots + \binom{p}{p}\left(b_l\sqrt{\ }\ \right)^{p-1}\right) = (x, p) = 1,$$

and thus, (7) gives $x^{\frac{m-1}{2}} \mid b_l$, and so $b_l \geqslant x^{\frac{m-1}{2}}$. However, $p \geqslant 3$ and $a_l > 1$, by (6) we have

$$2ax^m - 1 > \binom{p}{2} a_l^{p-2}(b_l\sqrt{\ }\ )^2 > b_l^2 \cdot 4abx(abx-1) > b_l^2 \cdot 2ax - 1,$$

i. e. $b_l^2 < x^{m-1}$, which is a contradiction since $b_l \geqslant x^{\frac{m-1}{2}}$.

If $p | x$, and let $x = p^\beta \cdot x_2$, $p \nmid x_2$, $\beta > 0$, then (6) and (7) imply $p = 3$ as follows.

Suppose $p > 3$, we have $m > 3$ in (6). Thus for (7) take modulus p^2:

$$0 \equiv pb_l (\bmod\ p^2),$$

and so $b_l \equiv 0 (\bmod\ p)$. Let $p^\alpha \| b_l$, $\alpha \geqslant 1$. By $p > 3$, we have

$$\frac{1}{p}\left[\binom{p}{1} a_l^{p-1} + \cdots + \binom{p}{p}(b_l\sqrt{\ }\ )^{p-1}\right] \equiv 1 (\bmod\ x).$$

Thus, (7) gives $\frac{m-1}{2}\beta = \alpha + 1$, $x_2^{\frac{m-1}{2}} | b_l$, and so $b_l \geqslant p^\alpha \cdot x_2^{\frac{m-1}{2}}$. Now, by $p > 3$, $b_l \geqslant p^\alpha \cdot x_2^{\frac{m-1}{2}}$, from (6) we have

$$2ax^m - 1 > \binom{p}{4} a_l^{p-4} b_l^4 \sqrt{\ }\ ^4 > 4b_l^4 \cdot a^2b^2x^2 > 4p^{4\alpha} \cdot x_2^{2(m-1)} \cdot a^2b^2x^2,$$

which is impossible since $4\alpha \geqslant 2\alpha + 2 = (m-1)\beta$ and $x = p^\beta \cdot x_2$.

Hence $p = 3$. Since p is any prime divisor of n, thus $n = 3^u$, $u \in \mathbb{N}$, and so (6) and (7) give respectively

$$2ax^m - 1 = a_l^3 + 3a_l(b_l\sqrt{\ }\ )^2, \tag{9}$$

$$x^{\frac{m-1}{2}} y = b_l(3a_l^2 + b_l^2 \cdot \sqrt{\ }\ ^2), \tag{10}$$

where $l = 3^{u-1}$ and $3 | x$. If $u > 1$, then $3 | b_l$ by (8). Thus $3 \| (3a_l^2 + b_l^2 \cdot \sqrt{\ }\ ^2)$. Let $3^\beta \| x$, $x = 3^\beta \cdot x_3$, $\beta \geqslant 1$, $3 \nmid x_3$. Then $x_3^{\frac{m-1}{2}} \mid b_l$, $3^{\beta\frac{m-1}{2}-1} \| b_l$, and so $b_l \geqslant 3^{\beta\frac{m-1}{2}-1} \cdot x_3^{\frac{m-1}{2}} = \frac{1}{3} x^{\frac{m-1}{2}}$. However, from (9) and $a_l \geqslant 2$, we have

$$2ax^m - 1 > 3a_l b_l^2 \cdot 4abx\,(abx-1) \geqslant 3a_l \cdot \frac{1}{9} x^m \cdot 4ab(abx-1) > 2ax^m,$$

which is impossible. Hence $u = 1$, $l = 3^{u-1} = 1$, $a_1 = 2abx - 1$, $b_1 = 1$, and by substituting $l_1 = 1$, $a_1 = 2abx - 1$, $b_1 = 1$ into (9) and (10), we have

$$2ax^m - 1 = (2abx-1)^3 + 3(2abx-1) \cdot 4abx(abx-1), \tag{11}$$

$$x^{\frac{m-1}{2}} y=3(2abx-1)^2+4abx(abx-1). \tag{12}$$

From (12), $x|3$. Hence by $3|x$ we have $x=3$, and by (11) it is verified that

$$\begin{aligned} 2a\cdot 3^m &= (6ab-1)^3+3(6ab-1)\cdot 12ab(3ab-1)+1 \\ &= 2a\cdot 3^2(48a^2b^3-24ab^2+3b) \\ &= 2a\cdot 3^3\cdot b(16a^2b^2-8ab+1), \end{aligned}$$

i.e. $3^{m-3}=b(4ab-1)^2$. Let $b=3^{s_1}$, $0\leqslant s_1\leqslant m-3$. Then

$$3^{m-3-s_1}=(4a\cdot 3^{s_1}-1)^2. \tag{13}$$

Clearly, (13) implies $m-3-s_1=0$ or $s_1=0$, and (13) is impossible if $m-3-s_1=0$. Thus $s_1=0$, and (13) gives $4a-1=3^{\frac{m-3}{2}}$, $m\equiv 1 \pmod 4$ and $b=1$. By (12), we have

$$\begin{aligned} 3^{\frac{m-1}{2}} y &= 3(6a-1)^2+12a(3a-1)=3(48a^2-16a+1) \\ &= 3[3(4a-1)^2+2(4a-1)]=3^{\frac{m-1}{2}}\left(3\cdot 3^{\frac{m-3}{2}}+2\right), \end{aligned}$$

which gives $y=3^{\frac{m-1}{2}}+2$. By $m\equiv 1 \pmod 4$, $m>1$, we have $m=4s+1$, $s\in\mathbb{N}$, and thus, $a=\frac{1}{4}(3^{2s-1}+1)$, $b=1$, $x=3$, $y=3^{2s}+2$. It is easy to prove that these are the solution of Eq. (1). Q. E. D.

Proof of Theorem 2 repeats completely the process of the proof of Theorem 1.

References

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