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SEMI-BLIND DECONVOLUTION OF FINITE LENGTH SEQUENCE (I)

——LINEAR PROBLEM

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ABSTRACT

In this paper, a class of so-called semi-blind deconvolution (SBD for short) problems is proposed. The linear SBD problem of finite-length sequence is studied. Several theorems are given to show the conditions for a number of usual SBD problems, which are linear solvable and have unique solution. An algorithm for linear SBD is developed and the results of computer simulations are shown. Some conclusions of this paper are applicable to the problem of signal reconstruction from spectral magnitude and partial sample points.

I. INTRODUCTION

In many cases, a signal may be regarded as the output z of a system y with an input x. A problem of solving x and y from z is called an inverse problem which is often encountered in many physical and engineering areas, e.g. in optics, astronomy, seismic prospecting, image restoration, reconstruction from projection, communication, automatic control, and speech signal processing. An inverse problem is called deconvolution problem if y is a linear time-invariant concentrated system, since in this case, z is the convolution of x with y, i.e. $z = x * y$. For a usual deconvolution problem, not only z but also one of x, y is known, the problem can be solved by commonly used inverse filter under the noiseless situation or by the least square inverse filter[1], iterative algorithm[2,3] under the noise situation. For a class of so-called blind deconvolution (BD for short) problems[4], only z is known. It is quite difficult to get the solutions without making any assumption on x and/or y[5].

In some application areas, besides the known z, a part of x and/or y may be obtained by some other means too. For example, for seismic signal $s = w * \xi$, it is possible to pick out a part of wavelet w on seismogram around the less interfered horizons with large intervals, and a part of reflection coefficients ξ on the data of sonic logs or the deconvolved seismogram by prediction deconvolution. Similar situations may also be encountered in other areas such as speech signal processing[6]. To distinguish such a class of problems of solving the whole x, y by knowing z and partial x, y from those of usual and blind deconvolution, we call them semi-blind decovolution (SBD) problems. Since some assumptions must be imposed on x, y for BD, SBD is more objective than BD and has wide potential applications.

Generally, x, y cannot be uniquely specified only from z even under the noiseless situation. One of the key problems for SBD is that by knowing how much of x, y and by which method the whole x, y can be uniquely specified from z.

As a first-stage investigation, this paper only considers the SBD on finite-length sequences, i.e.

$$\{x(n)/x(n) = 0,\ \forall n \notin [1,\ N_1],\ \text{and } x(1) \neq 0,\ x(N_1) \neq 0\},$$
$$\{y(n)/y(n) = 0,\ \forall n \notin [1,\ N_2],\ \text{and } y(1) \neq 0,\ y(N_2) \neq 0\}, \tag{1}$$

thus,

$$\{z(n)/z(n) = 0,\ \forall n \notin [1, N_1 + N_2 - 1], \text{and } z(1) \neq 0, z(N_1 + N_2 - 1) \neq 0\},$$

where

$$z(n) = x(n) * y(n) = \sum_{k=\max(1,n+1-N_1)}^{\min(N_2,n)} x(n + 1 - k)y(k),\ n = 1, 2, \cdots, N_1 + N_2 - 1. \qquad (2)^{1)}$$

From (2), it follows that to specify $x(n)$, $y(n)$ from $z(n)$ is a problem of solv ing a simultaneous 2nd-order nonlinear equations of multivariables. (For the detail discussion of the nonlinear problem, see Ref. [8].) But if a part of $x(n)$, $y(n)$ are known, in (2) some 1st-order linear equations will appear. If the number of these linear equations is equal to or larger than the number of unknown points, then, the original problem is reduced into the one of solving a simultaneous linear equations. In this paper, we focus our attention to how many sample points of $x(n)$, $y(n)$ are needed to be known so as to reduce the original problem of (2) into a linear problem, and to how and under what conditions the linear problem can be uniquely solved.

First, we turn a number of usual problems into six canonical types. Thus, several theorems are given to show the conditions for them being linearly solvable and with unique solution. Finally, an algorithm for linear SBD is developed and the results of computer simulations are shown.

Moreover, it is worth mentioning that if we let $y(n) = x(-n)$, then $z(n) = x(n) * x(-n)$ is the autocorrelation sequence of $x(n)$. So it is not difficult to see that the problem of reconstructing signal from some of its samples and spectral magnitude discussed in Refs. [6, 7] is only one of the special cases of the problems discussed in this paper.

II. A Number of Usual Problems and Their Canonical Types

1. *Lemmas*

Lemma 1. *Given $x(n)$, $y(n)$ can be uniquely specified from $z(n)$.*

This lemma can easily obtained from (2).

Lemma 2. *Given $x(n)$, $n = 1,2,\cdots,k_1 \leqslant N_1$, $y(n)$, $n = 1,2,\cdots$, $\min[k_1,N_2]$*

1) Due to the symmetry property of convolution, if $x(n)$, $y(n)$ are interchanged, all the conclusions in this paper are still true.

Table

A Number of Usual Problems and

	I	II	III
Canonical type	$m1$ $m2$ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ $m1$ $m2$ For 3(a) & 4: $m1 = \max[k1, k1']$ For 3(b) & 4: $m2 = \max[k2, k2']$.	m a. ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 m b. m 0 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ m For a: $m = \max[k1, k1']$ For b: $m = \max[k2, k2']$	m 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
Problems	1. $m1$ $m2$ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 0 0 0 0 0 0 2. $m1$ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ $m2$ 3. $k1$ $m2$ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ a. ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 $k1'$ $m1$ $k2$ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ b. 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ $k2'$ 4. $k1$ $k2$ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 ⊙ ⊙ $k1'$ $k2'$	1. m ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 0 0 a. 0 0 0 0 0 0 0 0 0 0 0 0 m 0 0 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ b. 0 0 0 0 0 0 0 0 0 0 0 0 0 2. $k1$ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 0 0 a. ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 $k1'$ $k2$ 0 0 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ b. 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ $k2'$	The same as the above

Note. 1. If $x(n)$, $y(n)$ are interchanged, a class of similar problems are obtained. Since the symmetry
2. $m1$, $m2$, m, $k1$, $k2$, $k1'$, $k2'$ are all positive integers, and each denotes the length of its
3. $\max[m, m1+m2, l] < \min[N1, N2]$, otherwise the problem is already solved by Lemma 1, the
4. "⊙" denotes the known point, "0" denotes the unkown point.

can be uniquely specified from $z(n)$.

Proof. From (2), it follows that

$$\begin{bmatrix} x(1) & 0 & \cdots\cdots & 0 \\ x(2) & x(1) & & \vdots \\ \vdots & & \ddots & 0 \\ x(k_1) & \cdots & \cdots & x(1) \end{bmatrix} \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(k_1) \end{bmatrix} = \begin{bmatrix} z(1) \\ z(2) \\ \vdots \\ z(k_1) \end{bmatrix}. \tag{3}$$

1

Their Canonical Types

IV	V	VI
$k1$ m 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 $k1'$ l 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0	$m1$ $m2$ a. ⊙ ⊙ ⊙ ⊙ 0 0 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 $m1$ b. $m1$ $m2$ 0 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ $m2$ For 3(a): $m1 = \max[k1, k1']$ For 3(b): $m2 = \max[k2, k2']$	a. "⊙" and "0" Space each other, e. g. 0 ⊙ 0 ⊙ 0 ⊙ 0 ⊙ 0 0 ⊙ 0 ⊙ 0 ⊙ 0 ⊙ 0 ⊙ 0 b. 2 m m $2m$ m 0 0 0 0 ⊙ ⊙ 0 0 0 ⊙ ⊙ 0 0 0 0 m $2m$ m $2m$ 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 ⊙ ⊙ ⊙ ⊙ 0 0
The same as the above	1. $m2$ 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 0 a. ⊙ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 0 $m1$ $m1$ 0 0 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 b. 0 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ $m2$ 2. $m1$ $m2$ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 ⊙ ⊙ ⊙ 0 0 0 a. 0 0 0 0 0 0 0 0 0 0 0 $m1$ $m2$ 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 0 0 ⊙ ⊙ ⊙ ⊙ b. 0 0 0 0 0 0 0 0 0 3. $k1$ $m2$ ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ ⊙ a. ⊙ ⊙ ⊙ ⊙ 0 0 0 0 0 0 0 $k1'$ $m1$ $k2$ 0 0 0 ⊙ ⊙ ⊙ ⊙ 0 0 0 0 ⊙ ⊙ ⊙ ⊙ b. 0 0 0 0 0 0 0 0 ⊙ ⊙ ⊙ ⊙ $k2'$	The same as the above

property of convolution, we only need to consider the problems in Table 1.

correspondent segment.

further consideration is need not.

And remember that $x(1) \neq 0$. The lemma is proved. Q.E.D.

Corollary 1. *When $k_1 \geqslant N_2$, the whole $x(n)$, $y(n)$ can be uniquely specified.* Similar to Lemma 2, we also have

Lemma 3. *Given $x(n)$, $n = N_1 - k_2, \cdots, N_1$, $k_2 \leqslant N_1$, $y(n)$, $n = \max[0, N_2 - k_2], \cdots, N_2$ can be uniquely specified from $z(n)$.*

Corollary 2. *When $k_2 \geqslant N_2$, the whole $x(n)$, $y(n)$ can be uniquely specified.*

2. *A Number of Usual Problems and Their Canonical Types*

In this section, we discuss a number of usual SBD problems with the feature that the known part of $x(n)$, $y(n)$ is a segment of the whole signal or is the sparsely resampled signal of $x(n)$, $y(n)$ (see the second row of Table 1). From the above lemmas these problems are turned into six canonical types as shown in the first row of Table 1. For example, for case 1 (column 1, row 2), it follows from Lemma 2 that by knowing the first m_1 points of $x(n)$, the first m_1 points of $y(n)$ can be specified. And similarly, the last m_2 points of $y(n)$ can be specified by knowing the last m_2 points of $x(n)$. Consequently, case 1 is turned into canonical type I.

III. The Conditions on Linear Solvability and Uniqueness

1. *The Derived Simultaneous Linear Equations and the Derived Matrix*

Definition 1. Under the condition that $z(n)$ and partial sample points of $x(n)$, $y(n)$ are known, all the equations which are reduced into linear equations are collected to form (4)

$$A\vec{X}=\bar{b}, \tag{4}$$

where $\vec{X}$ is an unknown vector consisting of all unknown sample points, if N_x is the number of the unknown points, then $\vec{X}$ is an N_x-dimensional vector. $\bar{b}$ is an l-dimensional constant vector which is derived from the known points of $z(n)$, $x(n)$ and $y(n)$, l is the number of the linear equations. A is an $l\times N_x$-dimensional matrix with its elements being the known sample points of $x(n)$, $y(n)$. We call Eq. (4) matrix A and constant vector $\bar{b}$ the derived simultaneous linear equations, the derived matrix and the derived constant vector, respectively, under the known condition.

The different orders of the variables in $\vec{X}$ and of the equations in (4) will result in the different matrix A. For convenience, in this paper, the equations in (4) are ordered according to the index of $z(n)$, and the variables in $\vec{X}$ are ordered in such a way that firstly the unknown points of $x(n)$ are ordered according to its original index, then continuously, those of $y(n)$ are ordered according to its original index.

For example, for canonical type I, when $N_1=N_2=N$, $m_1+m_2\geqslant[(N_1+N_2+2)/3]$, where $[c]$ denotes the integral part of c (see Theorem 1 below), and $m_1\geqslant N-(m_1+m_2)$, $m_2\geqslant N-(m_1+m_2)$. The derived simultaneous linear equations under such a special condition are given by (5).

$$\left|\begin{array}{cccc}
y(1) & 0 & \cdots\cdots & 0\\
y(2) & y(1) & & 0\\
\cdot & & & \\
\cdot & & & 0\\
y(N-m_1-m_2) & y(N-m_1-m_2-1) & \cdots\cdots\cdots & y(1)\\
\cdot & & & \cdot\\
\cdot & & & \cdot\\
y(m_1) & y(m_1-1) & \cdots\cdots & y(2m_1+m_2+1-N)\\
\hline
y(2N-2m_2-m_1) & y(2N-2m_2-m_1-1) & \cdots\cdots & y(N-m_2+1)\\
\cdot & & & \cdot\\
\cdot & & & \cdot
\end{array}\right.$$

$$
\begin{array}{|cccc}
y(N) & y(N-1) & \cdots\cdots & y(m_1+m_2+1) \\
0 & & & \cdot \\
\cdot & & & \cdot \\
\cdot & y(N) & & y(N-1) \\
0 \; \cdots \; \cdots \; \cdots & 0 & & y(N)
\end{array}
$$

$$
\begin{array}{cccc|}
x(1) & 0 & \cdots\cdots & 0 \\
x(2) & x(1) & & 0 \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
 & & & 0 \\
x(N-m_1-m_2) & x(N-m_1-m_2-1) & \cdots\cdots & x(1) \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
x(m_1) & x(m_2-1) & \cdots\cdots & x(2m_1+m_2+1-N) \\
\hdashline
x(2N-2m_2-m_1) & x(2N-2m_2-m_1-1) & \cdots\cdots & x(N-m_2+1) \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
x(N) & x(N-1) & \cdots\cdots & x(m_1+m_2+1) \\
0 & & & \cdot \\
\cdot & & & \cdot \\
\cdot & & x(N) & x(N-1) \\
0 & \cdots \quad \cdots \quad \cdots & 0 & x(N)
\end{array}
$$

$$
\cdot \left|\begin{array}{c}
x(m_1+1) \\
x(m_2+2) \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
x(N-m_2) \\
\hdashline
y(m_1+1) \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
y(N-m_2-1) \\
y(N-m_2)
\end{array}\right| = \left|\begin{array}{c}
b(1) \\
b(2) \\
\cdot \\
\cdot \\
b(N-m_1-m_2) \\
\cdot \\
\cdot \\
b(m_1) \\
\hdashline
b(m_1+1) \\
\cdot \\
\cdot \\
\cdot \\
b(2m_1+2m_2+1-N) \\
\cdot \\
\cdot \\
b(m_1+m_2-1) \\
b(m_1+m_2)
\end{array}\right|, \tag{5}
$$

where

(i) for $1 \leqslant j \leqslant N - m_1 - m_2$, $b(j) = 2(j + m_1) - \sum_{i=j+1}^{m_1} x(j) y(m_1 + j + 1 - i)$;

(ii) for $N - m_1 - m_2 < j \leqslant m_1$, $b(j) = z(j + m_1) - \Big\{ \sum_{i=j+1}^{m_1} x(j) y(m_1 + j + 1 - i)$

$$+ \sum_{i=1}^{j+m_1+m_2-N} [x(i)y(m_1+j+1-i) + y(i)x(m_1+j+1-i)]\Big\};$$

(iii) for $1 \leqslant j \leqslant m_1 + 2m_2 - N$, $b(m_1+j) = z(2N-2m_2-1+j)$

$$-\Big\{\sum_{i=N-m_2+1}^{N-m_2+j-1} x(i)\cdot y(2N-2m_2+j-i)$$
$$+\sum_{i=N-2m_2+j}^{m_1} [x(i)y(2N-2m_2+j-i)$$
$$+ y(i)x(2N-2m_2+j-i)]\Big\};$$

(iv) for $m_2 \geqslant j > m_1 + 2m_2 - N$, $b(m_1+j) = z(2N-2m_2-1+j)$

$$-\sum_{i=N-m_2+1}^{N-m_2+j-1} x(i)y(2N-2m_2+j-i).$$

Similarly, we can write out the derived matrix A and vector $\bar{b}$ under a more general condition, but it is too intricate and complicated. Fortunately, A, $\bar{b}$ can be easily obtained by a computer program. An algorithm for doing this is given in Appendix 1.

2. *The Conditions on Linear Solvability and Uniqueness*

Theorem 1. *For canonical type I*

(1) *If* $m_1 + m_2 \geqslant \min[N_1, N_2]$, *then* $x(n)$, $y(n)$ *can be uniquely specified.*

(2) *If* $3(m_1+m_2) \geqslant N_1 + N_2$, *or* $(m_1+m_2) \geqslant [(N_1+N_2+2)/3]$, *and on its derived simultaneous linear equation*, Rank $[A:\bar{b}] = N_x$, *where* Rank$[\cdot]$ *denotes the rank of a matrix*, $N_x = N_1 + N_2 - 2l$ *is the number of unknown points*, $l = m_1 + m_2$ *is the number of equations.*

Then $x(n)$, $y(n)$ *can be uniquely specified by solving the simultaneous linar equations.*

Proof. (1) It is easily proved from the lemmas in Sec. II, 1.

(2) $N_1 + N_2 - 1$ equations as (6) are given by (2). For convenience, it is assumed that $N_1 > N_2$ which does not lose generality, where $x(m_1+1), \cdots, x(N_1 - m_2)$, and $y(m_1+1), \cdots, y(N_2 - m_2)$ are unknown points, its number is $N_x = N_1 + N_2 - 2(m_1+m_2)$.

$$z(1) = x(1)y(1),$$
$$z(2) = x(1)y(2) + x(2)y(1),$$
$$\cdot$$
$$\cdot$$
$$z(N_2) = x(1)y(N_2) + x(2)y(N_2-1) + \cdots\cdots\cdots + x(N_2)y(1),$$
$$z(N_2+1) = x(2)y(N_2) + x(3)y(N_2-1) + \cdots\cdots\cdots + x(N_2+1)y(1),$$
$$\cdot \qquad \cdot \qquad (6)$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$z(N_1) = x(N_1 - N_2 + 1)y(N_2) + x(N_1 - N_2 + 2)y(N_2 - 1) + \cdots + x(N_1)y(1),$$

$$z(N_1 + 1) = x(N_1 - N_2 + 2)y(N_2) + x(N_1 - N_2 + 3)y(N_2 - 1) + \cdots + x(N_1)y(2),$$

$$\cdot$$

$$\cdot$$

$$z(N_2 + N_1 - 2) = x(N_1 - 1)y(N_2) + x(N_1)y(N_2 - 1),$$

$$z(N_2 + N_1 - 1) = x(N_1)y(N_2).$$

After analysis, it can be discovered that the above equations consist of the following three groups.

(i) The m_1 equations from $z(1)$ to $z(m_1)$ and the m_2 equations from $z(N_1 + N_2 - m_2)$ to $z(N_1 + N_2 - 1)$ are all reduced into the constant identities.

(ii) The m_1 equations from $z(m_1 + 1)$ to $z(2m_1)$ and the m_2 equations from $z(N_1 + N_2 - 2m_2)$ to $z(N_1 + N_2 - m_2 - 1)$ are all reduced into the linear equations.

(iii) The $(N_1 + N_2 - 1) - 2(m_1 + m_2)$ equations from $z(2m_1 + 1)$ to $z(N_1 + N_2 - 2m_1 - 1)$ are still the 2nd-order nonlinear equations.

By (ii) the number of linear equations is $(m_1 + m_2)$, to make these equations uniquely solvable, it is required that $(m_1 + m_2) \geqslant N_x = N_1 + N_2 - 2(m_1 + m_2)$.

Thus, $3(m_1 + m_2) \geqslant N_1 + N_2$ or $m_1 + m_2 \geqslant (N_1 + N_2)/3$.

Because $m_1 + m_2$ can only be integral, we have $m_1 + m_2 \geqslant [(N_1 + N_2 + 2)/3]$.

Furthermore, it is known from (2) that the simultaneous linear equations are surely consistent, i. e.

$$\text{Rank}[A|\bar{b}] = \text{Rank}[A].$$

To ensure the unique solution, it is required that

$$\text{Rank}[A|\bar{b}] = N_x = N_1 + N_2 - 2(m_1 + m_2). \quad \text{Q.E.D.}$$

The above procedure for finding linear equations is somewhat complicated, but it may be replaced by a concise way (see Fig. 1).

Fig. 1 illustrates the procedure of $x(n) * y(n)$ for canonical type I, where, for each equation, the two overlapped ⊚ points correspond to a constant item, the two overlapped points with one ⊚ and one 0 to a 1st-order item, and the two overlapped 0 points to a 2nd-order item. After $y(n)$ is convolved into $y(-n)$, it is shifted point by point from the left to the right, each shifted position corresponds to an equation. The leftest one is $z(1) = x(1)y(1)$, the rightest one is $z(N_1 + N_2 - 1) = x(N_1)y(N_2)$. Consequently, Figs. 1 (a) and 1 (e) correspond to the part (i) of the above proof, Figs. 1(b) and 1(d) to the part (ii), and Fig. 1(c) to the part (iii).

Similarly, we can prove the following theorems.

Theorem 2. *For canonical type* II, *if* $m \geqslant [(N_1 + N_2 + 2)/3]$, *and on its derived simultaneous linear equations* $\text{Rank}[A:\bar{b}] = N_x$, *where* $N_x = N_1 + N_2 - 2l$ *is the number of the unknown points,* $l = m$ *is the number of equations.*

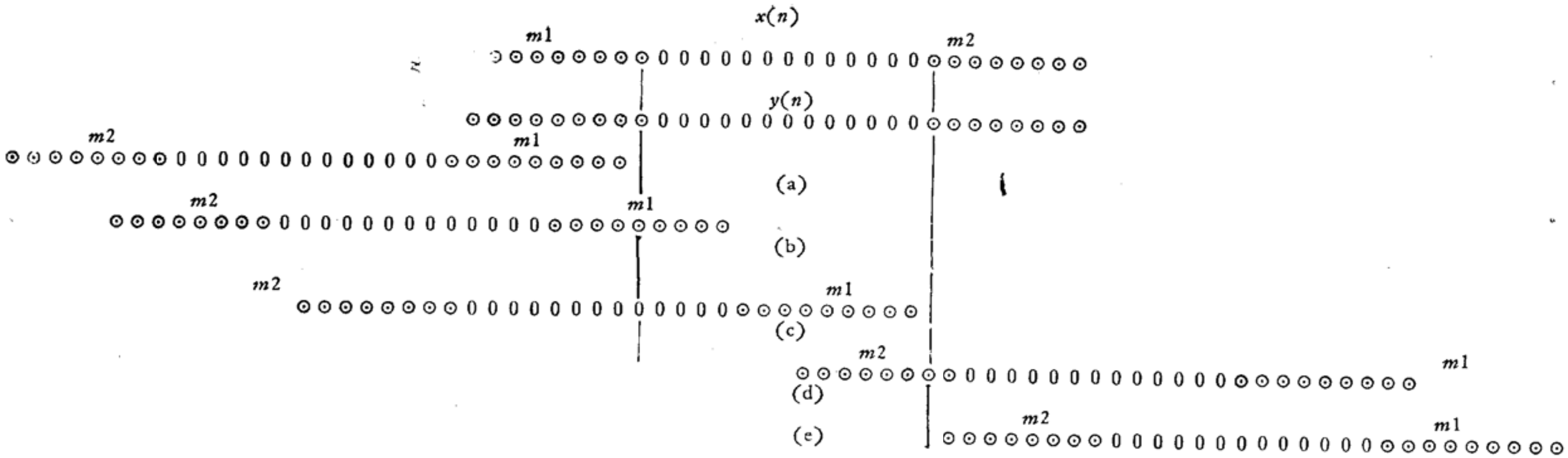

Fig. 1. The procedure of $x(n)*y(n)$ (cononical type I).

⊙, the known point; 0, the unknown point.

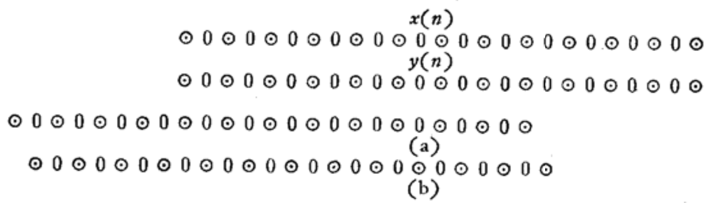

Fig. 2. The procedure of $x(n)*y(n)$ (cononical type VI).

⊙, the known point; 0, the unknown point.

Then $x(n)$, $y(n)$ can be uniquely specified by solving the simultaneous linear equations.

Theorem 3. *For canonical tpye* III, *if* $m \geqslant N_1 + [(N_1+1)/2]$, *and on its derived simultaneous linear equations* Rank $[A|\bar{b}] = N_x = N_1 + N_2 - m$, (*in this case, the number of equations is* $l = m_1 - N_2$).

Then $x(n)$, $y(n)$ can be uniquely specified by solving the simultaneous linear equations.

Theorem 4. *For canonical type* IV, *if the inequalities* $m + l > \max[N_2 - k_1', N_1 - k_1]$, $l > k_1$ $m > k_1'$ *hold simultaneously, and if*

(1) $3l + 2m \geqslant 2N_1 + N_2$, *when* $l - m \leqslant k_1 - k_1' \leqslant N_1 - N_2$,

(2) $3l + 2m \geqslant N_1 + 2N_2 + (k_1 - k_1')$, *when* $k_1 - k_1' \geqslant l - m$ *and* $k_1 - k_1' > N_1 - N_2$,

(3) $3m + 2l \geqslant 2N_1 + N_2 - (k_1 - k_1')$, *when* $k_1 - k_1' < l - m$ *and* $k_1 - k_1' \leqslant N_1 - N_2$,

(4) $3m + 2l \geqslant N_1 + 2N_2$, *when* $k_1 - k_1' < l - m$ *and* $k_1 - k_1' > N_1 - N_2$,

and on its derived simultaneous linear equations $\text{Rank}[A|\bar{b}] = N_x = N_1 + N_2 - (l - m)$, *then $x(n)$, $y(n)$ can be uniquely specified by solving the simultaneous linear equations.*

Theorem 5. *For canonical type* V, *if*

(1) $3m_1 + m_2 \geqslant N_1 + N_2$ *for* $V(a)$ *and* $3m_2 + m_1 > N_1 + N_2$ *for* $V(b)$, *when* $N_2 > m_1 + m_2$,

(2) $4m_1 + 2m_2 \geqslant N_1 + 2N_2$ *for* $V(a)$ *and* $4m_2 + 2m_1 \geqslant N_1 + 2N_2$ *for* $V(b)$, *when* $N_2 \leqslant m_1 + m_2$,

and on its derived simultaneous linear equations $\text{Rank}[A|\bar{b}] = N_x = N_1 + N_2 - 2m_1 - m_2$, *then $x(n)$, $y(n)$ can be uniquely specified by solving the simultaneous linear equations.*

Theorem 6. *For canonical type* VI

(1) *If at least one of the four points $x(1)$, $x(N_1)$, $y(1)$, $y(N_2)$ is unknown, then $x(n)$ $y(n)$ can be uniquely specified recursively.*

(2) *If $x(1)$, $x(N_1)$, $y(1)$, $y(N_2)$ are all known, and on its derived simultaneous linear equations* $\text{Rank}[A|\bar{b}] = N_x = (N_1 + N_2 - 2)/2$, *then $x(n)$, $y(n)$ can be uniquely specified by solving the simultaneous linear equations.*

Proof. (1) Here, we only consider the case where $x(1)$ is an unknown point, it is similar to other cases.

Since $z(1)$, $y(1)$ are known and $z(1) = x(1)y(1)$, we get $x(1) = z(1)/y(1)$. In turn, from $z(2) = x(2)y(1) + y(2)x(1)$ and $x(1)$, $x(2)$, $y(1)$, we get $y(2) = (z(2) - x(2)y(1))/x(1)$, $\cdots$ recursively, $x(n)$, $y(n)$ can be specified.

(2) As shown in Fig. 2, for each position at which $\odot$ points and 0 points are just overlapping the correspondent equations in (2) are reduced into linear ones.

On the one hand, the number of the unknown points is $N_x = (N_1 - 1)/2 + (N_2 - 1)/2 = (N_1 + N_2 - 2)/2$. (where N_x is surely an integer, $\because$ for $x(n)$, the number of the unknown points cannot be simultaneously even or odd, which leads N_1 to be odd, similarly N_2 to be odd, therefore, $N_1 + N_2 - 2$ is even.)

On the other hand, the number of the linear equations in (2) is

$$[(N_1 + N_2 - 1) - 1]/2 = (N_1 + N_2 - 2)/2 \text{ which equals } N_x.$$

And we also have $\text{Rank}[A|\bar{b}] = N_x$. Consequently, $x(n)$, $y(n)$ can be uniquely specified by solving the simultaneous linear equations. Q. E. D.

For canonical type VI(b), similarly, $x(n)$, $y(n)$ may also be specified by combining the recursive way and the way of solving the derived simultaneous linear equations.

IV. Algorithm and Simulations

1. *Algorithm*

Step 1. According to the lemmas in Section II, 1, recurisively turn the problem into one of six canonical types.

Step 2. Use the algorithm in Appendix 1 to obtain the derived Matrix A and the derived constant vector $\bar{b}$, thus the derived simultaneous linear equations $A\bar{X} = \bar{b}$.

Step 3. Solve this simultaneous equations in one of the following two ways.

(a) Among all the equations in $A\bar{X} = \bar{b}$, take out N_x equations and solve them by one of usual methods for solving simultaneous linear equations.

(b) Enhance the anti-interference ability, make the full use of all the equations and obtain the solution by least square method, i.e. $\bar{X} = (A^TA)^{-1}A^T\bar{b}$.

Except the directly solving method mentioned above, an iterative method may also be used. The solution of the direct method may be taken as the initial for iteration. In the noise situation, by direct method, due to the recursion in Step 1, it is possible to make the errors unevenly distributed, which may possibly be lessened by iterative method. Further investigation is needed along this direction.

2. *Simulations*

Simulation 1. $x(n)$ is given as Fig. 3(b), $N_1 = 25$. $y(n)$ is given as Fig. 3(c), $N_2 = 30$. The convolution $z(n)$ of $x(n)$, $y(n)$ by (2) is given as Fig. 3(a), its length is $N_1+N_2-1 = 54$. Now, assume that we know the whole $z(n)$ and the partial sample points of $x(n)$, $y(n)$ which are circled by the dotted lines in Figs. 3(b) and (c). It is not difficult to see that the problem can be turned into the one of canonical type I. From Theorem 1, the problem is linear solvable if $m_1 + m_2 \geqslant [(N_1 + N_2 + 2)/3] = [(25 + 30 + 2)/3] = 19$. Now, given that $m_1 = 12$ for $y(n)$, $m_2 = 7$ for $x(n)$, $x(n)$, $y(n)$ are obtained by the algorithm in Section IV, 1. Figs. 3(d) and (e) are the results by usual method (i.e. Step 3(a)), Figs. 3(f) and (g) are the results by least square method (i.e. step 3(b)).

Simulation 2. $x(n)$, given as Fig. 4(b) with $N_1 = 21$, is a mixture phase signal, $y(n)$, given as Fig. 4(c) with $N_2 = 25$, is a minimum phase signal. The convolution

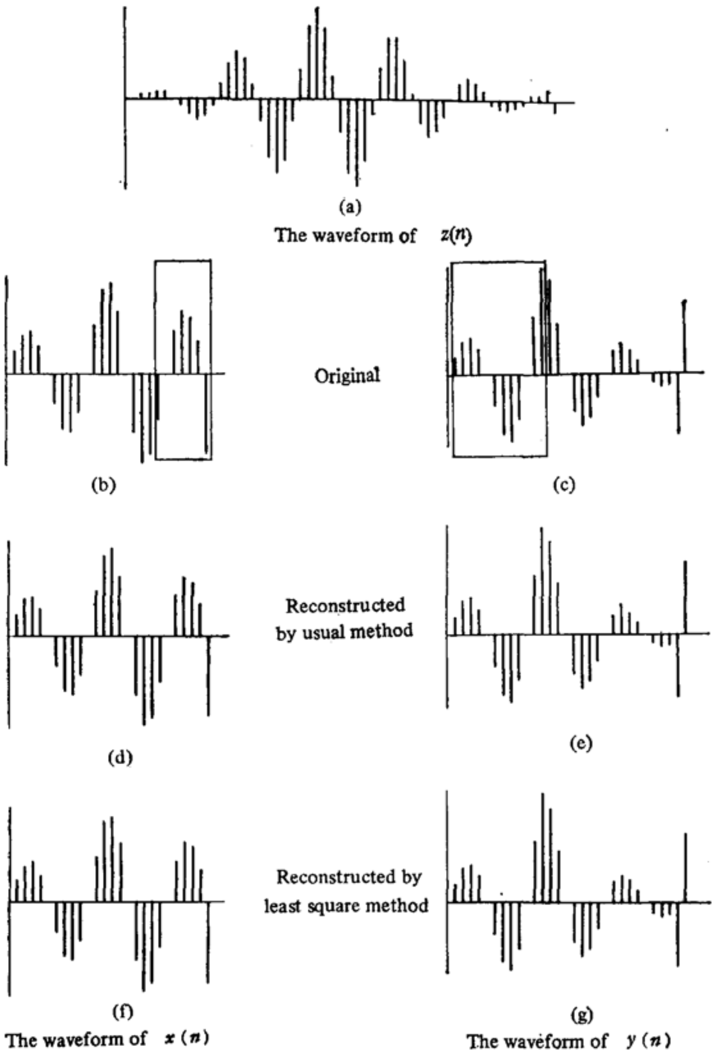

Fig. 3. L near semi-blind deconvolution (simulation 1).

$z(n) = x(n) * y(n)$ is given as Fig. 4(a), its length is $N_1 + N_2 - 1 = 54$. Now, assume that we know the whole $z(n)$ and the partial sample points of $x(n)$, $y(n)$ which are circled by the dotted lines in Figs. 4(b) and (c). It is not difficult to see that the problem can also be turned into the one of canonical type I. From Theorem 1, the problem is linear solvable if $m_1 + m_2 \geqslant [(21 + 25 + 2)/3] = 16$. Now, $m_1 = 8$ for $y(n)$, $m_2 = 8$ for $x(n)$, $x(n)$, $y(n)$ are obtained by the algorithm in Sec. IV, 2. Figs. 4(d) and (e) are the results by a usual method (i.e. Step 3 (a)), Figs. 4(f) and (g) are the results by least square method (i.e. Step 3(b)).

V. Summary

A class of deconvolution problem called SBD has heen proposed in this paper. The linear SBD problem of finite length sequence has been investigated. Firstly, we have

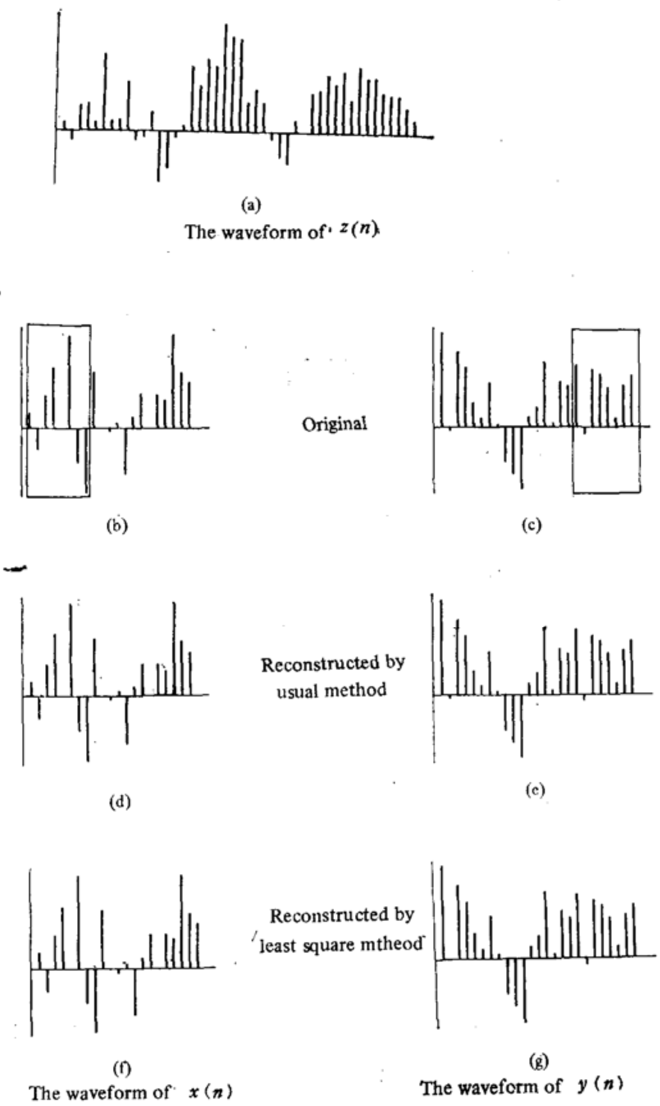

Fig. 4. Linear semi-blind deconvolution (simulation 2).

turned a number of usual problems into six canonical types, then several theorems have been given to show the conditions for the six types being linear solvable and with unique solution. Lastly, an algorithm for linear SBD is presented. The results of computer simulation have verified these theorems and the effectivenees of the algorithm.

Besides the six canonical types discussed in this paper, SBD in other problems may also be made in the same way, i.e. the algorithms similar to those in Sec. IV, 1 and Appendix 1 may be developed. However, as the situation is quite complicated, further investigation is needed.

Finally, it deserves to be pointed out that some conclusions of this paper are applicable to not only the deconvolution problem but also to the problem of the signal reconstruction from its partial samples and spectral magnitude. For example, (i) the problems corresponding to canonical type I is just the problem discussed in Refs. [6, 7] if $y(n) = x(-n)$, $m_1 = m_2 = m$, $N_1 = N_2 = N$. Let us put this condition into Theorem 1. The condition part of Theorem 1 is reduced to the one of $3m \geqslant N$ or $m > [(N+2)/3]$, then, Theorem 1 is just the same as Theorem 1 in Ref. [7]. The problem investigated in [6,7] may be considered as a special case of the problems investigated in this paper. (ii) In Theorem 6, let $y(n) = x(-n)$, then for finite time-duration discrete signal, from its spectral magnitude, the signal can be uniquely reconstructed from the sparsely resampled signal. This is a new result different from those of [6, 7], and may possibly be used for data compression in storing and transmitting the mixture phase signals.

Appendix 1

The Algorithm for the Derived Matrix A and the Derived Constant Vector B

Only the algorithm for canonical type I is given here. The algorithms for other canonical types as well as a general algorithm may be developed in the same way.

Initialization. 1) Array $X(n)$, $n = 1, 2, \cdots, N_1$ and $Y(n)$, $n = 1, 2, \cdots, N_2$ are prepared to store the sample points of $x(n)$, $y(n)$ respectively, and put the known points of $x(n)$, $y(n)$ at the corresponding positions of the two arrays. Another array $Z(n)$, $n = 1, 2, \cdots, L_Z$ is used to store all the samples of $z(n)$, where $L_Z = N_1 + N_2 - 1$ is the length of $z(n)$.

2) Create two integer array $IX(n)$, $n = 1, 2, \cdots, N_1$ and $IY(n)$, $n = 1, 2, \cdots, N_2$ respectively, e.g. if the sample point at $X(i)$ (or $Y(i)$) is known, then set $IX(i) = 1$ (or $IY(i) = 1$), otherwise set $IX(i) = 0$ (or $IY(i) = 0$).

3) An $L_Z \times N$-dimensional matrix A and an array $B(N_x)$ are prepared for storage of the derived matrix A and the derived constant vector $\bar{b}$ respectively, where N_x is the number of the unknown points.

4) Set up an indication variable IFG, it is automatically set that IFG $= 1$ if the unmber of equations is equal to or larger than that of the unkown points; otherwise, IFG $= 0$.

Step 1. Set $i_k = 1$, $i_1 = 1$, input m_1, m_2.

Step 2. Loop 1. $k = 1, 2, \cdots, L_Z$.

If $k > N_1$ then $kk = N_1$ else $kk = k$.

If $k > N_2$ then $i_1 - i_1 + 1$.

Loop 2. $i = i_1, \cdots, kk$

If $IX(i) = 0$ and $Y(k+1-i) = 0$ then go to Loop 1.

If $IX(i) = 0$ then $A(i_k, i - m_1) = Y(k_1 + 1 - i)$, go to Loop 2.

If $IY(k+1-i) = 0$ then $A(i_k, N_1 - 2m_1 - m_2 + k + 1 - i) = X(i)$, go to Loop 2.

$$Z(k) \leftarrow -X(i)Y(k+1-i) + Z(k).$$

$$B(i_k) = Z(k), \; i_k \leftarrow i_k + 1.$$

Step 3. If $(i_k - 1) \geqslant (N_1 + N_2 - 1)$ then IFG $= 1$, otherwise IFG $= 0$.

Step 4. Return, END.

After the algorithm stops, the first i_k row of Matrix A constitutes the derived matrix and the first i_k elements of Array B constitue the derived constant vector.

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