Homotopy formulas and $\bar{\partial}$-equation on local q-convex domains in Stein manifolds*

ZHONG Tongde (钟同德)

(Institute of Mathematics, Xiamen University, Xiamen 361005, China)

Received January 2, 1997

Abstract The homotopy formulas of (r, s) differential forms and the solution of $\bar{\partial}$-equation of type (r, s) on local q-convex domains in Stein manifolds are obtained. The homotopy formulas on local q-convex domains have important applications in uniform estimates of $\bar{\partial}$-equation and holomorphic extension of CR-manifolds.

Keywords: Stein manifold, Koppelman-Leray-Norguet formula, local q-convex domain, homotopy formula, $\bar{\partial}$-equation.

As is well known, Stein manifold is an important manifold, where there exist many non-constant holomorphic functions. $\mathbb{C}^n$ is a Stein manifold. It is natural that one wants to study complex analysis on Stein manifolds[1]. In this paper by using Hermitian metric and Chern connection[2—4] we obtain the homotopy formulas and the solution of $\bar{\partial}$-equation on local q-convex domains in Stein manifolds. Local q-convex domain is an extension of piecewise smooth pseudoconvex domain, so the homotopy formula obtained in this paper has its general meaning, which has important applications in uniform estimates of $\bar{\partial}$-equation and holomorphic extension of CR-manifolds. Moreover in this paper we discuss (r, s) differential forms on Stein manifold, which is different from (o, s) differential forms. In this case one cannot use Euclidean metric as in the case of $\mathbb{C}^n$, since Euclidean metric is not an invariant under holomorphic transformation on Stein manifold. In order to overcome this difficulty, we have introduced Hermitian metric and Chern connection[2] and constructed various integral kernels with respect to (r, s) differential forms under invariant metric on Stein manifolds, and thus obtained the above results.

Assume X to be an n-dimensional Stein manifold. Here we still use the definitions and notations in references [1—5].

Let $D \subset\subset X$ be a $C^{(1)}$ intersection. $(U_{\bar{D}}, \rho_1, \cdots, \rho_N)$ is a frame for D. Let ψ be a Leray map for the frame $(U_{\bar{D}}, \rho_1, \cdots, \rho_N)$. Then we set

$$\psi_{OK}(z, \zeta, \lambda) = \mathring{\lambda}(\lambda_0) \frac{\hat{S}(z, \zeta)}{|S(z, \zeta)|_\theta^2} + (1 - \mathring{\lambda}(\lambda_0))\psi_K(z, \zeta, \mathring{\lambda}) \tag{1}$$

for $K \in P'(N)$ and $(z, \zeta, \lambda) \in D \times S_K \times \Delta_{OK}$. Note that $1 - \mathring{\lambda}(\lambda_0) = 0$ for λ in the neighborhood $\Delta_{OK} \setminus \mathring{\Delta}_{OK}$ of Δ_0, and therefore ψ_{OK} is of class $C^{(2)}$.

Now for all $K \in P'(N)$, we define Bochner-Martinelli kernel:

$$\hat{B}(z, \zeta) = (-1)^{n-1}/(2\pi)^n \varphi^\nu(z, \zeta)\langle \hat{S}, D S\rangle \wedge (\langle \nabla'' \hat{S}, D S\rangle)^{n-1} / |S|_\theta^{2n} \tag{2}$$

* Project supported by the National Natural Science Foundation of China.

for all $(z, \zeta, \lambda) \in D \times S_K \times \Delta_{OK}$, and define Koppelman-Leray-Norguet kernel as

$$\hat{R}_K^\psi(z, \zeta, \lambda) = (-1)^{n-1}/(2\pi)^n \varphi^\nu(z, \zeta) \langle \psi_{OK}, D S \rangle \wedge (\langle \Delta'' \psi_{OK}, D S \rangle)^{n-1}. \tag{3}$$

And for all $(z, \zeta, \lambda) \in D \times S_K \times \Delta_K$, we define

$$\hat{L}_K^\psi(z, \zeta, \lambda) = (-1)^{n-1}/(2\pi)^n \varphi^\nu(z, \zeta) \langle \psi_K, D S \rangle \wedge (\langle \Delta'' \psi_K, D S \rangle)^{n-1}, \tag{4}$$

where $\nabla'' = \bar{\partial}_{z,\zeta}, \Delta'' = (\bar{\partial}_{z,\zeta} + d_\lambda)$, φ is a holomorphic function, ν is a suitable integer such that $\hat{B}(z, \zeta), \hat{R}_K^\psi(z, \zeta, \lambda), \hat{L}_K^\psi(z, \zeta, \lambda)$ are of continuous forms[1-4].

Set

$$\mathrm{d}\hat{R}_K^\psi(z, \zeta, \lambda) := \hat{Q}_K^\psi(z, \zeta, \lambda), \tag{5}$$

$$\bar{\partial}_{z,\zeta} \hat{B}(z, \zeta) := \hat{P}(z, \zeta). \tag{6}$$

Assume f to be a continuous (r, s)-form on $\bar{D}$. For all $K \in P'(N)$, we set

$$B_D f(z) = \int_{\zeta \in D} f(\zeta) \wedge \hat{B}(z, \zeta), \quad z \in D, \tag{7}$$

$$R_K^\psi f(z) = \int_{(\zeta, \lambda) \in S_K \times \Delta_{OK}} f(\zeta) \wedge \hat{R}_K^\psi(z, \zeta, \lambda), \quad z \in D, \tag{8}$$

$$L_K^\psi f(z) = \int_{(\zeta, \lambda) \in S_K \times \Delta_K} f(\zeta) \wedge \hat{L}_K^\psi(z, \zeta, \lambda), \quad z \in D, \tag{9}$$

$$Q_K^\psi f(z) = \int_{(\zeta, \lambda) \in S_K \times \Delta_{OK}} f(\zeta) \wedge \hat{Q}_K^\psi(z, \zeta, \lambda), \quad z \in D, \tag{10}$$

$$P_D f(z) = \int_{\zeta \in D} f(\zeta) \wedge \hat{P}(z, \zeta), \quad z \in D. \tag{11}$$

Then, for every continuous (r, s)-form f on $\bar{D}$, $0 \leqslant r, s \leqslant n$ such that $\bar{\partial} f$ is also continuous on $\bar{D}$. We have the following classical Koppelman-Leray-Norguet formula[3]:

$$(-1)^{r+s} f = \bar{\partial}_z B_D f - B_D \bar{\partial}_\zeta f + \sum_{K \in P'(N)} (L_K^\psi f + \bar{\partial}_z R_K^\psi f - R_K^\psi \bar{\partial}_\zeta f)$$
$$+ (-1)^{r+s+1} \Big(\sum_{K \in P'(N)} Q_K^\psi f + P_D f \Big). \tag{12}$$

In particular, if $C(T(X \times X)) = D^2 = 0$, then $\hat{Q}_K^\psi = 0$ and $\hat{P} = 0$. We have

$$(-1)^{r+s} f = \bar{\partial}_z B_D f - B_D \bar{\partial}_\zeta f + \sum_{K \in P'(N)} (L_K^\psi f + \bar{\partial}_z R_K^\psi f - R_K^\psi \bar{\partial}_\zeta f). \tag{13}$$

1 A Leray map for local q-convex domains

Let $D \subset\subset X$ be a domain and ρ a real $C^{(2)}$ function on D. Then we denote by $L_\rho(\zeta)$ the Levi form ρ at $\zeta \in D$, and by $F_\rho(\cdot, \zeta)$ the Levi polynomial of ρ at $\zeta \in D$, i.e.

$$L_\rho(\zeta) t = \sum_{j,k=1}^n \frac{\partial^2 \rho(\zeta)}{\partial \bar{\zeta}_j \partial \zeta_k} \bar{t}_j t_k, \qquad \zeta \in D, \ t \in \mathbb{C}^n,$$

$$F_\rho(z, \zeta) = 2 \sum_{j=1}^n \frac{\partial \rho(\zeta)}{\partial \zeta_j}(\zeta_j - z_j) - \sum_{j,k=1}^n \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k), \zeta \in D, z \in X.$$

Moreover,

$$\mathrm{Re} F_\rho(z, \zeta) = \rho(\zeta) - \rho(z) + L_\rho(\zeta)(\zeta - z) + o([\mathrm{dist}(z, \zeta)]^2).$$

Denote by $MO(n, q)$ the complex manifold of all $n \times n$-matrices which define an orthogonal projection from $\mathbb{C}^n$ to some q-dimensional subspace of $\mathbb{C}^n$.

Definition 1. A local q-convex domain, $0 \leqslant q \leqslant n-1$, is a $C^{(2)}$ intersection $D \subset\subset X$ for which one can find a $C^{(2)}$ frame $(U_{\bar{D}}, \rho_1, \cdots, \rho_N)$ satisfying the following two conditions.

(i) If $K = \{k_1, \cdots, k_l\} \in P'(N)$ and $U_{\bar{D}}^K := \{z \in U_{\bar{D}} : \rho_{k_1}(z) = \cdots = \rho_{k_l}(z)\}$, then $(d\rho_{k_1}(z) - d\rho_{k_2}(z)) \wedge \cdots \wedge (d\rho_{k_1}(z) - d\rho_{k_l}(z)) \neq 0$ for all $z \in U_{\bar{D}}^K$.

(ii) There exists a C^∞ map $Q: \Delta_{1\cdots N} \to MO(n, n-q-1)$ and constants $\alpha, A > 0$ such that $\operatorname{Re} F_{\rho_\lambda}(z, \zeta) \geqslant \rho_\lambda(\zeta) - \rho_\lambda(z) + \alpha|\zeta - z|^2 - A|Q(\lambda)(\zeta - z)|^2$ for all $\lambda \in \Delta_{1\cdots N}$ and $z, \zeta \in U_{\bar{D}}$, where $\rho_\lambda := \lambda_1\rho_1 + \cdots + \lambda_N\rho_N$.

Now we construct the Leray map ψ. Since $\rho_1, \cdots, \rho_N$ are of class $C^{(2)}$ and defined in the neighborhood of $\bar{U}_{\bar{D}}$, we can find C^∞ functions a_ν^{kj} $(\nu = 1, \cdots, N; k, j = 1, \cdots, n)$ on $U_{\bar{D}}$ such that for all $\zeta \in U_{\bar{D}}$,

$$\left| a_\nu^{kj}(\zeta) - \frac{\partial^2 \rho_\nu(\zeta)}{\partial \zeta_k \partial \zeta_j} \right| < \frac{\alpha}{2n^2}.$$

Set

$$a_\lambda^{kj} = \lambda_1 a_1^{kj} + \cdots + \lambda_N a_N^{kj}$$

for $\lambda \in \Delta_{1\cdots N}$. Then

$$\left| \sum_{k,j=1}^{n} \left(a_\lambda^{kj}(\zeta) - \frac{\partial^2 \rho_\nu(\zeta)}{\partial \zeta_k \partial \zeta_j} \right) t_k t_j \right| \leqslant \frac{\alpha}{2} |t|^2 \tag{14}$$

for all $\zeta \in U_{\bar{D}}$, $t \in C^{(n)}$ and $\lambda \in \Delta_{1\cdots N}$. Set

$$\tilde{F}_{\rho_\lambda}(z, \zeta) = 2\sum_{j=1}^{n} \frac{\partial \rho_\lambda(\zeta)}{\partial \zeta_j}(\zeta_j - z_j) - \sum_{k,j=1}^{n} a_\lambda^{kj}(\zeta)(\zeta_k - z_k)(\zeta_j - z_j) \tag{15}$$

for $(z, \zeta, \lambda) \in \mathbb{C}^n \times U_{\bar{D}} \times \Delta_{1\cdots N}$. Then it follows from (14) and condition (ii) in Definition 1 that

$$\operatorname{Re} \tilde{F}_{\rho_\lambda}(z, \zeta) \geqslant \rho_\lambda(\zeta) - \rho_\lambda(z) + \frac{\alpha}{2}[\operatorname{dist}(z, \zeta)]^2 - A|Q(\lambda)(\zeta - z)|^2 \tag{16}$$

for all $(z, \zeta, \lambda) \in U_{\bar{D}} \times U_{\bar{D}} \times \Delta_{1\cdots N}$. Set

$$\Psi(z, \zeta, \lambda) = \tilde{F}_{\rho_\lambda}(z, \zeta) + A|Q(\lambda)(\zeta - z)|^2 \tag{17}$$

for all $(z, \zeta, \lambda) \in X \times U_{\bar{D}} \times \Delta_{1\cdots N}$. Then it follows from (16) that

$$\operatorname{Re} \Psi(z, \zeta, \lambda) \geqslant \rho_\lambda(\zeta) - \rho_\lambda(z) + \frac{\alpha}{2}[\operatorname{dist}(z, \zeta)]^2 \tag{18}$$

for all $(z, \zeta, \lambda) \in U_{\bar{D}} \times U_{\bar{D}} \times \Delta_{1\cdots N}$. In particular, if $(z, \zeta, \lambda) \in D \times S_K \times \Delta_K$, $K \in P'(N)$, then $\psi(z, \zeta, \lambda) \neq 0$.

From definition of $\Psi(z, \zeta, \lambda)$ it follows that $\Psi(z, \zeta, \lambda)$ is a $C^{(1)}$-function for $(z, \zeta, \lambda) \in D \times S_K \times \Delta_K$ and has the following properties:

(i) $\Psi(z, \zeta, \lambda)$ is holomorphic in $z \in D$, (19)

(ii) $\Psi(z, \zeta, \lambda) \neq 0$ for $(z, \zeta, \lambda) \in D \times S_K \times \Delta_K$, (20)

(iii) $\Psi(z, z, \lambda) = 0$, for all $z \in D$. (21)

Therefore, by Corollary 4.9.4 of ref. [1] we can find a $T^*(X)$-valued $C^{(1)}$-map $S^*(z,\zeta,\lambda)$ defined for $(z,\zeta,\lambda)\in D\times S_K\times\Delta_K$ such that the following conditions are satisfied:

(1) $S^*(z,\zeta,\lambda)\in T_z^*(X)$ for $(z,\zeta,\lambda)\in D\times S_K\times\Delta_K$,

(2) $S^*(z,\zeta,\lambda)$ is holomophic in $z\in D$,

(3) $\varphi(z,\zeta)\Psi(z,\zeta,\lambda)=\langle S^*(z,\zeta,\lambda),S(z,\zeta,\lambda)\rangle$ for $(z,\zeta,\lambda)\in D\times S_K\times\Delta_K$.

By (20) $\Psi(z,\zeta,\lambda)\neq 0$, if $(z,\zeta,\lambda)\in D\times S_K\times\Delta_K$ we have

$$\frac{\varphi(z,\zeta)S^*(z,\zeta,\lambda)}{\langle S^*(z,\zeta,\lambda),S(z,\zeta,\lambda)\rangle}=\frac{S^*(z,\zeta,\lambda)}{\Psi(z,\zeta,\lambda)}. \tag{22}$$

Therefore, $(S^*(z,\zeta,\lambda),1)$ is a Leray section for (D,S,φ). If Stein manifold is $\mathbb{C}^n$, then

$$S_j^*(z,\zeta,\lambda)=2\frac{\partial\rho_\lambda(\zeta)}{\partial\zeta_j}-\sum_{k=1}^{n}a_\lambda^{kj}(\zeta)(\zeta_k-z_k)+A\sum_{k=1}^{n}\overline{Q_{kj}(\lambda)(\zeta_k-z_k)}. \tag{23}$$

Set

$$\psi_K(z,\zeta,\lambda)=\frac{S^*(z,\zeta,\lambda)}{\Psi(z,\zeta,\lambda)},(z,\zeta,\lambda)\in D\times S_K\times\Delta_K,\quad K\in P'(N). \tag{24}$$

Then $\psi_K(z,\zeta,\lambda)$ is a Leray map for the frame $(U_{\bar D},\rho_1,\cdots,\rho_K)$.

Definition 2. A map f defined on Stein manifold X will be called k-holomorphic if for each point $\xi\in X$ there exist holomorphic coordinates $h_1,\cdots,h_n$ in the neighborhood of ξ such that f is holomorphic with respect to $h_1,\cdots,h_k$.

Evidently in $\mathbb{C}^n$, since $Q(\lambda)$ is an orthogonal projection, it follows that for every fixed $(\zeta,\lambda)\in U_{\bar D}\times\Delta_{1\cdots N}$ map (23) and function $\Psi(z,\zeta,\lambda)$ and for every $K\in P'(N)$ and all fixed $(\zeta,\lambda)\in S_K\times\Delta_K$, map $\psi_K(z,\zeta,\lambda)$ is $(q+1)$-holomorphic in $z\in\mathbb{C}^n$. Similarly we have[5]

Lemma 1. (i) *For every fixed* $(\zeta,\lambda)\in U_{\bar D}\times\Delta_{1\cdots N}$, *the map* $S^*(z,\zeta,\lambda)$ *and the function* $\Psi(z,\zeta,\lambda)$ *are* $(q+1)$*-holomorphic in* $z\in X$.

(ii) *For each* $K\in P'(N)$ *and all fixed* $(\zeta,\lambda)\in S_K\times\Delta_K$, *the map* $\psi_K(z,\zeta,\lambda)$ *is* $(q+1)$*-holomorphic in* $z\in D$.

2 Homotopy formulas and the solution of $\bar\partial$-equation on local q-convex domains

Let $T^\psi=B_D+\sum_{K\in P'(N)}R_K^\psi$ and let $L^\psi=B_D+\sum_{K\in P'(N)}L_K^\psi$.

Theorem 1. *For* $n-q\leqslant s\leqslant n$, *and each continuous* (r,s)*-form* f *on* $\bar D$, *such that* $\bar\partial f$ *is also continuous on* $\bar D$, *then*

$$(-1)^{r+s}f=\bar\partial_zT^\psi f-T^\psi\bar\partial_\zeta f+(-1)^{r+s+1}\sum_{K\in P'(N)}Q_K^\psi+P_Df; \tag{25}$$

in particular, if $C(T(X\times X))=D^2=0$, *then* $\hat Q_K^\psi=0$, $\hat P=0$, *we have*

$$(-1)^{r+s}f=\bar\partial_zT^\psi f-T^\psi\bar\partial_\zeta f. \tag{26}$$

Proof. In view of the Koppelman-Leray-Norguet formulas (12) and (13) it is necessary to prove that for all $K\in P'(N)$, $L_K^\psi f=0$.

Fix $K \in P'(N)$ and denote by $\psi_K^1, \cdots, \psi_K^n$ the components of the map ψ_K. Since by Lemma 1(ii), map $\psi_K(z, \zeta, \lambda)$ is $(q+1)$-holomorphic in z, and since $r \geqslant n-q$, one has

$$\bar{\partial}_z \psi_K^{j_1}(z, \zeta, \lambda) \wedge \cdots \wedge \bar{\partial}_z \psi_K^{j_r}(z, \zeta, \lambda) \wedge dz_1 \wedge \cdots \wedge dz_r = 0$$

for all $1 \leqslant j_1, \cdots, j_r \leqslant n$. According to the definition of $L_K^{\psi} f$ (see ref. [4])

$$\langle \psi_K, D S \rangle \wedge (\langle \Delta'' \psi_K, D S \rangle)^{n-1}$$

$$= (-1)^{n(n-1)/2}(n-1)! \left[\sum_{j=1}^{n} (-1)^{j-1} \psi_K^j \bigwedge_{k \neq j} \Delta'' \psi_K^k \right] \bigwedge_{r=1}^{n} (du_r + ((H^{-1}\partial H) \wedge u)_r),$$

where u is assumed to be the local coordinate of $S(z, \zeta)$, and the local coordinate of ψ_K is also denoted by ψ_K, by definition, $L_K^{\psi} f = 0$.

Now we replace the integrals over the manifolds S_K in the homotopy formula (26) by integrals over certain submanifolds Γ_K of D.

2.1 The manifolds Γ_K [5]

For $K = (k_1, \cdots, k_l) \in P(N)$ if $k_1, \cdots, k_l$ are different in pairs, we set

$$U_{\bar{D}}^K = \{\zeta \in U_{\bar{D}} : \rho_{k_1}(\zeta) = \cdots = \rho_{k_l}(\zeta)\};$$

otherwise, we set $U_{\bar{D}}^K = \emptyset$. By condition (i) in Definition 1 each $U_{\bar{D}}^K$ is a closed $C^{(2)}$ submanifold of $U_{\bar{D}}$. We denote by ρ_K, $K \in P(N)$. The function on $U_{\bar{D}}^K$ is defined by

$$\rho_K(\zeta) = \rho_{k_\nu}(\zeta), \qquad (\zeta \in U_{\bar{D}}^K; \nu = 1, \cdots, l). \tag{27}$$

Now, for all $K \in P(N)$, we define

$$\Gamma_K = \{\zeta \in U_{\bar{D}}^K : \rho_j(\zeta) \leqslant \rho_K(\zeta) \leqslant 0 \text{ for } j = 1, \cdots, N\}. \tag{28}$$

Then it is easy to see that all Γ_K are $C^{(2)}$ submanifolds of $\bar{D}$ with piecewise $C^{(2)}$ boundary, and that

$$\bar{D} = \Gamma_1 \cup \cdots \cup \Gamma_N \tag{29}$$

$$\partial \Gamma_K = S_K \cup \Gamma_{K1} \cup \cdots \cup \Gamma_{KN}, \quad K \in P(N). \tag{30}$$

Lemma 2. $\partial \Gamma_k = S_K - \sum_{j=1}^{N} \Gamma_{Kj}, \quad K \in P(N).$

Lemma 3. $\sum_{K \in P'(N)} (-1)^{|K|} \partial(\Gamma_K \times \Delta_{OK}) = \bar{D} \times \Delta_0 + \sum_{K \in P'(N)} (-1)^{|K|} S_K \times \Delta_{OK} - \sum_{K \in P'(N)} \Gamma_K \times \Delta_K.$

2.2 The function $\Phi(z, \zeta, \lambda)$ and the map $\eta(z, \zeta, \lambda)$

Set

$$\rho_\lambda = \lambda_1 \rho_1 + \cdots + \lambda_N \rho_N, \lambda \in \Delta_{1 \cdots N},$$

$$\Phi(z, \zeta, \lambda) = \Psi(z, \zeta, \lambda) - 2\rho_\lambda(\zeta), (z, \zeta, \lambda) \in X \times U_{\bar{D}} \times \Delta_{1 \cdots N}. \tag{31}$$

Then it follows from (18) that $\mathrm{Re}\Phi(z, \zeta, \lambda) \geqslant -\rho_\lambda(\zeta) - \rho_\lambda(z) + \frac{\alpha}{2}[\mathrm{dist}(z, \zeta)]^2$ for all $(z, \zeta, \lambda) \in X \times U_{\bar{D}} \times \Delta_{1 \cdots N}$, where $\alpha > 0$ is a constant from condition (ii) in Definition 1. In particular, $\Phi(z, \zeta, \lambda) \neq 0$ if $(z, \zeta, \lambda) \in D \times \bar{D} \times \Delta_{1 \cdots N}$, and we can define $C^{(1)}$ map as

$$\eta(z, \zeta, \lambda) = \mathring{\lambda}(\lambda_0) \frac{\hat{S}(z, \zeta)}{|S(z, \zeta)|_\theta^2} + (1 - \mathring{\lambda}(\lambda_0)) \frac{S^*(z, \zeta, \mathring{\lambda})}{\Phi(z, \zeta, \mathring{\lambda})}. \tag{32}$$

Note that

$$\eta(z,\zeta,\lambda) = \frac{\hat{S}(z,\zeta)}{|S(z,\zeta)|_\theta^2}, \quad \text{if } 1/2 \leqslant \lambda_0 \leqslant 1, \tag{33}$$

$$\eta(z,\zeta,\lambda) = \frac{S^*(z,\zeta,\mathring{\lambda})}{\Phi(z,\zeta,\mathring{\lambda})}, \quad \text{if } 0 \leqslant \lambda_0 \leqslant 1/4, \tag{34}$$

$$\eta(z,\zeta,\lambda) = \frac{S^*(z,\zeta,\lambda)}{\Phi(z,\zeta,\lambda)}, \quad \text{if } \lambda_0 = 0. \tag{35}$$

Furthermore, we notice that, by (24), (31) and (1), for all $K \in P'(N)$ we have the relation:

$$\eta(z,\zeta,\lambda) = \psi_{OK}(z,\zeta,\lambda), \text{ if } (\zeta,\lambda) \in S_K \times \Delta_{OK}. \tag{36}$$

From Lemma 1 one immediately obtains the following lemma.

Lemma 4. *For fixed* $(\zeta,\lambda) \times U_{\bar{D}} \times \Delta_{1\cdots N}$, *the function* $\Phi(z,\zeta,\lambda)$ *is* $(q+1)$*-holomorphic in* $z \in X$, *and the map* $\eta(z,\zeta,\lambda)$ *is* $(q+1)$*-holomorphic in* $z \in D$.

2.3 The kernels $\hat{G}(z,\zeta,\lambda)$ and $\hat{H}(z,\zeta,\lambda)$

For all $(z,\zeta,\lambda) \in D \times \bar{D} \times \Delta_{01\cdots N}$ with $z \neq \zeta$ we introduce the continuous differential forms:

$$\hat{G}(z,\zeta,\lambda) = \frac{1}{n!\,(2\pi i)^n}\varphi^\nu(z,\zeta)\langle\eta(z,\zeta,\lambda) \wedge D\eta\rangle \wedge (\langle\Delta''\eta(z,\zeta,\lambda), DS\rangle)^{n-1}, \tag{37}$$

$$\hat{H}(z,\zeta,\lambda) = \frac{1}{n!\,(2\pi i)^n}\varphi^\nu(z,\zeta)(\langle\Delta''\eta(z,\zeta,\lambda), DS\rangle)^n. \tag{38}$$

Lemma 5. *If* $C(T(X \times X)) = D^2 = 0$, *then (for proof see Lemma 2.2 in ref.* [6])

$$d\hat{G} = \hat{H}. \tag{39}$$

By (33) *and the definition of Bochner-Martinelli kernel* (2) *there is*

$$\hat{G}\,|_{D\times\bar{D}\times\Delta_0} = \hat{B}. \tag{40}$$

By (36) *and the definition of Koppelman-Leray-Norguet kernel* (3) *it follows that for all* $K \in P'(N)$,

$$\hat{G}\,|_{D\times S_K\times\Delta_{OK}} = (-1)^{|K|}\hat{R}_K. \tag{41}$$

We omit the simple proof of the following lemma.

Lemma 6. *Denote by* $[\hat{G}(z,\zeta,\lambda)]_{\deg\lambda=k}$ *and* $[\hat{H}(z,\zeta,\lambda)]_{\deg\lambda=k}$ *the parts of the forms* $\hat{G}(z,\zeta,\lambda)$ *and* $\hat{H}(z,\zeta,\lambda)$, *respectively, which are of degree k in* λ. *Then the following statements hold*:

(i) *The singularity at* $z=\zeta$ *of the form* $[\hat{G}(z,\zeta,\lambda)]_{\deg\lambda=k}$ *is of order* $\leqslant 2n-2k-1$.

(ii) *The singularities at* $z=\zeta$ *of the first-order derivatives with respect to z of the coefficients of* $[\hat{G}(z,\zeta,\lambda)]_{\deg\lambda=k}$ *are of order* $\leqslant 2n-2k$.

(iii) *The singularity at* $z=\zeta$ *of the form* $[\hat{H}(z,\zeta,\lambda)]_{\deg\lambda=k}$ *is of order* $\leqslant 2n-2k+1$.

Lemma 7.

(i) *If* $f \in C^0_{r,s}(\bar{D})$, $n-q+1 \leqslant s \leqslant n$, *then* $\int_{(\zeta,\lambda)\in\Gamma_K\times\Delta_K} f(\zeta) \wedge \hat{G}(z,\zeta,\lambda) = 0.$

(ii) *If* $f \in C^0_{n,n-q}(\bar{D})$, *then* $\bar{\partial}_z \int_{(\zeta,\lambda)\in \Gamma_K \times \Delta_K} f(\zeta) \wedge \hat{G}(z,\zeta,\lambda) = 0$ *for all* $z \in D$ *and* $K \in P'(N)$.

Proof. Denote by $[\hat{G}(z,\zeta,\lambda)]^r_s$ the part of $\hat{G}(z,\zeta,\lambda)$ which is of degree r with respect to z and degree s with respect to $\bar{z}$. If $f \in C^0_{r,s}(\bar{D})$, $K \in P'(N)$, then

$$\int_{\Gamma_K \times \Delta_K} f(\zeta) \wedge \hat{G}(z,\zeta,\lambda) = \int_{\Gamma_K \times \Delta_K} f(\zeta) \wedge [\hat{G}(z,\zeta,\lambda)]^r_{s-1}, \quad z \in D.$$

On the orther hand, when $\lambda_0 = 0$, by Lemma 4 $\eta(z,\zeta,\lambda)$ is $(q+1)$-holomorphic in z, but by assumption $s \geqslant n-q+1$, therefore, $[\hat{G}(z,\zeta,\lambda)]^r_{s-1} = 0$, $(z,\zeta,\lambda) \in D \times \Gamma_K \times \Delta_K$, and (i) holds.

Similarly, when $\lambda_0 = 0$, $\eta(z,\zeta,\lambda)$ is $(q+1)$-holomorphic in z; therefore $\bar{\partial}_z[\hat{G}(z,\zeta,\lambda)]^r_{n-q-1} = 0$, $(z,\zeta,\lambda) \in D \times \Gamma_K \times \Delta_K$, and (ii) holds.

2.4 Operator H

Let $f \in B^\beta_{r,*}(D)$, $0 \leqslant \beta \leqslant 1$. Then, for all $K \in P'(N)$, we define

$$H_K f(z) = \int_{(\zeta,\lambda)\in \Gamma_K \times \Delta_{OK}} f(\zeta) \wedge \hat{H}(z,\zeta,\lambda), \quad z \in D. \tag{42}$$

It follows from Lemma 6(iii) that these integrals converge and the so-defined differential forms $H_K f$ are continuous on D. We set

$$Hf = \sum_{K \in P'(N)} (-1)^{|K|} H_K f \tag{43}$$

for $f \in B^\beta_{r,*}(D), 0 \leqslant \beta < 1$.

Now let $f \in B^\beta_{r,s}(D), 0 \leqslant \beta < 1, 0 \leqslant r, s \leqslant n$. Since $\hat{H}(z,\zeta,\lambda)$ is of degree $2n$ and contains the monomials of degree r in z, and since $\dim_{\mathbb{R}} \Gamma_K \times \Delta_{OK} = 2n+1$ such that $\hat{H}(z,\zeta,\lambda)$ only contains monomials of degree $2n+1-r-s (0 \leqslant r, s \leqslant n)$ in (ζ,λ), only monomials of $\hat{H}(z,\zeta,\lambda)$ of bidegree $(r, s-1)$ $(0 \leqslant r, s \leqslant n)$ in z contribute to integral (42). This implies that $H_K f = 0$, if $s = 0$ or $2n+1-r-s < |K| = \dim_{\mathbb{R}} \Delta_{OK}$.

Hence for $f \in B^\beta_{r,s}(D)$, $0 \leqslant \beta < 1$, $0 \leqslant r, s \leqslant n$, we have

$$\begin{cases} Hf = \sum\limits_{\substack{K \in P'(N) \\ |K| \leqslant 2n+1-r-s}} (-1)^{|K|} H_K f, \\ Hf = 0, \text{ if } s = 0; \; Hf \in C^0_{r,s-1}(D), \text{ if } 1 \leqslant s \leqslant n. \end{cases} \tag{44}$$

Theorem 2. *Assume* $D \subset\subset X$ *to be a local q-convex domain* $(0 \leqslant q \leqslant n-1)$ *in Stein manifold*, $C(T(X \times X)) = D^2 = 0$. *Let* $n-q \leqslant s \leqslant n, 0 \leqslant \beta \leqslant 1$. *Then, for all* $f \in B^\beta_{r,s}(D)$ *such that* $\bar{\partial} f \in B^\beta_*(D)$, *we have homotopy formula*:

$$f = \bar{\partial}_z Hf + H\bar{\partial}_\zeta f \text{ on } D. \tag{45}$$

In particular, if $\bar{\partial} f = 0$ *on* D, *then*

$$f = \bar{\partial}_z Hf; \tag{46}$$

that is,

$$u := Hf = \sum_{K \in P'(N)} (-1)^{|K|} H_K f \tag{47}$$

is a continuous solution of $\bar{\partial} u = f$ *on* D.

Proof. First we consider the case of closed domain $\bar{D}$. Let $g \in C^0_{r,s}(\bar{D})$, $0 \leqslant r, s \leqslant n$. Then by (39)

$$d_{\zeta,\lambda}(g \wedge \hat{G}) = dg \wedge \hat{G} - \bar{\partial}_z(g \wedge \hat{G}) + (-1)^{r+s} g \wedge \hat{H},$$

and it follows from Stokes' formula (which can be applied in view of Lemma 6) that

$$\int_{\partial(\Gamma_K \times \Delta_{OK})} g \wedge \hat{G} = \int_{\Gamma_K \times \Delta_{OK}} \bar{\partial}_\zeta g \wedge \hat{G} + \bar{\partial}_z \int_{\Gamma_K \times \Delta_{OK}} g \wedge \hat{G} + (-1)^{r+s} H_K g$$

for all $K \in P'(N)$. By Lemma 3 this implies that

$$\int_{D \times \Delta_0} g \wedge \hat{G} + \sum_{K \in P'(N)} (-1)^{|K|} \int_{S_K \times \Delta_{OK}} g \wedge \hat{G} - \sum_{K \in P'(N)} \int_{\Gamma_K \times \Delta_K} g \wedge \hat{G}$$
$$= \sum_{K \in P'(N)} (-1)^{|K|} \left(\int_{\Gamma_K \times \Delta_{OK}} \bar{\partial}_\zeta g \wedge \hat{G} + \bar{\partial}_z \int_{\Gamma_K \times \Delta_{OK}} g \wedge \hat{G} + (-1)^{r+s} H_K g \right).$$

Taking into account (40) and (41) as well as the definitions of T^ψ and H, this can be written as

$$T^\psi g - \sum_{K \in P'(N)} \int_{\Gamma_K \times \Delta_K} g \wedge \hat{G}$$
$$= \sum_{K \in P'(N)} (-1)^{|K|} \left(\int_{\Gamma_K \times \Delta_{OK}} \bar{\partial}_\zeta g \wedge \hat{G} + \bar{\partial}_z \int_{\Gamma_K \times \Delta_{OK}} g \wedge \hat{G} \right) + (-1)^{r+s} H g. \tag{48}$$

Now we consider a form $f \in C^0_{r,s}(\bar{D})$, $n - q \leqslant s \leqslant n$ such that $\bar{\partial} f$ is also continuous on $\bar{D}$. Setting $g = \bar{\partial} f$ and taking into account Lemma 7(i), we obtain

$$T^\psi \bar{\partial}_\zeta f = \sum_{K \in P'(N)} (-1)^{|K|} \bar{\partial}_z \int_{\Gamma_K \times \Delta_{OK}} \bar{\partial}_\zeta f \wedge \hat{G} + (-1)^{r+s+1} H \bar{\partial}_\zeta f.$$

Setting $g = f$ in (48), applying $\bar{\partial}_z$ to the resulting relation and taking into account Lemma 7 (ii), we obtain

$$\bar{\partial}_z T^\psi f = \sum_{K \in P'(N)} (-1)^{|K|} \bar{\partial}_z \int_{\Gamma_K \times \Delta_{OK}} \bar{\partial}_\zeta f \wedge \hat{G} + (-1)^{r+s} \bar{\partial}_z H f.$$

This implies that

$$\bar{\partial}_z T^\psi f - T^\psi \bar{\partial}_\zeta f = (-1)^{r+s} (\bar{\partial}_z H f + H \bar{\partial}_\zeta f),$$

and hence by (26) of Theorem 1 we have

$$f = \bar{\partial}_z H f + H \bar{\partial}_\zeta f \text{ on } D. \tag{49}$$

Finally for the general case of local q-convex domain $D \subset\subset X$, it may be proved by utilizing the preceeding result for closed local q-convex domain $\bar{D} \subset\subset X$ and the limiting process[5].

References

1 Henkin, G. M., Leiterer, J., *Theory of Functions on Complex Manifolds*, Berlin: Akademie-Verlag Berlin and Birkhäuser-Verlag Boston, 1984.

2 Demaily, J. P., Laurent-Thiebaut, C., Formules intégrales pour les formes différentielles de type (p, q) dans les variétés de Stein, *Ann Sci E'cole Norm. Sup.*, 1987, 20(4):579.

3 Zhong Tongde, Koppelman-Leray-Norguet formula on complex manifolds and applications, *Chinese Journal of Contemporary Mathematics*, 1995, 16(3): 235.

4 Zhong Tongde, Huang Sha, *Complex Analysis in Several Variables* (in Chinese), Shijiazhuang: Hebei Educational Press, 1990.

5 Laurent-Thiebaut, C., Leiterer, J., Uniform estimates for the Cauchy-Riemann equation on q-convex wedges, *Ann. Inst. Fourier* (Grenoble), 1993, 43(2):383.

6 Zhong Tongde, $\bar{\partial}$-equation on complex manifolds, *J. Math.*, 1997, 17(1):15.