

A note on finite abelian covers

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Abstract We prove that any abelian cover over a smooth variety is defined by some cyclic equations. From the defining equations, we compute explicitly the normalization, branch locus, ramification indices, global invariants, and the resolution of singularities. As an application, we construct a new algebraic surface which is the quotient of ball.

Keywords abelian cover, normalization, irregularity, the quotient of ball

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1 Introduction

Finite covers are widely used in algebraic geometry. They attract many authors to study the basic problems on finite covers: to find the construction data, to compute the branch locus, to resolve the singularities and to compute the global invariants.

In order to get the defining data for a finite cover, the modern method starts from characterizing the ring structure of a finite extension, i.e., finding all conditions for a finite module to be a finite algebra. For example, Miranda [14] and Pardini [15] did it for triple covers and abelian covers respectively. As a consequence of this method, Miranda [14] established the correspondence between the binary cubic forms on an algebraic variety and triple covers, which generalizes the classical correspondence between integral binary cubic forms and cubic number fields. On the other hand, when the degree of the cover is higher (even if in the abelian case), the construction data obtained by this method is too complicated to use as noted by several authors.

The classical method starts essentially with the defining equations of a finite cover. In fact, the theory of algebraic function fields is the classical language of the theory of finite covers. See, for example, [8] for double covers, [1] and [2] (or [4, p. 194]), [16] and [17] for triple covers, [6] for bi-double covers, [18] for B-J covers.

Double covers, triple covers and cyclic covers are well understood, and the basic problems of the covers have been solved starting with their defining equations.

Using the modern method, Pardini [15] gave a set of conditions for a free R -module A to be an R -algebra which is an abelian extension of R . As Bauer and Catanese [3] noted, the treatment by Pardini is very algebraic and not explicit. It is not convenient to use.

Our purpose of this note is to try to solve the basic problems about arbitrary finite abelian covers by using their defining equations. In fact, the defining equations are quite simple and all interesting

information (global invariants, singularities, branch locus, ramification indices) can be obtained explicitly from the equations.

A finite cover $\pi : Y \rightarrow X$ is said to be abelian with an abelian group G if the function field $\mathbb{C}(Y)$ of Y is a finite Galois extension of $\mathbb{C}(X)$ such that $\text{Gal}(\mathbb{C}(Y)/\mathbb{C}(X)) = G$. From the structure theorem about abelian field extensions (see [12], Corollary 1.16 at p. 208, and Theorem 6.2 at p. 289),

$$\mathbb{C}(Y) = \mathbb{C}(X)[\sqrt[n_1]{\xi_1}, \dots, \sqrt[n_k]{\xi_k}],$$

where $G \cong \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_k}$. Namely, the field extension is defined by $\eta_1^{n_1} = \xi_1, \dots, \eta_k^{n_k} = \xi_k$. It implies that any abelian G -cover $\pi : Y \rightarrow X$ over a smooth variety X is determined by some n_i -divisible effective divisors D_i , $i = 1, \dots, k$, i.e.,

$$D_1 \equiv n_1 L_1, \quad \dots, \quad D_k \equiv n_k L_k. \quad (1)$$

In fact, if $f_i \neq 0$ is the global section of $\mathcal{O}_X(n_i L_i)$ defining D_i , i.e., $D_i = \text{div}(f_i)$, then Y is the normalization of the variety Σ defined by the following equations,

$$z_1^{n_1} = f_1, \quad \dots, \quad z_k^{n_k} = f_k, \quad (2)$$

where z_i is the fiber coordinate of the line bundle $[L_i]$ associated with L_i . π is just the composition of the normalization $Y \rightarrow \Sigma$ with the natural projection $\Sigma \rightarrow X$. Therefore, (1) is the data determining the abelian cover π , and (2) is the defining equations of π . Because we did not find an appropriate reference, for the reader's convenience, we will present a proof of the known fact that any abelian cover can be constructed in this way.

The following formula can be used to compute some of the global invariants of Y , e.g., the geometric genus $p_g(Y)$, the irregularity $q(Y)$, and the Euler characteristic number $\chi(\mathcal{O}_Y)$.

Theorem 1.1. Denote by $[Z]$ the integral part of a $\mathbb{Q}$ -divisor Z and

$$L_g = -\sum_{i=1}^k g_i L_i + \left[\sum_{i=1}^k \frac{g_i}{n_i} D_i \right].$$

We have

$$\pi_* \mathcal{O}_Y = \bigoplus_{g \in G} \mathcal{O}_X \left(-\sum_{i=1}^k g_i L_i + \left[\sum_{i=1}^k \frac{g_i}{n_i} D_i \right] \right) = \bigoplus_{g \in G} \mathcal{O}_X(L_g).$$

When $k = 1$, the theorem is obtained by Esnault and Viehweg [7] and is used in Viehweg's proof of Kawamata-Viehweg theorem.

Actually, the proof of Theorem 1.1 is the computation of integral bases of the abelian extension $\mathcal{O}_Y(Y)/\mathcal{O}_X(X)$ of $\mathcal{O}_X$ -algebras.

The branch locus of π is the subset of $\{D_1, \dots, D_k\}$. Over each point p on the branch locus, we will compute also the ramification indices.

In order to resolve the singularities of Y , which lie over the singular points of the branch locus, we only need to resolve the singularities of $D = D_1 + \dots + D_k$. In fact, by blowing up X , $\sigma : X' \rightarrow X$, we can assume that $D' = \sigma^* D$ is a normal crossing divisor. Then the pullback abelian cover $\pi' : Y' \rightarrow X'$ is defined by

$$\sigma^* D_1 \equiv n_1 \sigma^* L_1, \quad \dots, \quad \sigma^* D_k \equiv n_k \sigma^* L_k.$$

Y' admits at worst Hirzebruch-Jung singularities, which can be resolved by the standard method (see [5, pp. 99–107]).

Because we know the ramification indices of the ramification divisors, by Hurwitz formula, we can compute the canonical divisor of the smooth model of Y . From the canonical resolution of the singularities, we have formulas to compute the first Chern class of Y .

To explain our method, we will construct a new surface of the general type whose Chern numbers satisfies $c_1^2 = 3c_2$. We also simplify Zuo and Ishida's computations of the invariants of some famous surfaces constructed by Hirzebruch (cf. [10, 19, 11]).

2 The defining equations of an abelian cover

We always assume that X is smooth. The data of an abelian cover over X with group $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ consist of k linear equivalence relations between divisors on X ,

$$D_1 \equiv n_1 L_1, \quad \dots, \quad D_k \equiv n_k L_k,$$

where $D_1, \dots, D_k$ are either effective divisors or zero on X .

Let $\mathcal{L}_i = \mathcal{O}_X(L_i)$. Then $D_i = \text{div}(f_i)$ is defined by a nonzero global section $f_i \in H^0(X, \mathcal{L}_i^{n_i})$. From these data, we can construct an abelian cover $\pi : Y \rightarrow X$ with the group $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$.

Denote by $V(\mathcal{L}_i) = \text{Spec } S(\mathcal{L}_i)$ the associated line bundle of $\mathcal{L}_i$, where $S(\mathcal{L}_i)$ is the symmetric $\mathcal{O}_X$ -algebra of $\mathcal{L}_i$. Let z_i be the fiber coordinate of $V(\mathcal{L}_i)$, i.e., a global section of $p^* \mathcal{L}_i$, where p is the bundle projection of $V(\mathcal{L}_i)$ to X . Thus we obtain a polynomial section $p_i(z_i) = z_i^{n_i} - f_i$ of $p^* \mathcal{L}_i^{n_i}$. Consider the subvariety $\Sigma \subset V := V(\mathcal{L}_1) \oplus \cdots \oplus V(\mathcal{L}_k)$ defined by

$$z_1^{n_1} - f_1 = 0, \quad \dots, \quad z_k^{n_k} - f_k = 0. \quad (3)$$

Then the composition of the normalization $Y \rightarrow \Sigma$ with the bundle projection defines an abelian cover $\pi : Y \rightarrow X$ with group G . The equations in (3) are called the defining equations of the abelian cover π .

Conversely, we will prove that any abelian cover $\pi : Y \rightarrow X$ can be constructed in this way. Let G be the Galois group. Assume that Y is normal, and $G = G_1 \times \cdots \times G_k$, where $G_i = \mathbb{Z}_{n_i}$ is a cyclic group of order n_i . The following theorem is based on the structure theorem about abelian field extensions (see [12], Corollary 1.16 at p. 208, and Theorem 6.2 at p. 289).

Theorem 2.1. *For any finite abelian cover $\pi : Y \rightarrow X$ with group $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, there exist effective divisors $D_1, \dots, D_k$ (maybe zero) on X , and some other divisors $L_1, \dots, L_k$ satisfying the equations*

$$D_1 \equiv n_1 L_1, \quad \dots, \quad D_k \equiv n_k L_k$$

such that $\pi : Y \rightarrow X$ is defined by (3). Furthermore, we can require that all the coefficients of the components in D_i are less than n_i .

Proof. Let $\pi^* : F = \mathbb{C}(X) \hookrightarrow K = \mathbb{C}(Y)$ be the function field extension. Then K/F is an abelian Galois extension. Let

$$H_i = G_1 \times \cdots \times G_{i-1} \times \{1\} \times G_{i+1} \times \cdots \times G_k,$$

and let $K_i = K^{H_i}$. Then K_i is a Galois extension of F , and

$$\text{Gal}(K_i/F) \cong G/H_i \cong G_i$$

which is a cyclic group of order n_i . Furthermore, $K_{i+1} \cap K_1 \cdots K_i = F$, and $K = K_1 \cdots K_k$ (see [12], Corollary 1.16 at p. 208, and Theorem 6.2 at p. 289).

Because $\mathbb{C} \subset F$ contains all roots of unity, the field extension K_i/F is generated by one element $\theta_i \in K_i \setminus F$ satisfying

$$\theta_i^{n_i} = \pi^* a_i, \quad a_i \in F,$$

i.e., $K_i = F[\theta_i]$. So $K = K_1 \cdots K_k = F[\theta_1, \dots, \theta_k]$.

Now, let L_i be any divisor (not necessarily effective) on X such that

$$n_i L_i + \text{Div}(a_i) \geq 0.$$

We can require also that L_i be minimal, i.e., for any prime divisor P , the divisor $n_i(L_i - P) + \text{Div}(a_i)$ is not effective, equivalently, the coefficients of the components of $n_i L_i + \text{Div}(a_i)$ are nonnegative integers less than n_i .

Let $\mathcal{L}_i = \mathcal{O}_X(L_i)$. Then L_i is defined by a nonzero rational section ℓ_i of $\mathcal{L}_i$. (If $L_i = A - B$ denotes the difference of two effective divisors A and B , then A (resp. B) is defined by a regular global section

a of $\mathcal{O}_X(A)$ (resp. b of $\mathcal{O}_X(B)$). One can see that $\ell_i = a/b$ is a rational section of $\mathcal{O}_X(L_i)$ which defines L_i .) Now, consider the following rational sections of $\mathcal{L}_i^{n_i}$ and $\pi^*\mathcal{L}_i$ respectively,

$$f_i = a_i \ell_i^{n_i}, \quad \tilde{\theta}_i = \pi^* \ell_i \cdot \theta_i.$$

The divisor of f_i is $\text{Div}(a_i) + n_i L_i$. By the choice of L_i , this is an effective divisor and f_i has no poles, so $f_i \in H^0(X, \mathcal{L}_i^{n_i})$. As sections of $\pi^*\mathcal{L}_i^{n_i}$, $i = 1, \dots, r$, we have

$$\tilde{\theta}_1^{n_1} = \pi^*(f_1), \quad \dots, \quad \tilde{\theta}_k^{n_k} = \pi^*(f_k). \quad (4)$$

Because of these equations, $\tilde{\theta}_i = \pi^*(\ell_i)\theta_i$ has no pole when viewed as a rational section of $\pi^*\mathcal{L}_i$ on Y . So $\tilde{\theta}_i \in H^0(Y, \pi^*\mathcal{L}_i)$.

On the other hand, we denote by $p_i : V(L_i) \rightarrow X$ the line bundle associated with L_i , and by $z_i \in H^0(V(L_i), p_i^*\mathcal{L}_i)$ the fiber coordinate of $V(L_i)$. Let $V = \bigoplus_i V(L_i)$, and $z = (z_1, \dots, z_k)$. Then the common zero set of the sections $z_i^{n_i} - \pi^*f_i \in H^0(Y, \pi^*\mathcal{L}_i^{n_i})$ defines a subvariety (maybe non-normal) $\Sigma \subset V$, which is an abelian cover of X with the group G , $p|_\Sigma : \Sigma \rightarrow X$. We say simply that Σ is defined by the following equations in V ,

$$z_1^{n_1} = f_1, \quad \dots, \quad z_k^{n_k} = f_k. \quad (5)$$

$\tilde{\theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_k)$ defines a section of the bundle $\tilde{p} : \bigoplus V(\pi^*\mathcal{L}_i) \rightarrow Y$ which is the pullback bundle of $p : V \rightarrow X$ under the base change $\pi : Y \rightarrow X$.

$$\begin{array}{ccc} \bigoplus V(\pi^*\mathcal{L}_i) & \xrightarrow{\tilde{\pi}} & V \\ \tilde{p} \downarrow \uparrow \tilde{\theta} & \nearrow \nu & \downarrow p \\ Y & \xrightarrow{\pi} & X. \end{array}$$

So π is lifted to a map $\nu = \tilde{\pi} \circ \tilde{\theta} : Y \rightarrow V$. Locally, $\nu(y) = \tilde{\pi}(y, \tilde{\theta}(y)) = (\pi(y), \tilde{\theta}(y))$, the fiber coordinate of $\nu(y)$ is $\tilde{\theta}(y)$, i.e., $z(\nu(y)) = \tilde{\theta}(y)$ and $\nu^*(z) = \tilde{\theta}$ as sections. Hence (4) is the pullback of (5) under ν^* , i.e.,

$$z_1(\nu(y))^{n_1} = f_1(\pi(y)), \quad \dots, \quad z_k(\nu(y))^{n_k} = f_k(\pi(y)). \quad (6)$$

Hence the image of ν is obviously Σ which is a (maybe non-normal) Galois cover of X with group G . Now we see that the birational finite map ν is nothing but the normalization of Σ and $\pi := p \circ \nu$. $\square$

Remark 2.1. If the divisors $D_1, \dots, D_k$ are all zero, then π is unramified. So π is determined by k torsion divisors. The converse is also true, i.e., if π is unramified, and if D_i contains no component whose multiplicity in D_i is n_i , then $D_1 = \dots = D_k = 0$. When $k = 1$, this is well-known. If $k > 1$, one can also prove it by using induction on k .

Corollary 2.1. π is unramified if and only if $D_1 = \dots = D_k = 0$.

3 The invariants of abelian covers

In this section, we use $[s/t]$ to denote the maximal integer $\leq s/t$. A k -tuple of integers $g = (g_1, \dots, g_k) \in \mathbb{Z}^k$ is said to be in $G = \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_k}$, if $0 \leq g_i < n_i$ for any i . For convenience, we introduce a new notation: if $f = \prod p_i^{a_i}$, $a_i \in \mathbb{Q}$, is the prime factorization, then we let $[f] = \prod p_i^{[a_i]}$.

From now on, we assume that the abelian cover $\pi : \Sigma \rightarrow X$ is defined by (2) and Y is the normalization of Σ . In order to compute the normalization, we can do it locally. Suppose $X = \text{Spec}(A)$, and $\Sigma = \text{Spec}(B)$,

$$B = A[\alpha_1, \dots, \alpha_k] \cong A[z_1, \dots, z_k]/(z_1^{n_1} - f_1, \dots, z_k^{n_k} - f_k),$$

where $\alpha_i^{n_i} - f_i = 0$. Because X is nonsingular, the local ring is always a UFD. So we can assume that A is a UFD and contains $\mathbb{C}$. Denote respectively by F and K the fraction fields of A and B . Then $K = F[\alpha_1, \dots, \alpha_k]$. If $g = (i_1, \dots, i_k) \in G$, then we denote

$$\alpha^g = \alpha_1^{i_1} \dots \alpha_k^{i_k}.$$

Proposition 3.1. $h = \sum_{g \in G} k_g \alpha^g \in K$ is integral over A if and only if $k_g \alpha^g$ is integral over A for any $g \in G$.

Proof. Denote by ξ_i the n_i -th root of unity, $\xi_i^{n_i} = 1$. Then for any $g = (g_1, \dots, g_k) \in G$, we have $g(\alpha_j) = \xi_j^{g_j} \alpha_j$, $j = 1, \dots, k$.

We prove the lemma by induction on k .

The case when $k = 0$ is trivial. We rewrite $h \in A[\alpha_2, \dots, \alpha_k][\alpha_1]$ as

$$h = h_0 + h_1 \alpha_1 + \dots + h_{n_1-1} \alpha_1^{n_1-1}.$$

Denote $\sigma_1 = (1, 0, \dots, 0) \in G = \text{Gal}(K/F)$. Note that h is integral over $\mathcal{O}_X$, so are $\sigma_1(h), \dots, \sigma_1^{n_1-1}(h)$ because σ_1 fixes A and F . Now we see that the elements

$$h, \quad \sigma_1(h), \quad \dots, \quad \sigma_1^{n_1-1}(h) \quad (7)$$

are linear transformations of

$$h_0, \quad h_1 \alpha_1, \quad \dots, \quad h_{n_1-1} \alpha_1^{n_1-1}, \quad (8)$$

and the transformation matrix is the Vandermonde matrix of $(1, \xi_1, \dots, \xi_1^{n_1-1})$. So (8) can be expressed linearly by (7). Hence $h_i \alpha_1^i$ is integral over K for any i .

We can do the same process for $h_i \alpha_1^i$ with respect to $\alpha_2, \dots$. It is easy to see that the proposition can be proved eventually. $\square$

Theorem 3.1. The integral closure $\tilde{B}$ of B in its fraction field K is freely generated over A by the generators

$$\left\{ \alpha_g = \frac{\prod_{i=1}^k \alpha_i^{g_i}}{[\prod_{i=1}^k f_i^{g_i/n_i}]} \mid g = (g_1, \dots, g_k) \in G \right\}.$$

Proof. According to Proposition 3.1, we see that $\tilde{B}$ is generated over A by the elements of the form $k_g \alpha^g$. Let $N = \prod_{i=1}^k n_i$. Then

$$(k_g \alpha^g)^N = k_g^N \prod_{i=1}^k \alpha_i^{g_i N} = k_g^N \prod_{i=1}^k f_i^{g_i N/n_i} \in K$$

is integral over A , so it must be in A because A is a UFD. Namely, for any prime p in A ,

$$\text{order}_p \left(k_g^N \prod_{i=1}^k f_i^{g_i N/n_i} \right) = N \cdot \text{order}_p(k_g) + N \cdot \text{order}_p \left(\prod_{i=1}^k f_i^{g_i/n_i} \right) \geq 0,$$

i.e.,

$$\text{order}_p(k_g^{-1}) \leq \text{order}_p \left(\prod_{i=1}^k f_i^{g_i/n_i} \right).$$

Because $k_g \alpha^g$ is a generator,

$$\text{order}_p(k_g^{-1}) = \left\lceil \text{order}_p \left(\prod_{i=1}^k f_i^{g_i/n_i} \right) \right\rceil,$$

we have $k_g^{-1} = [\prod_{i=1}^k f_i^{g_i/n_i}]$.

Therefore, the generators form an integral base of $\tilde{B}$ over A . $\square$

Theorem 1.1 is just the globalization of Theorem 3.1. By using Riemann-Roch theorem for surfaces, we get

Corollary 3.2. Assume that X is a smooth surfaces. Denote

$$L_g = - \sum_{i=1}^k g_i L_i + \left[\sum_{i=1}^k \frac{g_i}{n_i} D_i \right].$$

Then

$$\chi(\mathcal{O}_Y) = |G| \cdot \chi(\mathcal{O}_X) + \sum_{g \in G} \frac{1}{2} (L_g^2 - L_g K_X).$$

Theorem 3.2. Let P be an irreducible and reduced hypersurface in X , let $\bar{P} = \pi^{-1}(P)$ be the reduced preimage of P in Y , and let a_i be the multiplicity of P in $D_i = \text{div}(f_i)$. Then

$$\pi^* P = \frac{|G|}{d_P} \bar{P},$$

where

$$d_P = \gcd \left(|G|, |G| \frac{a_1}{n_1}, \dots, |G| \frac{a_k}{n_k} \right)$$

is the number of points in the preimage of a generic point on P .

Proof. At a generic point p of P , P is smooth and is defined locally by a coordinate $x = 0$. So the cover is defined locally by

$$z_1^{n_1} = x^{a_1}, \quad \dots, \quad z_k^{n_k} = x^{a_k}.$$

Now, we prove the theorem by using induction on k . In fact, we only need to prove that $\pi^{-1}(p)$ consists of d_P distinct points.

When $k = 1$, the result can be found in [7, p. 31, Lemma].

Assume that it is true for $k - 1$. We factorize the cover $\pi : Y \rightarrow X$ as $\pi : Y \xrightarrow{\pi'} Z \xrightarrow{\varphi} X$, where φ is defined by $z_1^{n_1} = x^{a_1}$, and π' is defined by

$$z_2^{n_2} = \varphi^*(x)^{a_2}, \quad \dots, \quad z_k^{n_k} = \varphi^*(x)^{a_k}.$$

Note that $\varphi^*(x) = y^{n_1/d_1}$, where $d_1 = \gcd(n_1, a_1)$. $p \in P$ is generic, so $\varphi^{-1}(p)$ consists of d_1 smooth points of $\varphi^{-1}(P) = P'$. Hence the number of points in $\pi^{-1}(p)$ is $d_1 \cdot d_{P'}$. By the induction hypothesis,

$$\begin{aligned} d_{P'} &= \gcd \left(\frac{|G|}{n_1}, \frac{|G|}{n_1} \frac{a_2 n_1 / d_1}{n_2}, \dots, \frac{|G|}{n_1} \frac{a_k n_1 / d_1}{n_k} \right) \\ &= \gcd \left(\frac{|G|}{n_1}, |G| \frac{a_2}{d_1 n_2}, \dots, |G| \frac{a_k}{d_1 n_k} \right), \end{aligned}$$

so

$$d_P = d_1 d_{P'} = \gcd \left(|G|, |G| \frac{a_1}{n_1}, |G| \frac{a_2}{n_2}, \dots, |G| \frac{a_k}{n_k} \right).$$

This proves the formula. □

4 Applications

In this section, we give some applications of our method. At first, we construct a new ball quotient surface. See Example 4.1. Secondly, we compute the irregularity of Hirzebruch's example which is the only one of Hirzebruch's ball quotient surfaces whose irregularity has never been computed as far as the author knows. See Example 4.2. Thirdly, we can also compute the invariants of the surface constructed by Hirzebruch [9, 10], which has been computed by Ishida [11], Libgober [13] and Zuo [19] by using different methods.

Example 4.1. This new ball quotient surface is an abelian cover over $\mathbb{P}^2$ whose branch locus is the following line configurations as Figure 1.

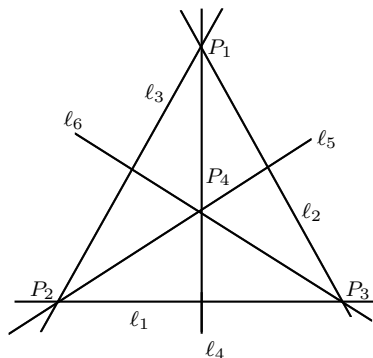


Figure 1 Branch locus

The surface Y is defined by the following equations,

$$z_1^5 = \ell_2 \ell_3 \ell_5 \ell_6^2, \quad z_2^5 = \ell_1 \ell_2 \ell_4 \ell_5^2, \quad z_3^5 = \ell_1 \ell_3 \ell_4 \ell_6^2.$$

Let $\sigma : X \rightarrow \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at P_1, P_2, P_3 and P_4 , $\widetilde{L}_i = \text{div}(\ell_i)$ be the strict transform of L_i , and E_i be the exceptional curve over P_i . Then we get 15 double points. At each double point, the cover is defined locally by

$$z_1^5 = x^{a_1} y^{b_1}, \quad z_2^5 = x^{a_2} y^{b_2}, \quad z_3^5 = x^{a_3} y^{b_3}.$$

It is easy to see that there exist some i, j such that $a_i b_j - a_j b_i \equiv 1 \pmod{5}$. So one can prove that Y is smooth.

Denote by $\tilde{\ell}_i$ the strict image of ℓ_i and by $e_i = 0$ the defining equation of the exceptional curve E_i . The abelian cover $\pi : Y \rightarrow X$ is defined by the following equations,

$$z_1^5 = \tilde{\ell}_2 \tilde{\ell}_3 \tilde{\ell}_5 \tilde{\ell}_6^2 e_1^2 e_2^2 e_3^3 e_4^3, \quad z_2^5 = \tilde{\ell}_1 \tilde{\ell}_2 \tilde{\ell}_4 \tilde{\ell}_5^2 e_1^2 e_2^3 e_3^2 e_4^3, \quad z_3^5 = \tilde{\ell}_1 \tilde{\ell}_3 \tilde{\ell}_4 \tilde{\ell}_6^2 e_1^3 e_2^2 e_3^2 e_4^3.$$

According to Theorem 3.2, the ramification indices of the curves over $\tilde{H}_i = \text{div}(\tilde{\ell}_i)$ and E_i are 5. So

$$\begin{aligned} K_Y &= \pi^* \left(\sigma^*(-3H) + \sum_{i=1}^4 E_i + \left(1 - \frac{1}{5}\right) \sum_{i=1}^6 \tilde{H}_i + \left(1 - \frac{1}{5}\right) \sum_{i=1}^4 E_i \right) \\ &= \pi^* \left(\frac{9}{5} \sigma^*(H) - \frac{3}{5} \sum_{i=1}^4 E_i \right). \end{aligned}$$

It implies $K_Y^2 = 225$. Due to Corollary 3.2, we get $\chi(\mathcal{O}_Y) = 25$. By Noether's formula, $\chi_{\text{top}}(\mathcal{O}_Y) = 75$. So the surface Y satisfies $c_1^2 = 3c_2$.

Via Theorem 1.1, we can compute L_g , for all $g \in \mathbb{Z}_5^{\oplus 3}$, where L_g is the same notation as Theorem 1.1. Then

$$\begin{aligned} L_{(g_1, g_2, g_3)} &= - \sum_{i=1}^3 g_i L_i + \left[\sum_{i=1}^3 \frac{g_i}{n_i} D_i \right] \\ &\equiv - (g_1 + g_2 + g_3) \sigma^* H + \left[\frac{g_2 + g_3}{5} \right] (\sigma^* H - E_2 - E_3) \\ &\quad + \left[\frac{g_1 + g_2}{5} \right] (\sigma^* H - E_1 - E_3) + \left[\frac{g_1 + g_3}{5} \right] (\sigma^* H - E_1 - E_2) \\ &\quad + \left[\frac{g_2 + 2g_3}{5} \right] (\sigma^* H - E_1 - E_4) + \left[\frac{g_1 + 2g_2}{5} \right] (\sigma^* H - E_2 - E_4) \\ &\quad + \left[\frac{2g_1 + g_3}{5} \right] (\sigma^* H - E_3 - E_4) + \left[\frac{2g_1 + 2g_2 + 3g_3}{5} \right] E_1 \end{aligned}$$

Table 1 Classification of g

Type	$K_X - L_g \equiv$	$\chi(\mathcal{L}_g)$	$h^1(X, \mathcal{L}_g)$
1)	0	1	0
2)	E_i	0	0
3)	$-E_i$	0	0
4)	$\sigma^*(H) - E_i - E_j - E_k$	0	0
5)	$-\sigma^*(H) + E_i$	0	0
6)	$-\sigma^*(H) + E_i + E_j$	0	0
7)	$-\sigma^*(H) + E_i + E_j + E_k$	0	0
8)	$-\sigma^*(H) + E_i + E_j + E_k - E_l$	-1	1
9)	$-2\sigma^*(H) + E_i + E_j + E_k$	0	0
10)	$-2\sigma^*(H) + \sum_{i=1}^4 E_i$	0	0
11)	$-3\sigma^*(H) + \sum_{i=1}^4 E_i$	1	0

$$\begin{aligned}
& + \left[\frac{2g_1 + 3g_2 + 2g_3}{5} \right] E_2 + \left[\frac{3g_1 + 2g_2 + 2g_3}{5} \right] E_3 \\
& + \left[\frac{3g_1 + 3g_2 + 3g_3}{5} \right] E_4,
\end{aligned}$$

where $\equiv$ means linear equivalent.

Let $\mathcal{L}_g = \mathcal{O}_X(L_g)$. It is easy to check that L_g is not effective, so $h^0(X, \mathcal{L}_g) = 0$ for all $g \neq 0$, $g \in \mathbb{Z}_5^{\oplus 3}$. And $L_0 = 0$, therefore, $h^0(X, \mathcal{L}_0) = 1$, $h^2(X, \mathcal{L}_0) = 0$ and $\chi(\mathcal{L}_0) = 1$. So $h^1(X, \mathcal{L}_0) = 0$.

Using Riemann-Roch theorem, we can compute $\chi(\mathcal{L}_g)$. Then

$$h^1(X, \mathcal{L}_g) = h^0(X, \mathcal{L}_g) + h^2(X, \mathcal{L}_g) - \chi(\mathcal{L}_g) = h^0(X, \mathcal{L}_g) + h^0(X, \mathcal{O}_X(K_X - L_g)) - \chi(\mathcal{L}_g).$$

After the computation, the elements $g \in G$ can be classified as Table 1, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

We take $g = (1, 2, 3) \in G$ for example. By the computation, $L_{(1,2,3)} \equiv -2\sigma^*(H) + 2E_1$, then

$$K_X - L_{(1,2,3)} \equiv -\sigma^*(H) - E_1 + E_2 + E_3 + E_4.$$

So $g = (1, 2, 3)$ is type 8).

If g is of the types 1), 2) or 3), obviously $h^0(X, \mathcal{O}_X(K_X - L_g)) = 1$.

For the type 4), $h^0(X, \mathcal{O}_X(K_X - L_g)) = 0$ because no 3 triple points are collinear. Then $h^1(X, \mathcal{L}_g) = 0$. For E_i being the exceptional curve, it cannot be moved. So $K_X - L_g$ is not effective for the types 5)–11). Therefore, $h^0(X, \mathcal{O}_X(K_X - L_g)) = 0$. Then $h^1(X, \mathcal{L}_g) = 1$ for the type 8) and $h^1(X, \mathcal{L}_g) = 0$ for other types.

In addition, there are 6 elements of the type 8). So

$$q(Y) = h^1(Y, \mathcal{O}_Y) = h^1(X, \pi_* \mathcal{O}_Y) = \sum_{g \in G} h^1(X, \mathcal{L}_g) = 6.$$

As far as the author knows, only one of Hirzebruch's example satisfies $c_1^2 = 3c_2 = 225$. We will show that the irregularity of Hirzebruch's surface is different from ours. So they are not isomorphic.

Example 4.2. In [9, p. 68, Example 9], Hirzebruch constructed a surface with $c_1^2 = 3c_2$ by using Kummer covers of degree 5³ ramified over the configuration in Figure 1. He gave a geometrical description of his example. We can see that it is defined by the following equations,

$$z_1^5 = \ell_1 \ell_2^4 \ell_3 \ell_4^2 \ell_5 \ell_6, \quad z_2^5 = \ell_1^4 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6^2, \quad z_3^5 = \ell_1 \ell_2 \ell_3^4 \ell_4 \ell_5^2 \ell_6.$$

Using the same method as the above, we get $c_1^2 = 3c_2 = 225$.

In this case, $g \in G$ can be classified as Table 2.

Table 2 Classification of g

Type	$K_X - L_g \equiv$	$\chi(\mathcal{L}_g)$	$h^1(X, \mathcal{L}_g)$
1)	0	1	0
2)	E_i	0	0
3)	$E_i - E_j$	0	0
4)	$-E_i$	0	0
5)	$\sigma^*(H) - E_i - E_j$	1	0
6)	$\sigma^*(H) - E_i - E_j - E_k$	0	0
7)	$-\sigma^*(H) + E_i$	0	0
8)	$-\sigma^*(H) + E_i + E_j$	0	0
9)	$-\sigma^*(H) + E_i + E_j + E_k$	0	0
10)	$-2\sigma^*(H) + E_i + E_j + E_k$	0	0
11)	$-2\sigma^*(H) + \sum_{i=1}^4 E_i$	0	0
12)	$-3\sigma^*(H) + \sum_{i=1}^4 E_i$	1	0

Similarly, $h^1(X, \mathcal{L}_g) = 0$ for all types. So

$$q = \sum_{g \in G} h^1(X, \mathcal{L}_g) = 0.$$

Example 4.3. The authors of [11], [13] and [19] computed the irregularities of Hirzebruch's examples. The surface Y is an abelian covering of degree 5^5 branched over the former line configurations on $\mathbb{P}^2$ (see Figure 1). Y is defined by the following equations,

$$z_1^5 = \ell_1 \ell_6^4, \quad z_2^5 = \ell_2 \ell_6^4, \quad z_3^5 = \ell_3 \ell_6^4, \quad z_4^5 = \ell_4 \ell_6^4, \quad z_5^5 = \ell_5 \ell_6^4.$$

Using the same method, we get $c_1^2 = 3c_2 = 5625$, and

$$q = \sum_{g \in G} h^1(X, \mathcal{L}_g) = 30.$$

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