Birkhoff's conjecture and almost periodic motions on torus T^2 *

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Abstract The relation between rotation number and almost periodic motion for almost all C^5 systems on T^2 which have no critical points is established. The result that every solution of such systems is a Liapunov stable and almost periodic motion is proved.

Keywords: **recurrent motion, almost periodic motion, rotation number, Liapunov stable, minimal set.**

1 Main results

Birkhoff[1] first introduced the concept of recurrent motion, and conjectured the existence of an analytical differential equation that has recurrent motions but has no almost periodic motions. Ding[2] constructed a system on T^2 of the following form:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{1}{F(u,v)}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\lambda}{F(u,v)} \end{cases} \tag{1}$$

that verifies Birkhoff's conjecture affirmatively. Here, we may call number λ a Birkhoff number. More precisely, λ is a Birkhoff number if there is an analytical function F such that system (1) on T^2 has recurrent motions but no almost periodic motion. On the other hand, it is well known that if the C^2 system on T^2

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = g(u,v), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = g(u,v)f(u,v) \end{cases} \tag{2}$$

has neither critical points nor periodic motions, then every motion of (2) is recurrent.

From the above, it seems natural to ask the following questions:

(i) How many Birkhoff numbers are there ? What is the relation between Birkhoff number

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and Liouville number1)?

(ii) Almost periodic motion, as the motional form between periodic motion and recurrent motion, exists widely in smooth systems (2) on T^2, doesn't it?

In this paper, we prove the following theorems.

Theorem 1. *For almost all real numbers* λ (*in the sense of Lebesgue measure*), *if* f, $g \in C^5$, $g \neq 0$ *and the rotation number* $\mu(f)$ *of*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = 1, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = f(u, v) \end{cases} \tag{3}$$

is λ, *then every motion of* (2) *is Liapunov stable and almost periodic.*

Theorem 2. *A Birkhoff number must be a Liouville number, and the set of Birkhoff numbers is of Lebesgue measure zero.*

Remark. A real algebraic number is not a Liouville number[3], so it is not a Birkhoff number either.

2 Proofs of main results

Lemma 1[4]**.** *Suppose M is a compact minimal set of a dynamical system on M. Then a necessary and sufficient condition for M to be an almost periodic minimal set is that every motion in M is Liapunov stable for M.*

Lemma 2[5,6]**.** *If* f, $g \in C^2$, $g \neq 0$ *and* $\mu(f) \in \mathbb{R} \setminus \mathbb{Q}$, *then* T^2 *is the minimal set of* (2) *and* (3).

Let $v = H(u; p)$ be the expression for the orbit of (2) and (3) passing through a point $p = (\tilde{u}, \tilde{v})$. From ref. [7], we know that there exists a unique function (mod 1) $h: \mathbb{R} \to \mathbb{R}$ which is continuous, increasing, such that for all u, v, p,

$$h(v + 1) = h(v) + 1, \tag{4}$$

$$h \circ H(1; (0, v)) = \lambda + h(v), \tag{5}$$

$$H(u; p) = \lambda u + h \circ H(0; p) + W(u, \lambda u + h \circ H(0; p)), \tag{6}$$

where $\lambda = \mu(f)$ and $W(y, z)$ is periodic 1 in y, z, in fact

$$W(y, z) = H(y; (0, h^{-1}(z - \lambda y))) - z.$$

Define the subsets of irrational numbers A, D_δ and D as follows:

1) A real number λ is called a Liouville number[3] if for every integer $\delta \geq 1$ there exist integers p and q with $q > 0$ such that $\left| \lambda + \dfrac{p}{q} \right| < \dfrac{1}{q^\delta}$.

$$A = \left\{ \lambda \,\middle|\, \lim_{N_2 \to +\infty} \lim_{N_1 \to +\infty} \sup \left[\left(\sum_{\substack{a_i \geqslant N_2 \\ 1 \leqslant i \leqslant N_1}} \log(1 + a_i) \right) \left(\sum_{1 \leqslant i \leqslant N_1} \log(1 + a_i) \right)^{-1} \right] = 0, \right.$$

$$\left. \text{where } [a_0, a_1, \cdots] \text{ is the non-terminating continued fraction of } \lambda \right\};$$

$$D_\delta = \left\{ \lambda \,\middle|\, \exists\, C = C(\lambda)(> 0) \text{ such that} \left| \lambda + \frac{p}{q} \right| \geqslant \frac{C}{q^\delta} \text{ for all integers } p \text{ and } q \right.$$

$$\left. \text{with } q > 0, \text{ where } p, q \in \mathbb{Z} \text{ are relatively prime} \right\};$$

$$D = \bigcap_{\delta > 2} D_\delta .$$

Lemma 3[8]**.** (i) *A is a subset of D, and the Lebesgue measure of $\mathbb{R} \setminus A$ is zero*;

(ii) *For $n \geqslant 3$, $\delta > 1$ and $\mu(f) \in A$, h is of class $C^{n-\delta}$.*

Lemma 4. *Suppose $\delta \geqslant 1$, $\lambda \in D_\delta$ and $k(\delta) = [\delta] + 1 + [2\{\delta\}]$, where $[\delta]$ and $\{\delta\}$ denote the integer part and the fractional part of δ, respectively. If $F(x, y)$ is of class $C^k (k = k(\delta))$ and periodic 1 in x, y, then*

$$\lim_{(x, y) \to (x_0, y_0)} \int_0^\theta [F(x + s, y + \lambda s) - F(x_0 + s, y_0 + \lambda s)] ds = 0 \tag{7}$$

uniformly for $\theta \in \mathbb{R}$.

Proof. By the definition of k, we know that $k \geqslant 2$, $2(\delta - k) < -1$, therefore

$$\sum_{n \neq 0} | n |^{2(\delta - 1 - k)} < + \infty, \tag{8}$$

$$\sum_{m \neq 0} \sum_{n \neq 0} | n |^{2(\delta - k)} | m |^{-2} < + \infty . \tag{9}$$

Denote $a_{mn} = \int_0^1 \int_0^1 F(x, y) e^{-2\pi i (mx + ny)} dx dy$. Then[9]

$$\sum_m \sum_n | a_{mn} m^l n^{k-l} |^2 < + \infty, \quad l = 0, 1, \cdots, k. \tag{10}$$

Since

$$\sum_{m \neq 0} \left| \frac{a_{m0}}{m} \right| = \sum_{m \neq 0} | a_{m0} m^k | \, | m |^{-(k+1)}$$

$$\leqslant \left[\sum_{m \neq 0} | a_{m0} m^k |^2 \right]^{\frac{1}{2}} \cdot \left[\sum_{m \neq 0} | m |^{-2(k+1)} \right]^{\frac{1}{2}},$$

$$\sum_m \sum_n | a_{mn} | \, | n |^{\delta - 1}$$

$$= \sum_m \sum_{n \neq 0} | a_{mn} | \, | n |^{\delta - 1} = \sum_{n \neq 0} | a_{0n} | \, | n |^{\delta - 1} + \sum_{m \neq 0} \sum_{n \neq 0} | a_{mn} | \, | n |^{\delta - 1}$$

$$\leqslant\left[\sum_{n\neq 0}|a_{0n}n^k|^2\right]^{\frac{1}{2}}\left[\sum_{n\neq 0}|n|^{2(\delta-1-k)}\right]^{\frac{1}{2}}$$
$$+\left[\sum_{m\neq 0}\sum_{n\neq 0}|a_{mn}mn^{k-1}|^2\right]^{\frac{1}{2}}\left[\sum_{m\neq 0}\sum_{n\neq 0}|m|^{-2}|n|^{2(\delta-k)}\right]^{\frac{1}{2}}$$

and

$$\sum_{(m,n)\neq(0,0)}\left|\frac{a_{mn}}{m+\lambda n}\right| = \sum_{m\neq 0}\left|\frac{a_{m0}}{m}\right| + \sum_{m}\sum_{n\neq 0}\left|\frac{a_{mn}}{m+\lambda n}\right|$$
$$\leqslant \sum_{m\neq 0}\left|\frac{a_{m0}}{m}\right| + \frac{1}{C(\lambda)}\sum_{m}\sum_{n\neq 0}|a_{mn}||n|^{\delta-1},$$

using (10), we have

$$\sum_{m\neq 0}\left|\frac{a_{m0}}{m}\right| < +\infty, \tag{11}$$

using (8)—(10), we have

$$\sum_{m}\sum_{n}|a_{mn}||n|^{\delta-1} < +\infty, \tag{12}$$

and using (11) and (12), we get

$$\sum_{(m,n)\neq(0,0)}\left|\frac{a_{mn}}{m+\lambda n}\right| < +\infty. \tag{13}$$

On the other hand

$$\left|\int_0^\theta[F(x+s,y+\lambda s)-F(x_0+s,y_0+\lambda s)]\mathrm{d}s\right|$$
$$=\left|\sum_m\sum_n a_{mn}[\mathrm{e}^{2\pi\mathrm{i}(mx+ny)}-\mathrm{e}^{2\pi\mathrm{i}(mx_0+ny_0)}]\int_0^\theta \mathrm{e}^{2\pi\mathrm{i}(m+\lambda n)s}\mathrm{d}s\right|$$
$$=\left|\sum_{(m,n)\neq(0,0)}\frac{a_{mn}}{2\pi(m+\lambda n)}[\mathrm{e}^{2\pi\mathrm{i}(mx+ny)}-\mathrm{e}^{2\pi\mathrm{i}(mx_0+ny_0)}][\mathrm{e}^{2\pi\mathrm{i}(m+\lambda n)\theta}-1]\right|$$
$$\leqslant\frac{1}{\pi}\sum_{(m,n)\neq(0,0)}\left|\frac{a_{mn}}{m+\lambda n}\right||\mathrm{e}^{2\pi\mathrm{i}(mx+ny)}-\mathrm{e}^{2\pi\mathrm{i}(mx_0+ny_0)}|$$

and

$$|\mathrm{e}^{2\pi\mathrm{i}(mx+ny)}-\mathrm{e}^{2\pi\mathrm{i}(mx_0+ny_0)}|\leqslant 2,\quad \lim_{(x,y)\to(x_0,y_0)}|\mathrm{e}^{2\pi\mathrm{i}(mx+ny)}-\mathrm{e}^{2\pi\mathrm{i}(mx_0+ny_0)}|=0,$$

hence, using (13), we obtain the lemma.

Proof of Theorem 1. By Lemmas 1—3, we only need to prove that system (2) is Liapunov stable for T^2 and for $\lambda\in A$.

Denote $(U(t;p), V(t;p))$ the motion of the system with $(U(0;p), V(0;p)) = p$. Hence we have

$$V(t;p) = H(U(t;p);p), \tag{14}$$

and by Lemma 3, we have $\tilde{g}(u,v) =: \dfrac{1}{g(u,v+W(u,v))}$ is C^3 and periodic 1 in u, v, so by (6), (14) and Lemma 4, we have

$$\frac{1}{g(u+s,H(u+s;p))}=\tilde{g}(u+s,\lambda(u+s)+h\circ H(0;p)) \tag{15}$$

and

$$\lim_{(\tilde{u},\tilde{v})\to(\tilde{u}_0,\tilde{v}_0)}\int_0^\theta\left[\frac{1}{g(\tilde{u}+s,H(\tilde{u}+s;p))}-\frac{1}{g(\tilde{u}_0+s,H(\tilde{u}_0+s;p_0))}\right]ds=0 \tag{7'}$$

uniformly for $\theta\in\mathbb{R}$, where $p_0=(\tilde{u}_0,\tilde{v}_0)$, $p=(\tilde{u},\tilde{v})$.

Let $T(\theta;p)$ be the time along the orbit of the system from point $p=(\tilde{u},\tilde{v})$ to point $(\tilde{u}+\theta,H(\tilde{u}+\theta;p))$. Obviously, we have

$$T(U(t;p)-\tilde{u};p)\equiv t \tag{16}$$

for all $p\in T^2$, and

$$T(\theta;p)=\int_0^\theta\frac{1}{g(\tilde{u}+s,H(\tilde{u}+s;p))}ds. \tag{17}$$

Thus, by (17) and (7′), we have

$$|T(\theta_1;p)-T(\theta_2;p)|\geqslant\frac{1}{\max\limits_{T^2}|g|}|\theta_1-\theta_2|, \tag{18}$$

$$\lim_{p\to p_0}|T(\theta;p)-T(\theta;p_0)|=0 \tag{19}$$

uniformly for $\theta\in\mathbb{R}$, and by (16) and (18), we have

$$\begin{aligned}&|U(t;p)-U(t;p_0)|\\ \leqslant&|\tilde{u}-\tilde{u}_0|+|[U(t;p)-\tilde{u}]-[U(t;p_0)-\tilde{u}_0]|\\ \leqslant&|\tilde{u}-\tilde{u}_0|+\max_{T^2}|g|\cdot|T(U(t;p)-\tilde{u};p_0)-T(U(t;p_0)-\tilde{u}_0;p_0)|\\ =&|\tilde{u}-\tilde{u}_0|+\max_{T^2}|g|\cdot|T(U(t;p)-\tilde{u};p_0)-T(U(t;p)-\tilde{u};p)|,\end{aligned}$$

therefore we have

$$\begin{aligned}&|U(t;p)-U(t;p_0)|\\ \leqslant&|\tilde{u}-\tilde{u}_0|+\max_{T^2}|g|\cdot|T(U(t;p)-\tilde{u};p)-T(U(t;p)-\tilde{u};p_0)|.\end{aligned} \tag{20}$$

Hence, by (19) and (20), we have

$$\lim_{p\to p_0}|U(t;p)-U(t;p_0)|=0 \tag{21}$$

uniformly for $t\in\mathbb{R}$, and by (6), (14) and (21), we have

$$\lim_{p\to p_0}|V(t;p)-V(t;p_0)|=0 \tag{22}$$

uniformly for $t\in\mathbb{R}$. (21) and (22) indicate that every motion of the system is Liapunov stable for T^2. The proof is completed.

Remark. For $\forall\,\alpha\in(0,1)$, the smoothness of f and g can be reduced to $C^{4,\alpha}$ and C^3, respectively.

Proof of Theorem 2. Let λ_0 be a Birkhoff number, i.e. there exists an analytical function $F_0(u,v)$ $(\neq 0)$ such that for system

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{1}{F_0(u, v)}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\lambda_0}{F_0(u, v)}, \end{cases} \tag{1}$$

T^2 is a minimal set but not an almost periodic minimal set. We prove that λ_0 is a Liouville number. Suppose that λ_0 is not a Liouville number, then there exists a number δ_0 ($\geqslant 1$) such that $\lambda_0 \in D_{\delta_0}$. Similarly to the proof of Theorem 1 (here $h(x) \equiv x$, $W(x, y) \equiv 0$, F_0 analytic), we can prove that system (1′) is Liapunov stable for T^2. Using Lemma 1, this implies that the system has T^2 as its almost periodic minimal set, yielding a contradiction to the fact that the system has T^2 as its non-almost periodic minimal set. Similarly, we can prove that the set of Birkhoff numbers is a subset of $\mathbb{R} \setminus D$. So by Lemma 3, we know that the set of Birkhoff numbers is Lebesgue measure zero. Now the proof is completed.

Let X be any system on T^2 of class C^n ($n \geqslant 2$) with no critical points. From refs. [7, 10, 11], we know that X is C^n-conjugate to system (2), where $g \neq 0$. We call $\mu(f)$ a rotation number of X. Since T^2 is compact, one can easily prove that:

(i) if (2) has T^2 as its minimal set, so does X;

(ii) if (2) has a motion which is Liapunov stable for T^2, so does X.

By Theorem 1 and the above, we get the following theorem.

Theorem 3. *For almost all real numbers λ (in the sense of Lebesgue measure), T^2 is the almost periodic minimal set of system X, where X is of class C^n ($n \geqslant 5$) with no critical points and λ is a rotation number of X.*

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