Complete Lie algebras

MENG Daoji

Department of Mathematics, Nankai University, Tianjin 300071, China

Abstract A remark on complete Lie algebras is given. Since the theory of complete Lie algebras is still developing, this remark cannot be complete.

Keywords: complete Lie algebra, derivation, center.

IT is well known that, in the theory of finite-dimensional Lie algebras, the simple (semisimple) Lie algebras, solvable Lie algebras and nilpotent Lie algebras have been the object of concern all the way. Among them, the theory on simple (semisimple) Lie algebras is the perfect.

In recent years, more and more attention has been paid to complete Lie algebras; i.e. the Lie algebras with trivial center and only inner derivations. The conception of complete Lie algebras is the corresponding conception of complete groups. The conception of complete groups, whose center is unit and all of whose automorphisms are inner, arose in the 1930s. But the conception of complete Lie algebras appeared in the 1940s[1]. From the 1940s to the 1950s, many famous scholars on Lie groups and Lie algebras had been studying these Lie algebras and obtained some important results corresponding to those of complete groups. For example, the semisimple Lie algebras over a field of characteristic 0 are complete. Over any field, the Lie algebras with nondegenerate Killing form are complete. But at that time, few complete Lie algebras were known except these complete Lie algebras[2]. So people could not understand complete Lie algebras and their importance very well. From the late fifties to the beginning of eighties papers relating to complete Lie algebras were few, but many results on complete Lie algebras were included in the studies on derivation algebras. Because of the close relationship between the automorphism groups and the derivation algebras, the latter is always regarded as a major subject in the study on Lie algebras[3—6]. Since the late 1980s, the study on complete Lie algebras has been more and more active and some interesting results have been achieved[7—22].

1 Range of complete Lie algebras

Over a field of characteristic 0, complete Lie algebras include all finite-dimensional semisimple Lie algebras, some solvable Lie algebras and some other Lie algebras which are neither solvable nor semisimple.

Because nilpotent Lie algebras have nontrivial center and some outer derivations, they are not complete. But complete Lie algebras have close relations with nilpotent Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra. Set

$$\mathfrak{h}(\mathfrak{g}) = \mathfrak{g} \dotplus \mathrm{Der}\,\mathfrak{g},$$

and define the bracket in $\mathfrak{h}(\mathfrak{g})$ by

$$[x_1 + D_1, x_2 + D_2] = [x_1, x_2] + D_1(x_2) - D_2(x_1) + [D_1, D_2],$$

where $x_1, x_2 \in \mathfrak{g}$, D_1, $D_2 \in \mathrm{Der}\,\mathfrak{g}$. It is easy to prove that $\mathfrak{h}(\mathfrak{g})$ is a Lie algebra which is called the holomorph of $\mathfrak{g}$.

It is clear that all finite-dimensional complete Lie algebras over a field of characterisitic 0 are the subalgebras of the holomorphs of some nilpotent Lie algebras.

Over a field of characteristic $p(>0)$, it is different. The sample Lie algebras may not be complete, and it is not clear whether a complete Lie algebra is a subalgebra of the holomorph of some nilpotent Lie algebra, but we know that the simple (semisimple) Lie algebras with non-degenerate Killing forms are still complete.

2 Direct sum decomposition of complete Lie algebras

Killing-Cartan theorem shows that, over a field of characteristic 0, the finite-dimensional Lie algebras are semisimple if and only if they can be decomposed into the direct sum of their simple ideals and this decomposition is unique up to the order of these ideals. This theorem makes us reduce the study on semisimple Lie algebras to that on simple Lie algebras. But the proof of this theorem is dependent on the fact that a Lie algebra over a field of characteristic 0 is semisimple if and only if its Killing form is non-degenerate. So the method of proving of Killing-Cartan theorem does not work at these Lie algebras with degenerate Killing forms. Especially, the semisimple Lie algebras over a field of characteristic $p(>0)$ have not corresponding result. We cannot reduce the study on these semisimple Lie algebras to that on simple Lie algebras.

In the late 1980s, we proved the result that any complete Lie algebra over an arbitrary field can be decomposed into the direct sum of simple complete ideals and this decomposition is unique up to the order of these ideals. A complete Lie algebra is called simple complete if it is indecomposable; equivalently, anyone of its non-trivial ideal is not complete. Now we can also reduce the study on complete Lie algebras to the study on simple complete Lie algebras.

Compared with the proof about the existence of the decomposition, the proof on uniqueness of the decomposition is much more difficult. In order to prove this, we introduce the notion of $\mathfrak{g}$-endomorphism. An endomorphism ϕ of $\mathfrak{g}$ is called a $\mathfrak{g}$-endomorphism if

$$\mathrm{ad}x \cdot \phi = \phi \cdot \mathrm{ad}x \qquad \forall x \in \mathfrak{g}.$$

In other words, ϕ is not only an endomorphism of $\mathfrak{g}$ but also an adjoint $\mathfrak{g}$-module endomorphism.

Using the properties about $\mathfrak{g}$-endomorphism, we can prove that if a Lie algebra with trivial center can be decomposed into the direct sum of its indecomposable ideals, then the decomposition is unique up to the order of these ideals. From this, we obtain the uniqueness of decomposition of complete Lie algebras. Because all of semisimple Lie algebras over a field of characteristic 0 are complete, the uniqueness of the decomposition of semisimple Lie algebras can be easily obtained from this[7,8].

Whether acting as Lie algebra endomorphisms or as module endomorphisms, the set of all $\mathfrak{g}$-endomorphisms of a Lie algebra $\mathfrak{g}$ forms a monoid, which is worth studying. But in some sense, the study on this monoid still remains untouched up to now.

If we apply the idea of $\mathfrak{g}$-endomorphism to other algebra systems, the uniqueness of decomposition of the algebras with "trivial center" can be proved. So the notion of $\mathfrak{g}$-endomorphism goes beyond the Lie algebras.

3 Structure of complete Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0, then $\mathfrak{g}$ has the Levi decomposition $\mathfrak{g} = \mathfrak{s} \dotplus \mathfrak{r}$, where $\mathfrak{s}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{r}$ the radical of $\mathfrak{g}$. On the other hand, let $\mathfrak{g}$ be a complete Lie algebra; then for all D in $\mathrm{Der}\mathfrak{g}$, the semisimple part D_s and the nilpotent part D_n in the Jordan decomposition of D are also in $\mathrm{Der}\mathfrak{g}$. So any x in $\mathfrak{g}$ can be decomposed into the sum of its semisimple part x_s and nilpotent part x_n.

From this, we can obtain the Levi decomposition of any complete Lie algebra $\mathfrak{g}$ over algebraically closed field of characteristic 0, i.e. $\mathfrak{g} = \mathfrak{s} \dotplus \mathfrak{t} \dotplus \mathfrak{n}$, where $\mathfrak{s}$, $\mathfrak{r} = \mathfrak{t} \dotplus \mathfrak{n}$, $\mathfrak{n}$ are a Levi subalgebra, the radical and the nilpotent radical of $\mathfrak{g}$ respectively. Because all complete Lie algebras are algebraic Lie algebras, the above decomposition can also be obtained by the theory on algebraic Lie algebras[9].

Moreover, we know that $\mathfrak{n}$ is an $\mathfrak{s} \dotplus \mathfrak{t}$ module and certainly it is completely reducible. Let $\mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{s}$; then $\mathfrak{h} + \mathfrak{t}$ is the maximal toral subalgebra of $\mathfrak{g}$, and $\mathfrak{g}$ has the root spaces decomposition with respect to $\mathfrak{g} \dotplus \mathfrak{t}$:

$$\mathfrak{g} = \mathfrak{h} \dotplus t \dotplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}\alpha$ satisfies $[h + t, x] = \alpha(h + t)x$, $\forall h + t \in \mathfrak{h} + \mathfrak{t}, x \in \mathfrak{g}_\alpha$.

It is clear that this decomposition of complete Lie algebra is very similar to the root spaces decomposition of semisimple Lie algebra with respect to its Cartan subalgebra. As in the theory on semisimple Lie algebras, we can obtain many delicate results by the decomposition. So the decomposition is one of the most important results on complete Lie algebras.

For example, although the Killing form of a complete Lie algebra may be degenerate in general, but its restriction to a maximal toral subalgebra is nondegenerate. So we can identify the maximal toral subalgebra with its dual spaces, and the restriction of Killing form to the real space generated by root system is positive definite. Using these, we can obtain a criterion of simple complete Lie algebras[10,11].

Now we can say that the theories on algebraic Lie algebras, nilpotent Lie algebras, reductive Lie algebras and their representations are all very useful in the study on complete Lie algebras.

4 Some complete Lie algebras

During some period, the study of complete Lie algebras was dormant. One of the reasons is that people knew very few complete Lie algebras except the semisimple Lie algebras. With the development of the theory on Lie algebras, it was not until the late 1980s that more and more complete Lie algebras were known, and the general theory on complete Lie algebras has been set up along.

First, it has been proved that all the parabolic subalgebras of complex semisimple Lie algebras are complete, and they are simple complete if and only if they are the parabolic subalgebras of some simple Lie algebras[7,12].

Secondly, some methods of constructing complete Lie algebras were found at the beginning of the 1990s. Using these, we have constructed many complete Lie algebras such as complete Lie algebras whose nilpotent radical is abelian Lie algebras or the direct sum of some abelian Lie algebras and some Heisenberg algebras, solvable complete Lie algebras with maximal rank and some solvable complete Lie algebras with non-maximal rank[7,10,12—19].

Thirdly, we have constructed some complete Lie algebras from generalized Kac-Moody algebras and their subalgebras, Virasoro algebra and so on[20,21].

Fourthly, from the derivation algebras, holomorphs and the derivations of these holomorph algebras of some Lie algebras, we have also obtained some complete Lie algebras[13].

Some basic problems on the theory of Lie algebras have been understood very well through the study on complete Lie algebras; for example, we have known that all the derivation algebras of simple Lie algebras are simple complete. We can find a simple Lie algebra over a field of characteristic $p(>0)$ which is not complete but its derivation algebra is simple complete; further, we show that this derivation algebra is semisimple but not simple. Therefore the study on semisimple Lie algebras over a field of characteristic $p(>0)$ cannot reduce to that of simple Lie algebras[22].

In a word, among the theories of Lie algebras, the theory of complete Lie algebras is an important branch with abundant contents. Many important results on complete Lie algebras have been obtained. We can expect great progress in the studies on the structure, classifications, realization, representations and subalgebras of complete Lie algebras and on the relationship among complete Lie algebras, Lie groups and differential geometry, even though the theory on complete Lie algebras is far from perfect.

Acknowledgement This work was supported by the National Natural Science Foundation of China (Grant No. 19671045) and Doctoral Programme Foundation of Institution of Higher Education (Grant No. 97005511)

References

1 Goto, M., Faithful representation of Lie groups I, *Math. Japonicae*, 1948(1): 1.
2 Jacobson, N., *Lie Algebras*, New York: Wiley (Interscience), 1962.
3 Leger, G., Derivations of Lie algebras Ⅲ, *Duke Math. J.*, 1963, 30: 637.
4 Müller, D., Isometries of Bi-invariant pseudo-Riemannian metrics on Lie groups, *Geometric Dedicator*, 1989, 29: 65.
5 Stitzinger, E. L., On Lie algebras with only inner derivations, *J. of Algebras*, 1987, 105: 341.
6 Tôgô, S., Outer derivations of Lie algebras, *Trans. Amer. Math. Soc.*, 1967, 128: 264.
7 Meng, D. J., On complete Lie algebras, *Act. Sci. Nat. Univ. Nankai*, 1985, 18: 9.
8 Meng, D. J., The uniqueness of the decomposition of the complete Lie algebras, *Act. Sci. Nat. Univ. Nankai*. (in Chinese), 1990, 23: 23.
9 Meng, D. J., Wang, S. P., On the construction of complete Lie algebras, *Journal of Algebra*, 1995, 176: 621.
10 Meng, D. J., Wang, S. P., The Killing form and maximal toral subalgebra of the complete Lie algebra, *Science in China*, Ser. A, 1996, 39: 477.

11 Meng, D. J., The maximal toral subalgebras of complete Lie algebras, *Chinese Annual of Math*. (in Chinese), 1995, 16A: 723.
12 Meng, D. J., Some results on complete Lie algebras, *Communications in Algebra*, 1994, 22:5457.
13 Jiang, C., Meng, D. J., Zhang, S. Q., Some complete Lie algebras, *Jounal of Algebra*, 1996, 186:807.
14 Meng, D. J., Complete Lie algebras and Heisenberg algebras, *Communications in Algebra*, 1994, 22: 5509.
15 Meng, D. J., Zhu, L. S., Solvable complete Lie algebras I, *Communications in Algebra*, 1996, 24: 4184.
16 Zhu, L. S., Meng, D. J., The classification of complete Lie algebras with low dimensions, *Algebra Colloquium*, 1997, 4: 95.
17 Jiang, C. B., Meng, D. J., The realization and struction of complete Lie algebras whose nilpotent radicals are Heisenberg algebras, *Chinese Science Bulletin*, 1998, 43: 383.
18 Meng, D. J., The complete Lie algebras with abelian nilpotent radicals, *Acta Math. Sinica* (in Chinese), 1991, 34: 191.
19 Zhu, L. S., Meng, D. J., Solvable complete Lie algebras Ⅰ, Ⅱ, *Chinese Science Bulletin* (in Chinese), 1996 (6): 670.
20 Zhu, L. S., Meng, D. J., Generalized Kac-Moody algebras and infinite dimensional complete Lie algebras, *Act. Sci. Nat. Univ. Nankai* (in Chinese), 1995, 28: 5.
21 Zhu, L. S., Meng, D. J., The derivations of generalized Kac-Moody algebras, *Chinese Annual of Math*. (in Chinese), 1996, 17A: 271.
22 Yang, Q., Meng, D. J., Simple Lie algebras and complete Lie algebras, *Northeast Math. Journal*, 1996, 12: 381.

(*Received October*, 23, 1998)