HIRZEBRUCH SUM AND CLASS NUMBER OF THE QUADRATIC FIELDS*

LU HONG-WEN (陆洪文)

(*Department of Mathematics, University of Science and Technology of China, Hefei* 230026, *PRC*)

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1. The purpose of this note is to give the proof of some results in my preprint [1].

For a real quadratic irrational number β, let

$$\Psi(\beta)=\begin{cases}\sum_{j=1}^{k}(-1)^{j+s}a_j, & \text{if } k \text{ even};\\ 0, & \text{if } k \text{ odd},\end{cases}$$

where β has a development of the simple continued fractions

$$\beta=[\hat{a}_0,\cdots,\hat{a}_s,\overline{a_1,\cdots,a_k}]$$

with the basic period $\overline{a_1,\cdots,a_k}$.

We call $\Psi(\beta)$ the Hirzebruch sum of β.

The following three theorems have been stated in [1].

Theorem 1. *Let both $d>1$ and $-k<-1$ be the fundamental discriminants such that g. c. d. $(2d, k)=1$. Then we have*

$$48J\cdot h(-k)h(-kd)$$

$$=\delta_d W_{-k}W_{-kd}\sum_{\{A=[a,\frac{b+\sqrt{d}}{2}]\}}\ \sum_{\substack{nu=k\\ n,u\geqslant 1}}\chi_u(a)\sum_{m\,(\mathrm{mod}\,n)}\chi_n(am^2+bm+c)\Psi\left(\frac{u}{n}\left(m+\frac{b+\sqrt{d}}{2a}\right)\right),$$

where J is a positive integer such that the least solution of the Pell's equation $x^2-dk^2y^2=4$ is ε_+^J with the total positive fundamental unit ε_+ of the real quadratic field $Q(\sqrt{d})$; $h(x)$ denotes the class number of the quadratic field $Q(\sqrt{x})$; a, b and c are integers such that $d=b^2-4ac$, $|b|\leqslant a\leqslant -c$ and g. c.d.$(a,b,c)=1$; $A=[a,\frac{b+\sqrt{d}}{2}]$ runs through a complete set of representatives of the (wide) ideal classes in $Q(\sqrt{d})$;

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$$\delta_d = \begin{cases} 1, \text{ if } N(\varepsilon) = -1, \text{ i. e. } \varepsilon_+ = \varepsilon^2, \\ 2, \text{ if } N(\varepsilon) = 1, \text{ i. e. } \varepsilon_+ = \varepsilon, \end{cases}$$

with the fundamental unit ε *of* $Q(\sqrt{d}\,)$; χ_u *and* χ_n *are real primitive characters* mod u *and* mod n *respectively; and finally for a negative fundamental discriminant* $-D$,

$$W_{-D} = \begin{cases} 2, & \text{if } -D < -4, \\ 4, & \text{if } -D = -4, \\ 6, & \text{if } -D = -3. \end{cases}$$

Theorem 2. *Let both* $d > 1$ *and* $-k < -1$ *be the fundamental discriminants such that* $4 \parallel k$ *and g. c. d.* $(d, k) = 1$. *Then we have*

$$48J \cdot h(-k)h(-kd) = \delta_d W_{-k} W_{-kd}$$
$$\times \sum_{\{A = [a, \frac{b+\sqrt{d}}{2}]\}} \left(\sum_{\substack{4nu=k \\ n, u \geqslant 1}} \chi u(a) \sum_{m \pmod{4n}} \chi_{4n}(am^2 + bm + c) \Psi\left(\frac{u}{4n} \left(m + \frac{b+\sqrt{d}}{2a} \right)\right)\right.$$
$$+ \sum_{\substack{4nu=k \\ n, u \geqslant 1}} \chi_{4u}(a) \sum_{m \pmod{n}} \chi_n(am^2 + bm + c) \left(\Psi\left(\frac{u}{n} \left(m + \frac{b+\sqrt{d}}{2a} \right)\right)\right.$$
$$\left.\left. - 3\Psi\left(\frac{2u}{n} \left(m + \frac{b+\sqrt{d}}{2a} \right)\right) + 2\Psi\left(\frac{4u}{n} \left(m + \frac{b+\sqrt{d}}{2a} \right)\right)\right)\right),$$

where all notations are the same as those in Theorem 1.

Theorem 3. *Let both* $d > 1$ *and* $-k < -1$ *be the fundamental discriminants such that* $8 \parallel k$ *and g. c. d.* $(d, k) = 1$. *Then we have*

$$48J \cdot h(-k)h(-kd) = \delta_d W_{-k} W_{-kd}$$
$$\times \sum_{\{A = [a, \frac{b+\sqrt{d}}{2}]\}} \left(\sum_{\substack{8nu=k \\ n, u \geqslant 1}} \chi_u(a) \sum_{m \pmod{8n}} \chi_{8n}(am^2 + bm + c) \Psi\left(\frac{u}{8n} \left(m + \frac{b+\sqrt{d}}{2a} \right)\right)\right.$$
$$+ \frac{1}{2} \sum_{\substack{8nu=k \\ n, u \geqslant 1}} \chi_u(a) \sum_{m \pmod{8n}} \chi_{8n}(am^2 + 2bm + 4c) \Psi\left(\frac{u}{2n} \left(\frac{m}{2} + \frac{b+\sqrt{d}}{2a} \right)\right)$$
$$+ \sum_{\substack{8nu=k \\ n, u \geqslant 1}} \chi_{8u}(a) \sum_{m \pmod{n}} \chi_n(am^2 + bm + c) \left(\Psi\left(\frac{2u}{n} \left(m + \frac{b+\sqrt{d}}{2a} \right)\right)\right.$$
$$\left.\left. - 3\Psi\left(\frac{4u}{n} \left(m + \frac{b+\sqrt{d}}{2a} \right)\right) + 2\Psi\left(\frac{8u}{n} \left(m + \frac{b+\sqrt{d}}{2a} \right)\right)\right)\right),$$

where all notations are the same as those in Theorem 1, *and both the compositions of* χ_u and χ_{8n}, *or* χ_{8u} *and* χ_n *are* χ *which is the real primitive character* mod k *with* $\chi(-1) = -1$.

Example. Let $p\equiv 1 \pmod 8$ be a prime such that $h(p)=1$. Then we have $3h(-4p)=\Psi(\sqrt{4p})$, which is similar to a result due to *F*. Hirzebruch[2] : If $p\equiv 3 \pmod 4$ is a prime $\geqslant 7$ such that $h(p)=1$, then we have $3h(-p)=\Psi(\sqrt{p})$.

2. We are going to give the proof for Theorem 1. Firstly, under the assumption of Theorem 1, we have

$$\sum_{\substack{nu=k\\ n,u\geqslant 1}}\chi_u(a)\sum_{m \,(\mathrm{mod}\; n)}\chi_n(am^2+bm+c)=\sum_{\substack{nu=k\\ n,u\geqslant 1}}\chi_u(a)\mu(n)\chi_n(a)=\chi_k(a)\sum_{n\mid k}\mu(n)=0, \qquad (*)$$

by using the following identity (cf. [3])

$$\sum_{m \,(\mathrm{mod}\; n)}\chi_n(am^2+bm+c)=\mu(n)\chi_n(a),$$

where μ is the Möbius function.

According to $(*)$, Theorems 2 , 5 , 6′ in [3] and the case (1. 18) of the theorem in [4], it is not difficult to know that for proving our Theorem 1, we only need to show that

$$\text{g.c.d.}\ ((2am+b)k,(am^2+bm+c)ku/n,\ akn/u)=1, \qquad (1)$$

if

$$nu=k \text{ and } \text{g.c.d.}(u,a)=\text{g.c.d.}(n,am^2+bm+c)=1.$$

Since k has no square factors, hence the left hand side of (1) is

$$\text{g.c.d.}((2am+b)um,(am^2+bm+c)u^2,an^2)$$
$$=\text{g.c.d.}(2am+b,am^2+bm+c,a)=\text{g.c.d.}(a,b,c)=1,$$

which proves (1). Therefore the proof of our theorem 1 has been completed.

3. Theorems 2 and 3 of the present note can be proved by using a similar method.

4. For example, we take $k=4$ and $d=p$ in Theorem 2, and consider that all Hirzebruch sums which are involved in the example are zero except $\Psi(\sqrt{4p})$. So we get the statement of the example.

References

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