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Analytical solution for spatially axisymmetric problem of thick-walled cylinder subjected to different linearly varying pressures along the axis and uniform pressures at two ends

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To our best knowledge, in the open literature, there is no analytical solution of thick-walled cylinder subjected to uniform pressures at two ends and different inner- and outer- surface pressures that are constant circumferentially but vary linearly at different rates along the axis. We now present such a solution. After repeated trials, we have finally succeeded in finding a necessary new displacement function. Based on A. E. H. Love method, the stress, displacement and volume strain formulas are derived by using the new displacement function. The present results include the Lamé's formulas as special cases. Furthermore, the results obtained here can be applied to not only the thick-walled cylinders subjected to uniform pressures on the inner and outer surface of the thick-walled cylinder, respectively, but also the cylinders subjected to uniform pressures at two ends and different inner- and outer-surface pressures that are constant circumferentially but vary linearly at different rates along the axis, respectively. Finally we give a numerical example to compare our exact method with the approximate method.

thick-walled cylinder, new displacement function, spatially axisymmetric, three-dimensional analytical solution

1 Introduction

Owing to the ever-increasing industrial demand for axisymmetric pressure vessels in chemistry, nuclear, fluid transmitting plants, power plants and military equipments, the attention of designers has been concentrated on the elastic-plastic analysis of thick-walled cylinders[1]. These elastic-plastic analyses were performed mainly based on the Lamé's formulas[2−5]. The Lamé's formulas are suitable to thick-walled cylinders subjected to uniformly distributed pressures on the

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inner and outer surface of the thick-walled cylinder, respectively. However, it cannot be applied to thick-walled cylinders subjected to linearly distributed pressures on the inner and outer surface of the thick-walled cylinder, respectively. Lin[6] (1997) applied the Kantorovich method to the three-dimensional axial-symmetric stress analysis, and the Euler equations and boundary conditions were derived for the thick-walled cylinder with finite length. Lin[7] (1999) developed a Kantorovich-type stress variation method of higher approximation for the finite thick-walled cylinders under the axisymmetric load. Variation and integration of the stress expressions are conducted and the Euler equation and the corresponding end conditions are established. Up to now, unfortunately, we have not seen the analytical solution to uniform pressures at two ends and different inner- and outer- surface pressures that are constant circumferentially but vary linearly at different rates along the axis in the open literatures. This article presents three-dimensional analytical solutions for the thick-walled cylinder mentioned above by making use of Love potential functions. It can be shown that the present analysis is very simple and clear. Numerical comparison is made with refs. [6,7].

2 Basic equations for axisymmetric problems

In the case of axisymmetric spatial problem without the effect of body forces, the stress components, according to A. E. H. Love method, can be expressed in terms of a potential function ϕ as

$$\begin{cases} \sigma_r = \dfrac{\partial}{\partial z}\left(\mu\nabla^2\phi - \dfrac{\partial^2\phi}{\partial r^2}\right), \\ \sigma_\theta = \dfrac{\partial}{\partial z}\left(\mu\nabla^2\phi - \dfrac{1}{r}\dfrac{\partial\phi}{\partial r}\right), \\ \sigma_z = \dfrac{\partial}{\partial z}\left((2-\mu)\nabla^2\phi - \dfrac{\partial^2\phi}{\partial z^2}\right), \\ \tau_{rz} = \dfrac{\partial}{\partial r}\left((1-\mu)\nabla^2\phi - \dfrac{\partial^2\phi}{\partial z^2}\right), \end{cases} \tag{1}$$

where σ_r, σ_θ, σ_z, and τ_{rz} are the stress components; G is the shear modulus and μ is the Poisson's ratio. The potential function φ in eq. (1) satisfies the following bi-harmonic equation:

$$\nabla^2\nabla^2\phi = 0, \tag{2}$$

where ∇^2 is the three-dimensional Laplace's operator, which reads in the axisymmetric case as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \tag{3}$$

the volume strain is

$$\theta = \varepsilon_r + \varepsilon_\theta + \varepsilon_z = \frac{1-2\mu}{E}\Theta, \tag{4}$$

where Θ is the sum of the three normal stress components $\Theta = \sigma_r + \sigma_\theta + \sigma_z$.

3 Displacement function and stress components

Consider a thick-walled cylinder with inner diameter $2r_1$, outer diameter $2r_2$ and length l subjected to different inner- and outer-surface pressures that are constant circumferentially but vary linearly at different rates along the axis and uniform pressures at two ends as shown in Figure 1.

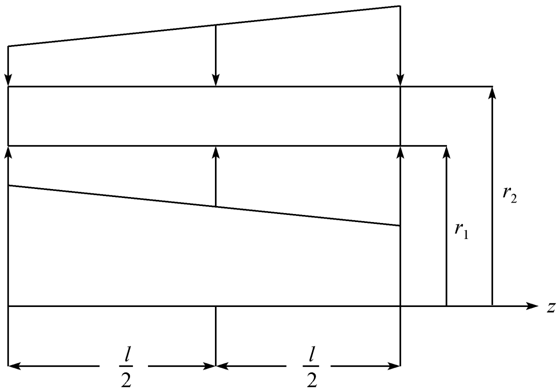

Figure 1 The thick-walled cylinder subjected to different linearly varying pressures along the axis and uniform pressures at two ends.

The boundary conditions corresponding to the thick-walled cylinder mentioned above are

$$\begin{cases} r = r_1: \ \sigma_r = p_1^*\left(\dfrac{z}{l}\right) + p_1, \ \tau_{rz} = 0, \\ r = r_2: \ \sigma_r = p_2^*\left(\dfrac{z}{l}\right) + p_2, \ \tau_{rz} = 0, \\ z = 0, l: \ \sigma_z = q, \ \tau_{rz} = 0, \end{cases} \tag{5}$$

where p_1 and p_2 are the inner- and outer- surface pressures at $z = 0$, respectively. p_1^*/l and p_2^*/l are the rate of the linearly varying pressures on the inner- and outer-surface along the axis, respectively.

As the geometrical shape of the body is concerned, the condition of constraint and the external loads are all symmetrical with respect to any plane passing through the axis *z,* the stress, strain and displacement components will have the same symmetry. In the cylindrical coordinates, the stress, strain and displacement components will be functions of only two coordinates *r* and *z.*

For the solution of the axisymmetric spatial problem, we take the following displacement function with 8 terms:

$$\phi = A_1 z^4 + A_2 r^4 + A_3 z^3 + A_4 z^2 r^2 + A_5 z^2 \ln r + A_6 z r^2 + A_7 r^2 \ln r + A_8 z \ln r, \tag{6}$$

where A_1, A_2, A_3, A_4, A_5, A_6, A_7 and A_8 are unknown constants to be determined by the boundary conditions. Substitution of eq. (6) into eq. (1) yields:

$$\begin{cases} \sigma_r = \left[24\mu A_1 + 4(2\mu - 1)A_4\right]z + 2A_5 \cdot \dfrac{z}{r^2} + A_8 \cdot \dfrac{1}{r^2} + 6\mu A_3 + 2(2\mu - 1)A_6, \\ \sigma_\theta = \left[24\mu A_1 + 4(2\mu - 1)A_4\right]z - 2A_5 \cdot \dfrac{z}{r^2} - A_8 \cdot \dfrac{1}{r^2} + 6\mu A_3 + 2(2\mu - 1)A_6, \\ \sigma_z = \left[24(1-\mu)A_1 + 8(2-\mu)A_4\right]z + 6(1-\mu)A_3 + 4(2-\mu)A_6, \\ \tau_{rz} = \left[32(1-\mu)A_2 - 4\mu A_4\right]r + \left[4(1-\mu)A_7 - 2\mu A_5\right] \cdot \dfrac{1}{r}. \end{cases} \tag{7}$$

Substitution of eq. (7) into eq. (5) gives

$$BA = f, \tag{8}$$

where

$$A = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}^{\mathrm{T}},$$

$$f = \left\{0, 0, 0, q, p_1^*, p_2^*, p_1/l, p_2/l\right\}^{\mathrm{T}},$$

$$c = 1 - \mu,\ d = 2 - \mu,\ e = 2\mu - 1,$$

$$B = \begin{bmatrix} 24c & 0 & 0 & 8d & 0 & 0 & 0 & 0 \\ 0 & 8c & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 6c & 0 & 0 & 4d & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\mu & 0 & 4c & 0 \\ 0 & 0 & 6\mu & 0 & 0 & 2e & 0 & 1/r_1^2 \\ 0 & 0 & 6\mu & 0 & 0 & 2e & 0 & 1/r_2^2 \\ 24\mu & 0 & 0 & 2e & 2/r_1^2 & 0 & 0 & 0 \\ 24\mu & 0 & 0 & 2e & 2/r_2^2 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions of eq. (8) are

$$\begin{cases} A_1 = \dfrac{2-\mu}{12(1+\mu)} \cdot \dfrac{p_2^* r_2^2 - p^*{}_1 r_1^2}{l\left(r_2^2 - r_1^2\right)}, \\ A_2 = -\dfrac{\mu}{32(1+\mu)} \cdot \dfrac{p_2^* r_2^2 - p_1^* r_1^2}{l\left(r_2^2 - r_1^2\right)}, \\ A_3 = \dfrac{2-\mu}{3(1+\mu)} \cdot \dfrac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2} + \dfrac{1-2\mu}{6(1+\mu)} \cdot q, \\ A_4 = -\dfrac{1-\mu}{4(1+\mu)} \cdot \dfrac{p_2^* r_2^2 - p_1^* r_1^2}{l\left(r_2^2 - r_1^2\right)}, \\ A_5 = -\dfrac{r_1^2 r_2^2 \left(p_2^* - p_1^*\right)}{2l\left(r_2^2 - r_1^2\right)}, \\ A_6 = -\dfrac{1-\mu}{2(1+\mu)} \cdot \dfrac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2} + \dfrac{\mu}{2(1+\mu)} \cdot q, \\ A_7 = -\dfrac{\mu}{1-\mu} \cdot \dfrac{r_1^2 r_2^2 \left(p_2^* - p_1^*\right)}{4l\left(r_2^2 - r_1^2\right)}, \\ A_8 = -\dfrac{r_1^2 r_2^2 \left(p_2 - p_1\right)}{r_2^2 - r_1^2}. \end{cases} \tag{9}$$

Substitution of eq. (9) into eq. (6) yields the displacement function ϕ, and it is easy to find that the function ϕ satisfies eq. (2).

Substitution of eq. (9) into eq. (7) yields

$$\begin{cases}\sigma_r = \dfrac{p_2^* r_2^2 - p_1^* r_1^2}{l\left(r_2^2 - r_1^2\right)} z - \dfrac{r_1^2 r_2^2 \left(p_2^* - p_1^*\right)}{l\left(r_2^2 - r_1^2\right)} \cdot \dfrac{z}{r^2} + \dfrac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2} - \dfrac{r_1^2 r_2^2 \left(p_2 - p_1\right)}{r_2^2 - r_1^2} \cdot \dfrac{1}{r^2}, \\ \sigma_\theta = \dfrac{p_2^* r_2^2 - p_1^* r_1^2}{l\left(r_2^2 - r_1^2\right)} z + \dfrac{r_1^2 r_2^2 \left(p_2^* - p_1^*\right)}{l\left(r_2^2 - r_1^2\right)} \cdot \dfrac{z}{r^2} + \dfrac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2} + \dfrac{r_1^2 r_2^2 \left(p_2 - p_1\right)}{r_2^2 - r_1^2} \cdot \dfrac{1}{r^2}, \\ \sigma_z = q, \\ \tau_{rz} = 0. \end{cases} \tag{10}$$

4 The special case when $l \to \infty$

When $l \to \infty$, we can obtain the following expressions for A_1, A_2, A_3, A_4, A_5, A_6, A_7 and A_8 from eq. (9):

$$\begin{cases} A_1 = A_2 = A_4 = A_5 = A_7 = 0, \\ A_3 = \dfrac{2-\mu}{3(1+\mu)} \cdot \dfrac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2} + \dfrac{1-2\mu}{6(1+\mu)} \cdot q, \\ A_6 = -\dfrac{1-\mu}{2(1+\mu)} \cdot \dfrac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2} + \dfrac{\mu}{2(1+\mu)} \cdot q, \\ A_8 = -\dfrac{r_1^2 r_2^2 \left(p_2 - p_1\right)}{r_2^2 - r_1^2}. \end{cases} \tag{11}$$

If $q = 0$, eq. (10) reduces to

$$\begin{cases} \sigma_r = \dfrac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2} - \dfrac{r_1^2 r_2^2 \left(p_2 - p_1\right)}{r_2^2 - r_1^2} \cdot \dfrac{1}{r^2}, \\ \sigma_\theta = \dfrac{p_2 r_2^2 - p_1 r_1^2}{r_2^2 - r_1^2} + \dfrac{r_1^2 r_2^2 \left(p_2 - p_1\right)}{r_2^2 - r_1^2} \cdot \dfrac{1}{r^2}, \\ \sigma_z = 0, \\ \tau_{rz} = 0. \end{cases} \tag{12}$$

These are the well-known Lamé's formulas for the stresses in a hollow circular cylinder subjected to uniform pressures.

Take $p_1^* = p_2^* = 0$, $q = 0$, eq. (10) reduces to the well-known Lamé's formulas once again.

5 Displacement components and volume strain

The displacement components, according to A. E. H. Love method, can be expressed in terms of a potential function ϕ as

$$\begin{cases} u_r = -\dfrac{1}{2G} \dfrac{\partial^2 \phi}{\partial r \partial z}, \\ w = \dfrac{1}{2G} \left[2(1-\mu)\nabla^2 - \dfrac{\partial^2}{\partial z^2} \right] \phi, \end{cases} \tag{13}$$

where u and w are the displacements in r- and z- directions, respectively.

Substitution of eq. (6) and (9) into eq. (13) yields

$$\begin{cases} u_r = \dfrac{1}{2G}\left[\left(\dfrac{1-\mu}{1+\mu}\cdot\Delta_1\cdot r+\Delta_3\cdot\dfrac{1}{r}\right)z+\left(\dfrac{1-\mu}{1+\mu}\cdot\Delta_2\cdot r+\Delta_4\cdot\dfrac{1}{r}\right)\right]+\dfrac{\mu rq}{E}, \\ w=-\dfrac{1}{2G}\left[\dfrac{\mu}{1+\mu}\cdot\Delta_1\cdot z^2-\dfrac{\mu-1}{2(1+\mu)}\cdot\Delta_1\cdot r^2+\dfrac{2\mu}{1+\mu}\cdot\Delta_2\cdot z+\Delta_3\cdot\ln r+2\mu\Delta_3\right]+\dfrac{qz}{E}, \end{cases} \tag{14}$$

where

$$\Delta_1=\frac{p_2^* r_2^2-p_1^* r_1^2}{l\left(r_2^2-r_1^2\right)},\ \Delta_2=\frac{p_2 r_2^2-p_1 r_1^2}{r_2^2-r_1^2},\ \Delta_3=\frac{r_1^2 r_2^2\left(p_2^*-p_1^*\right)}{l\left(r_2^2-r_1^2\right)},\ \Delta_4=\frac{r_1^2 r_2^2\left(p_2-p_1\right)}{r_2^2-r_1^2}.$$

The volume strain can be expressed in terms of the displacement function ϕ as

$$e=\frac{1-2\mu}{2G}\cdot\frac{\partial}{\partial z}\nabla^2\phi.$$

Substitution of eq. (6) and (9) into this equation yields

$$e=\frac{1-2\mu}{E}\cdot\left(2\Delta_1 z+2\Delta_2+q\right), \tag{15}$$

where

$$\Delta_1=\frac{p_2^* r_2^2-p_1^* r_1^2}{l\left(r_2^2-r_1^2\right)},\ \Delta_2=\frac{p_2 r_2^2-p_1 r_1^2}{r_2^2-r_1^2}.$$

6 Example

The dimensions of the thick-walled cylinder in our example are: outer diameter $4r_1$, thickness r_1, length $10r_1$. The pressure on the inner surface is zero and on the outer surface is hydrostatic, varying from zero at the top ($z = 10r_1$) to the maximum at the bottom ($z = 0$). We computed the circumferential stresses at a cross-section $3r_1$ from the bottom. The comparison of dimesionless circumferential stresses σ_θ/q computed by this paper and by refs. [6,7] as shown in Table 1.

Table 1 Comparison of dimensionless circumferential stresses σ_θ/q

	r/r_2					
	1	1.2	1.4	1.6	1.8	2.0
Ref. [6]	–1.2821	–1.5211	–1.6132	–1.5512	–1.3278	–0.9359
Ref. [7]	–1.7320	–1.6834	–1.5001	–1.3017	–1.2098	–1.3487
This paper	–1.8667	–1.5815	–1.4095	–1.2979	–1.2214	–1.1667

7 Conclusions

(1) The necessary new displacement function which satisfied the bi-harmonic equation and the corresponding boundary condition was successfully found. Based on the A. E. H. Love method, analytical solutions, for thick-walled cylinders subjected to linearly distributed pressures on the inner and outer surface along the axis and uniform pressures at two ends, were derived by using the new displacement function. The stress, displacement and volume strain formulas were obtained.

(2) Two special cases need to be pointed out: Letting $l \to \infty$, we can get $A_1 = A_2 = A_4 = A_5 = A_7 =$ 0, and the radial stress and circumferential stress formulas obtained in this paper reduce to the famous Lamé's formulas; letting $p_1^* = p_2^* = 0$, $q = 0$, we can also obtain the famous Lamé's formulas from the results obtained in this paper. It is shown that the present results include the Lame formulas as special cases. Results obtained in this paper can be applied to not only the thick-walled cylinders subjected to uniform pressures on the inner and outer surface of the thick-walled cylinder, respectively, but also the cylinders subjected to uniform pressures at two ends and different inner- and outer-surface pressures that are constant circumferentially but vary linearly at different rates along the axis.

(3) From Table 1, it can be seen that the dimensionless circumferential stresses σ_θ / q calculated by ref. [7] are more close to the analytical solutions than by ref. [6]. The results derived by our exact method are not only more accurate but also much simpler and faster.

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