QUANTUM GRAVITATIONAL BOUND STATES

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I. INTRODUCTION

The Dirac equations of curved spacetime had already been put forward and discussed by the end of the 30's, but all of them were in the conformal invariant form. Because it was difficult to treat γ-matrix, the Dirac equation for $m = 0$ in form of Cartan moving frame was not obtained until 1957[1]. Furthermore, Brill and Cohen[2] wrote out the Dirac equation for $m \neq 0$ for curved spacetime and got the plane-wave solution in 1966. Recently, Soffel[3] obtained the Dirac equation in the Schwarzschild metric and discussed the situations near the black hole by the method of numerical solutions.

Since Hawking[4] suggested the possibility that there may exist small black holes of the order of 10^{-13}cm, the problem has been discussed from various aspects. The possibility of the existence of gravitational quantum bound states seems quite likely. Recently, the bound states of bosons (Klein-Gordon equation) in the neighbourhood of a Kerr black hole have been discussed[5]. As all the stable elementary particles are fermions, it is more proper to discuss the bound states of the fermions around small black holes. It is also necessary to solve the Dirac equation for this purpose. Because the Dirac equation has not been derived in the Kerr metric for the time being and it is necessary to solve the equation in the Schwarzschild metric with Newton approximation to obtain the energy levels and wave funtions.

From [3], the radial Dirac equation in Schwarzschild spacetime is written as follows:

$$\frac{dg(r)}{dr} = -e^{\lambda/2}\frac{k}{r}g(r) + c^{\lambda/2}[e^{-\frac{\nu}{2}}(E - V(r)) + m]f(r) \tag{1A}$$

and

$$\frac{df(r)}{dr} = e^{\frac{\lambda}{2}}\frac{k}{r}f(r) - e^{\lambda/2}[e^{-\frac{\nu}{2}}(E - V(r)) - m]g(r), \tag{1B}$$

where $g(r)$ and $f(r)$ forming $R(r) = \begin{pmatrix} g(r) \\ f(r) \end{pmatrix}$ are the radial wave functions; e^{ν} and e^{λ} represent the components of the Schwarzschild metric g_{00} and g_{11} respectively; E and m are the energy and mass of the Dirac particle; $V(r)$ is a potential function (here potential is caused by various fields except gravity). If only pure gravitational action is considered, then $V(r) = 0$; k is the angular quantum number. Eq. (1) is solved as follows.

To make the results of computation easy to see, the C. G. S System is used. Setting

$f(r) = \phi(r)e^{-\alpha r}$; $g(r) = \chi(r)e^{-\alpha r}$,

where

$$\alpha = \frac{\sqrt{m^2c^4 - E^2}}{\hbar c}$$

and

$$\rho = \alpha r, \quad \frac{E}{\hbar c\alpha} = c_1, \quad \frac{mc^2}{\hbar c\alpha} = c_2,$$

we obtain

$$\left.\begin{aligned} &e^{-\frac{\lambda}{2}}(\varphi' - \varphi) - \frac{k\varphi}{\rho} + (e^{-\frac{\nu}{2}}c_1 - c_2)\chi = 0, \\ &e^{-\frac{\lambda}{2}}(\chi' - \chi) + \frac{k\chi}{\rho} - (e^{-\frac{\nu}{2}}c_1 + c_2)\varphi = 0. \end{aligned}\right\} \quad (2)$$

These are a set of differential equations with argument ρ, where $e^{-\lambda} = e^{\nu} = 1 - \frac{2KM}{c^2r}$, K is a gravitation constant, c the speed of light and M the mass of the black hole.

The equations cannot be solved strictly, so we solve them approximately in two cases.

(1) Newtonian limit $e^{-\frac{\lambda}{2}} = 1$, $e^{-\frac{\nu}{2}} = 1 + \frac{KM\alpha}{c^2\rho}$.

(2) The neighbourhood of black hole

$$\frac{2KM}{c^2r} \to 1, \qquad \therefore \quad 1 - \frac{2KM\alpha}{c^2\rho} \to 0,$$

that is $\frac{k}{\rho} e^{\nu/2} \ll c_1$. Eq. (2) turns to

$$\left.\begin{aligned} &e^{-\frac{\lambda-\nu}{2}}(\varphi' - \chi') - \frac{k}{\rho} e^{\nu/2}(\varphi + \chi) + c_1(\varphi + \chi) = 0, \\ &e^{-\frac{\lambda-\nu}{2}}(\varphi' - \chi') - \frac{k}{\rho} e^{\frac{\nu}{2}}(\varphi - \chi) - c_1(\varphi - \chi) = 0. \end{aligned}\right\} \quad (3)$$

II. The Solution of the Dirac Equation in Newtonian Limit

Substituting the Newtonian limit into (2), we obtain

$$\varphi' - \varphi - \frac{k\varphi}{\rho} + \left[\left(1 + \frac{KM\alpha}{c^2\rho}\right)c_1 - c_2\right]\chi = 0, \quad (4A)$$

and

$$\chi' - \chi + \frac{k\chi}{\rho} - \left[\left(1 + \frac{KM\alpha}{c^2\rho}\right)c_1 + c_2\right]\varphi = 0. \quad (4B)$$

We then substitute the power series $\varphi = \sum_{\nu=0}^{\infty} a_\nu b^{\nu+s}$, $\chi = \sum_{\nu=0}^{\infty} b_\nu \rho^{\nu+s}$ into (4).

Equating coefficients of the same power of ρ are given for coefficients of $\rho^{\nu+s-1}$,

$$(\nu + s - k)a_\nu - a_{\nu-1} + \frac{\alpha KMc_1}{c^2} b_\nu - (c_2 - c_1)b_{\nu-1} = 0, \tag{5A}$$

$$(\nu + s + k)b_\nu - b_{\nu-1} - \frac{\alpha KMc_1}{c^2} a_\nu - (c_2 + c_1)a_{\nu-1} = 0. \tag{5B}$$

From (5A) and (5B) we can easily obtain

$$\left(\nu + s - k + \beta \frac{KM\alpha}{c^2} c_1\right)a_\nu + \left[-(\nu + s + k)\beta + \frac{KM\alpha}{c^2} c_1\right] b_\nu = 0, \tag{6}$$

where

$$\beta = \sqrt{\frac{mc^2 - E}{mc^2 + E}} = c_2 - c_1 = \sqrt{\frac{c_2 - c_1}{c_2 + c_1}}.$$

In order to make the wave function finite, the series should be cut off in $a_{\nu+1}$ and $b_{\nu+1}$. By (5A) or (5B)

$$\alpha_\nu + \beta b_\nu = 0 \tag{7}$$

is obtained. From (6) and (7), it can be seen that the determinant of coefficients must be equal to zero, if a solution of the equation is to be found. Then,

$$2(\nu + s)\beta + \beta^2 \frac{KM\alpha c_1}{c^2} - \frac{KM\alpha}{c^2} c_1 = 0, \tag{8}$$

so that the energy of particles is

$$E = \pm \hbar c \sqrt{\frac{-1 + \sqrt{1 + 4\left(\frac{KM}{(\nu + s)c^2}\right)^2 \left(\frac{mc}{\hbar}\right)^2}}{2\left(\frac{KM}{(\nu + s)c^2}\right)^2}}. \tag{9}$$

In view of weak gravitational field, the gravitational bound energy (9) can be expanded, so that the bound energy would be

$$E' = E - mc^2 = -\frac{(KM)^2 m^3}{2(\nu + s)^2 \hbar^2}. \tag{10}$$

This is the energy spectrum of the gravitational bound state under the Newtonian approximation. It is similar to the energy spectrum of the Schrödinger equation with pure gravitational potential (see Appendix (A3)). From (10) it can be known that the energy spectrum of small black hole bound state is in the neighbourhood of the visible spectrum, and the radiant energy of spontaneous emission from small black hole is high energy photons (X-ray and γ-ray), so they are in two different spectra, and there is no question of their mutual suppression. Of course, the number of photons of the bound state is very small, and it is almost impossible to see them, unless there are groups of small black holes in some regions of space. As the coefficient of $\rho^{\nu+s}$ equals zero, it is easy to obtain

$$s = \pm\sqrt{k^2 - \left(\frac{KM\alpha c_2}{c^2}\right)^2}. \tag{11}$$

To make sure that the wave function will be finite when $\rho \to 0$, s should be positive. Further discussion is as follows.

1. *The Limitation of the Bound Particles Caused by Radius of the Black Hole*

From (11), in order to make s a real number, it is necessary that $k > \frac{\pi KM}{c^2} \cdot \frac{E}{2\hbar c}$. Because $\frac{2KM}{c^2}$ is the radius of black hole r_s, and $\frac{hc}{E}$ is the λ_c of the Compton wave length of the particle,

$$r_s < k\lambda_c/\pi. \tag{12}$$

Thus it follows that not all partlcies can be in the quantum bound state, at least not in the ground state, i.e. $k = 1$. For small black holes of the order of 10^{-13}cm, the upper limit of the mass of the particles that can be in the ground state is the mass of neutrons $\left(E \approx mc^2,\ m < \frac{2\hbar}{cr_s}\right)$.

Note. We do not mean that particles of large mass cannot form bound state, but they cannot form quantum bound state.

2. *Removed Degeneracy*

Substituting (11) into (9), we can easily see that the energy spectrum is relevant to k, that is, energy levels are split with the angular quantum number k. The comparison of energy levels with the solution (A2) of the Schrödinger equation shows that degeneracy has been removed.

3. *Comformity With the Schrödinger Eq. (A2) of Pure Gravitational Potential Under the Approximation of Low Speed and Weak Gravitation*

Substituting the Newtonian approximation in (1A), we obtain

$$f(r) = \frac{\left(\frac{d}{dr} + \frac{1+k}{r}\right)g(r)}{\frac{1}{\hbar c}\left(E - mc^2 + \frac{KME}{c^2 r} + 2mc^2\right)}. \tag{13}$$

As in low speed and weak field, we have $E - mc^2 = E' \ll mc^2$ and $\frac{KME}{c^2 r} \ll mc^2$. Substituting (13) into (1B), making the necessary operation and taking care that the difference between E and mc^2 should be a small quantity of the first order, we may reduce Eq. (13) to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg}{dr}\right) + \frac{2m}{\hbar^2}\left(E' + \frac{KME}{c^2 r}\right)g - \frac{k(1+k)}{r^2}g = 0. \tag{14}$$

This is similar to Eq. (A2) in form, i. e. it has returned to the Schrödinger equation.

III. THE NEIGHBOURHOOD OF THE BLACK HOLE

Setting $y = \phi + x$, $z = \varphi - x$, $1 - \frac{2KM\alpha}{c^2\rho} = A$ and as $\frac{k}{\rho} e^{\frac{\nu}{2}} \ll c$, Eq. (3) then may be reduced to

$$\left.\begin{aligned} Ay' + c_1 z &= 0, \\ Az' - c_1 y &= 0. \end{aligned}\right\} \tag{15}$$

Obviously, the small black hole is similar to the large one in that the equation has also a zero solution ($y = z = 0$), but it is different from the large black hole because besides this zero solution, it also has a non-zero solution caused by the quantum effect. The non-zero solution is

$$\left.\begin{aligned} \varphi = \frac{y+z}{2} &= \frac{D}{2}\left\{\sin\left[\frac{|E|}{\hbar c}(r - r_s + r_s \ln \alpha(r - r_s))\right]\right. \\ &\quad \left.+ \cos\left[\frac{|E|}{\hbar c}(r - r_s + r_s \ln \alpha(r - r_s))\right]\right\}, \\ x = \frac{y-z}{2} &= \frac{D}{2}\left\{\sin\left[\frac{|E|}{\hbar c}(r - r_s + r_s \ln \alpha(r - r_s))\right]\right. \\ &\quad \left.- \cos\left[\frac{|E|}{\hbar c}(r - r_s + r_s \ln \alpha(r - r_s))\right]\right\}, \end{aligned}\right\} \tag{16}$$

where D is a constant to be determined, and by definition above $\alpha = \frac{\sqrt{m^2c^4 - E^2}}{\hbar c}$, $\frac{2KM}{c^2} = r_s$. From (16) we know that φ and x are not bound states, but continuous spectra, that is, bound states do not exist.

APPENDIX

THE SCHRÖDINGER EQUATION OF PURE GRAVITATIONAL POTENTIAL AND ITS SOLUTION

If the gravitational potential is $V(r) = -\frac{KMm}{r}$, then the Schrödinger equation must be

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\varphi = E\varphi. \tag{A1}$$

The angular part of the wave function is the spherical harmonic function Y_{lm} and the radial part of the equation is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{2m}{\hbar^2}\left(E + \frac{KMm}{r}\right) - \frac{l(l+1)}{r^2}\right]R = 0. \tag{A2}$$

In a similar way to the solution of the hydrogen atom, we can obtain the energy levels as:

$$E_n = -\frac{K^2M^2m^3}{2\hbar^2n^2}. \tag{A3}$$

The wave function is

$$R(r) = N_{nl}e^{-\alpha r/2}(\alpha r)^l L_{n+l}^{2l+1}(\alpha r), \tag{A4}$$

where $\alpha = \frac{2KMm^2}{n\hbar^2}$, $N_{nl} = -\left\{\alpha^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}\right\}^{\frac{1}{2}}$ are normalized constants, L_{n+l}^{2l+1} are the associated Laguerre polynomials.

The radius of the ground state is

$$a_0 = \int_0^\infty R_{10}^* r R_{10}(r) r^2 dr = \frac{\hbar^2 \sqrt[3]{3}}{KMm^2}. \tag{A5}$$

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