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On some examples of obstructed irregular surfaces

Dedicated to Professor F. Catanese on the Occasion of his 60th Birthday

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Abstract We determine the base space of the Kuranishi family of some complete intersections in the product of an abelian variety and a projective space. As a consequence, we obtain new examples of obstructed irregular surfaces with ample canonical bundle and maximal Albanese dimension.

Keywords algebraic surfaces, deformation theory, Kuranishi family

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0 Introduction

A compact complex manifold is called *obstructed* if the base space of its semiuniversal deformation is a singular germ of complex space; equivalently, a manifold X is obstructed if and only if there exist elements of $H^1(X, T_X)$ not belonging to the image of the Kodaira-Spencer map of a deformation of X over a smooth base.

In their foundational work on deformation theory, Kodaira and Spencer [24, 25] proved that, for a compact complex manifold X, if the cup product $H^1(T_X) \otimes H^1(T_X) \longrightarrow H^2(T_X)$ is non-trivial, then X is obstructed. Using this criterion, it is very easy to produce examples of obstructed manifolds of dimension greater than of equal to 3: the simplest example is probably the product $A \times \mathbb{P}^n$, where $n > 0$ and A is a complex torus of dimension greater than of equal to 2 [25, p. 436].

The search for obstructed surfaces is a more challenging problem; in fact the first, examples of obstructed surfaces were given in 1966 by Kas [20], while the first examples of obstructed surfaces with ample canonical bundle were given in 1975 by Horikawa [16, 17].

The examples of Kas and Horikawa are basically different and, in some sense, are the ancestors of two classes of examples: almost all the known examples of obstructed surfaces belong to such classes. The first is the class of surfaces containing smooth rational curves with self-intersection -2. The effect of such curves on the deformations has been clarified by Burns and Wahl in [3], Kas in [21] and Pinkham in [35]. Other examples of obstructed surfaces with -2 curves are described in [2, 5, 26, 32]. Moreover, Catanese [5] produced examples of surfaces with -2 curves with non reduced Kuranishi family; also an immediate application of the results of Burns and Wahl shows also that the minimal resolutions of the surfaces described in Example 3.24 of [28] are obstructed.

The second class of examples are constructed by using Horikawa theorems and their refinements, in order to compare the deformation spaces of geometric objects related by some standard procedure, such as blow-up, general hyperplane section and branched covering. For example, starting from Mumford's famous example of an obstructed curve $C \subset \mathbb{P}^3$, Horikawa [17] proved that a sufficiently ample hyperplane section of the blow-up of $\mathbb{P}^3$ along C is an obstructed surface with very ample canonical bundle. The obstructed surfaces described in [9, 10] as abelian covering branched over "obstructed" building data belong to the same class of examples. Finally, following the same ideas, Vakil proved in [42] that the moduli space of regular surfaces satisfies "Murphy's law". More precisely, every singularity type defined over $\mathbb{Z}$ is obtained as the deformation space of a regular surface.

One of the main tools used in the second construction is Kodaira's stability theorem [23]: if $Z \subset X$ is a smooth submanifold with $H^1(N_{Z|X}) = 0$ (e.g., if $H^2(\mathcal{O}_X) = 0$ and Z is a sufficiently ample hyperplane section), then every deformation of X lifts to a deformation of the pair (X, Z).

The main theme of this paper is to use the theory of differential graded Lie algebras in order to compute the deformation space of sufficiently ample hyperplane sections in some cases when Kodaira's theorem does not apply; as a by-product, we obtain the following examples of obstructed irregular surfaces.

Example A. Let A be an abelian surface and S a smooth surface of general type contained in $A \times \mathbb{P}^1$. Then the base space of the Kuranishi family of S is analytically isomorphic to the germ at 0 of $\mathbb{C}^s \times C$, where $s = h^1(S, T_S) - 6$ and C is the affine cone over the Segre variety $\mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^5$ (Example 5.4).

It is a great pleasure to dedicate this paper to Fabrizio Catanese on the occasion of his sixtieth birthday: several years ago, I was beginning to work to my PhD thesis under the supervision of Fabrizio and, as usual, he suggested a lot of good ideas and interesting problems about moduli of algebraic surfaces. One of them was concerned with deformation theory and, in particular, with the last section of the famous paper of Palamodov [34]. As I remember, Fabrizio told me something similar to *"the Massey powers seem very powerful but until now nobody has used them to compute the universal family of a concrete example of an algebraic surface"*.

At that time, I had not really understood the Massey powers and this idea was not pursued. On the other hand, working with Massey powers is quite difficult; as well explained in [38] *"Massey product structures can be very helpful, though they are in general described in a form that is unsatisfactory"*. In the same papers Schlessinger and Stasheff gave the basis for more satisfying structures which are a refinement of the Massey products (see the following Remark 2.5). The goal of this paper is to use these refinements to compute the universal deformation for a particular class of obstructed irregular surfaces.

1 Singularity type of commuting varieties

We work over the field $\mathbb{C}$ of complex numbers; every complex manifold is assumed to be compact and connected.

Following Vakil [42], we shall say that two analytic singularities $(X, 0)$ and $(Y, 0)$ have the same *singularity type* if there exists a diagram

$$\begin{tikzcd} & (Z,0) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ (X,0) & & (Y,0), \end{tikzcd} \tag{1.1}$$

where f and g are smooth morphisms of analytic singularities. Since smoothness is stable under base change, having the same singularity type is an equivalence relation. The following lemma implies that every analytic singularity is determined up to isomorphism by its singularity type and the dimension of its Zariski tangent space.

Lemma 1.1. *Assume that* $(X, 0)$ *and* $(Y, 0)$ *have the same singularity type and* $\dim(X, 0) \geqslant \dim(Y, 0)$, *then there exists an isomorphism* $(X, 0) \cong (Y \times \mathbb{C}^k, 0)$ *for some* $k \geqslant 0$.

Proof. Let $(Z,0)\xrightarrow{f}(X,0)$ be a smooth map of analytic singularities. If $\dim(Z,0)=\dim(X,0)$, then f is an isomorphism. If $\dim(Z,0)-\dim(X,0)=k>0$, then there exists an isomorphism $(Z,0)\cong(X\times\mathbb{C}^k,0)$ and, if $(W,0)\subset(Z,0)$ is a generic hyperplane section, then the restriction $f\colon(W,0)\to(X,0)$ is smooth.

Taking possibly generic hyperplane sections of the singularity $(Z,0)$ in the diagram (1.1), we can assume that f is an isomorphism. □

Let L be a finite-dimensional complex Lie algebra and q a positive integer. The (possibly non reduced) affine scheme

$$C(q,L)=\{(a_1,\dots,a_q)\in L^{\oplus q}\mid [a_i,a_j]=0\ \text{ for every } i,j\}$$

is called the *q-th commuting variety* of L.

Example 1.2. Since two matrices in $\mathfrak{sl}(2,\mathbb{C})$ commute if and only if they are linearly dependent, the commuting variety $C(q,\mathfrak{sl}(2,\mathbb{C}))$ is isomorphic to determinantal variety of matrices $q\times 3$ of rank $\leqslant 1$, or equivalently, to the affine cone over the Segre variety $\mathbb{P}^{q-1}\times\mathbb{P}^2\subset\mathbb{P}^{3q-1}$.

Remark. The structure of the varieties $C(q,L)$ has been studied by several people. The case $L=\mathfrak{sl}(n,\mathbb{C})$ was studied in Gerstenhaber [11]; he proved, in particular, that $C(2,\mathfrak{sl}(n,\mathbb{C}))$ is irreducible for every n (this fact was also proved independently by Motzkin and Taussky [33]). It is a well-known open (and hard) problem to determine whether $C(2,\mathfrak{sl}(n,\mathbb{C}))$ (defined by the ideal generated by brackets) is a reduced scheme. Moreover, according to Richardson [36], the variety $C(2,L)$ is irreducible for every reductive Lie algebra L.

Proposition 1.4. *Let L be a finite dimensional complex Lie algebra with trivial center. Then, for every $q\geqslant 2$, the analytic singularity $(C(q,L),0)$ is minimal in its singularity type class.*

Proof. Since $C(q,L)$ is defined in $L^{\oplus q}$ by quadratic equations, its Zariski tangent space at 0 is equal to $L^{\oplus q}$. Let $a=(a_1,\dots,a_q)\in C(q,L)-\{0\}$ and assume for simplicity $a_1\neq 0$. Then there exists $b\in L$ such that $[a_1,b]\neq 0$ and thus the vector $(0,b,0,\dots,0)$ does not belong to the Zariski tangent space of $C(q,L)$ at the point a. Therefore,

$$\dim T_0C(q,L)>\dim T_aC(q,L),\qquad \forall\, a\neq 0,$$

and the singularity $(C(q,L),0)$ cannot be of the form $(X\times\mathbb{C},0)$. □

Corollary 1.5. *Assume that the analytic singularity $(X,0)$ has the same singularity type as $C(q,\mathfrak{sl}(n,\mathbb{C}))$ for some $q,n\geqslant 2$. Then there exists an isomorphism*

$$(X,0)\cong(\mathbb{C}^k\times C(q,\mathfrak{sl}(n,\mathbb{C})),0),\qquad k=\dim T_0X-q(n^2-1).$$

Proof. The center of $\mathfrak{sl}(n,\mathbb{C})$ is trivial for $n\geqslant 2$. □

2 Differential graded Lie algebras and deformations

A differential graded vector space is a pair (V,d), where $V=\bigoplus V^i$ is a graded vector space and d is a differential of degree $+1$.

Definition 2.1. *A* differential graded Lie algebra *(DGLA for short) is the data of a differential graded vector space (V,d) and a bilinear map $[-,-]\colon V\times V\to V$ (called bracket) of degree* 0 *such that:*

(1) (*graded skewsymmetry*) $[a,b]=-(-1)^{\deg(a)\deg(b)}[b,a]$.

(2) (*graded Jacobi identity*) $[a,[b,c]]=[[a,b],c]+(-1)^{\deg(a)\deg(b)}[b,[a,c]]$.

(3) (*graded Leibniz rule*) $d[a,b]=[da,b]+(-1)^{\deg(a)}[a,db]$.

A morphism of DGLA is a morphism of complexes commuting with the brackets.

The reader may consult [12,27,29,30] for a more detailed exposition of differential graded Lie algebras and their associated deformation functors.

In this paper, we are mainly interested in two examples of differential graded Lie algebras. Given a holomorphic vector bundle E on a complex manifold X, we denote by $\mathcal{A}^{p,q}_X(E)$ the sheaf of differentiable (p,q)-forms of X, with values in E and $A^{p,q}_X(E)=\Gamma(X,\mathcal{A}^{p,q}_X(E))$ the space of its global sections.

Example 2.2. The graded vector space

$$A_X^{0,*}(T_X) = \bigoplus_i A_X^{0,i}(T_X),$$

where T_X is the holomorphic tangent sheaf, has the natural structure of a DGLA, when endowed with the opposite of the Dolbeault differential and with the antiholomorphic extension of the standard bracket of $\mathcal{A}_X^{0,0}(T_X)$.

Example 2.3. For every line bundle L on a complex manifold X, we denote by $\mathcal{D}(L)$ the locally free sheaf of first order holomorphic differential operators on L. If $z_1, \ldots, z_n$ are local holomorphic coordinates on an open set U and $s \in \Gamma(U, L)$ is a nowhere vanishing section, then every $\xi \in \Gamma(U, \mathcal{D}(L))$ is written formally as

$$\xi = s\left(\sum_i \alpha_i \frac{\partial}{\partial z_i} + \beta\right)s^{-1},$$

and this means that for every $h \in \mathcal{O}_X(U)$ we have

$$\xi(sh) = s\left(\sum_i \alpha_i \frac{\partial h}{\partial z_i} + \beta h\right).$$

There exists an exact sequence of sheaves of Lie algebras on X,

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{i} \mathcal{D}(L) \xrightarrow{\sigma} T_X \longrightarrow 0, \tag{2.1}$$

where i is the inclusion and σ is the principal symbol. In local coordinates,

$$i(\beta) = s(\beta)s^{-1}, \qquad \sigma\left(s\left(\sum_i \alpha_i \frac{\partial}{\partial z_i} + \beta\right)s^{-1}\right) = \sum_i \alpha_i \frac{\partial}{\partial z_i}.$$

The sequence (2.1) is obtained by applying the functor $\mathcal{H}om(-, L)$ to Atiyah's extension of L [1, p. 194].

If X is compact Kähler, then according to [1, Propositions 3 and 12], the extension class of (2.1) is equal to $2\pi i c_1(L) \in H^1(X, \Omega_X^1) \cong \mathrm{Ext}^1(T_X, \mathcal{O}_X)$. In particular, for every $p \geqslant 0$, the morphism $H^p(T_X) \longrightarrow H^{p+1}(\mathcal{O}_X)$ induced by the exact sequence (2.1) is a multiple of the contraction with the first Chern class of L. Considering the Dolbeault resolution of the Atiyah extension of L, we get an exact sequence

$$0 \longrightarrow A_X^{0,*} \longrightarrow A_X^{0,*}(\mathcal{D}(L)) \xrightarrow{\sigma} A_X^{0,*}(T_X) \longrightarrow 0 \tag{2.2}$$

and $A_X^{0,*}(\mathcal{D}(L))$ carries a natural structure of DGLA such that σ is a morphism of differential graded Lie algebras.

Denoting by **Art** the category of local artinian $\mathbb{C}$-algebras and by **Set** the category of sets, for every differential graded Lie algebra V, we have the functors

$$\mathrm{MC}_V\colon \mathbf{Art} \to \mathbf{Set}, \qquad \mathrm{Def}_V = \frac{\mathrm{MC}_V}{\text{Gauge action}}\colon \mathbf{Art} \to \mathbf{Set}.$$

The functor MC_L is called the *Maurer-Cartan* functor and is defined as

$$\mathrm{MC}_V(A) = \left\{x \in V^1 \otimes \mathfrak{m}_A \;\middle|\; dx + \frac{1}{2}[x, x] = 0\right\},$$

where $\mathfrak{m}_A$ is the maximal ideal of A.

Two elements $x, y \in V \otimes \mathfrak{m}_A$ are said to be *gauge equivalent* if there exists an $a \in V^0 \otimes \mathfrak{m}_A$ such that

$$y = e^a * x := x + \sum_{n=0}^{\infty} \frac{[a, -]^n}{(n+1)!}([a, x] - da).$$

We have $e^a * (e^b * x) = e^{a\bullet b} * x$, where $\bullet$ is the Baker-Campbell-Hausdorff product [19,41] in the nilpotent Lie algebra $V^0 \otimes \mathfrak{m}_A$. The set

$$\exp(V^0 \otimes \mathfrak{m}_A) = \{e^a \mid a \in V^0 \otimes \mathfrak{m}_A\}$$

of formal exponentials of elements of the nilpotent Lie algebra $V^0 \otimes \mathfrak{m}_A$ has a group structure with unit e^0, with inverse $(e^a)^{-1} = e^{-a}$ and with product $e^a e^b = e^{a\bullet b}$; thus $*$ is an action of the exponential group $\exp(V^0 \otimes \mathfrak{m}_A)$ on the graded vector space $V \otimes \mathfrak{m}_A$, called the *gauge action*. It is not difficult to see that the set of solutions to the Maurer-Cartan equation is stable under the gauge action, and then it makes sense to consider the functor $\mathrm{Def}_V\colon \mathbf{Art} \to \mathbf{Set}$ defined as above.

It is known (see e.g. [13,27,30]) that the tangent space of Def_V is isomorphic to $H^1(V)$ and that $H^2(V)$ is an obstruction space. If $H^1(V)$ is finite-dimensional, then Def_V satisfies Schlessinger's conditions in order to have a hull [37]. Then, by the standard smoothness criterion of [8], a morphism $V \to W$ of DGLA induces a smooth morphism of functors $\mathrm{Def}_V \to \mathrm{Def}_W$ if $H^1(V) \to H^1(W)$ is surjective and $H^2(V) \to H^2(W)$ is injective.

Theorem 2.4. [38] *Let $V \to W$ be a morphism of differential graded Lie algebras. Assume that*
(1) $H^0(V) \to H^0(W)$ *is surjective*;
(2) $H^1(V) \to H^1(W)$ *is bijective*;
(3) $H^2(V) \to H^2(W)$ *is injective.*
Then $\mathrm{Def}_V \to \mathrm{Def}_W$ *is an isomorphism.*

Proof. See e.g., Theorem 5.71 of [30]. □

The importance of the above construction is motivated by the fact, or metatheorem, that in characteristic 0, every functor of Artin rings arising from a deformation problem is isomorphic to Def_V for some DGLA V defined up to quasi-isomorphism. For instance, if X is a complex manifold, then the functor Def_X of infinitesimal deformations of X is isomorphic to $\mathrm{Def}_{A^{0,*}_X(T_X)}$ (see e.g. [13]); notice that in this case, the Maurer-Cartan equation is essentially the Newlander-Niremberg integrability condition, as described in [4].

Let L be a line bundle on a complex manifold X; according to [40], a deformation of the pair (X, L) is the data of a deformation $\mathcal{X}$ of X and an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}_{|X} = L$. It is well known [6] that the DGLA $A^{0,*}_X(\mathcal{D}(L))$ governs the deformation of the pair (X, L). Moreover, via the isomorphisms

$$\mathrm{Def}_{A^{0,*}_X(T_X)} \cong \mathrm{Def}_X \quad \text{and} \quad \mathrm{Def}_{A^{0,*}_X(\mathcal{D}(L))} \cong \mathrm{Def}_{(X,L)},$$

the forgetful natural transformation

$$\mathrm{Def}_{(X,L)} \to \mathrm{Def}_X$$

is induced by the morphism σ of (2.2).

Remark 2.5. Every differential graded Lie algebra V carries a sequence of operations

$$H^1(V) \to \{\text{subsets of } H^2(V)\}, \quad \theta \mapsto [\theta^k], \quad k \geqslant 2$$

called *Massey powers* that are invariant under quasi-isomorphism and satisfying the following properties:
(1) $[\theta^2] = \frac{1}{2}[\theta, \theta]$ is the primary obstruction of θ,
(2) $[\theta^{k+1}]$ is not empty if and only if $0 \in [\theta^k]$.
For the classical definition of Massey powers we refer to [34,39]. Here, we clarify their role in deformation theory by deriving them from the functor Def_V.

For every integer $n > 0$ define $A_n = \frac{\mathbb{C}[t]}{(t^{n+1})}$; we have seen that there exists a canonical isomorphism $\mathrm{Def}_V(A_1) = H^1(V)$. Denote by

$$\pi_n\colon \mathrm{Def}_V(A_n) \to \mathrm{Def}_V(A_1)$$

the map induced by the natural projection $A_n \to A_1$. Next, we define for every n an *obstruction map*

$$o_n\colon \mathrm{Def}_V(A_n) \to H^2(V)$$

in the following way: for every element $\tilde{x} \in \mathrm{Def}_V(A_n)$ we choose a representative of it as a solution

$$x \in V^1 \otimes \mathfrak{m}_{A_n} = V^1 t + V^1 t^2 + \cdots + V^1 t^n$$

of the Maurer-Cartan equation. Then, we choose a lifting of x to an element $y \in V^1 \otimes \mathfrak{m}_{A_{n+1}}$ and we consider

$$dy + \frac{1}{2}[y, y] \in V^2 \otimes \mathfrak{m}_{A_{n+1}}.$$

Clearly, $dy + \frac{1}{2}[y, y] = ht^{n+1}$ for some $h \in V^2$, moreover (see e.g. [27, 31]), $dh = 0$ and its cohomology class $[h] \in H^2(V)$ depends only on $\tilde{x}$. This gives a map

$$o_n\colon \mathrm{Def}_V(A_n) \to H^2(V), \qquad \tilde{x} \mapsto [h],$$

and for every $\theta \in H^1(V)$, $n > 0$, we have

$$[\theta^{n+1}] = o_n(\pi_n^{-1}(\theta)).$$

3 Deformations of $\mathbb{C}^q/\Gamma \times \mathbb{P}^n$

If Z, Y are complex manifolds and $E \to Z$, $F \to Y$ are vector bundles, we denote

$$E \boxtimes F = p^*E \otimes q^*F,$$

where $p\colon Z \times Y \to Z$ and $q\colon Z \times Y \to Y$ are the projections. By the Künneth formula [15] we have

$$H^i(Z \times Y, E \boxtimes F) = \bigoplus_j H^j(Z, E) \otimes H^{i-j}(Y, F).$$

Notice that

$$T_{Z\times Y} = p^*T_Z \oplus q^*T_Y = (T_Z \boxtimes \mathcal{O}_Y) \oplus (\mathcal{O}_Z \boxtimes T_Y).$$

Let $\mathbb{C}^q/\Gamma$ be a complex torus of dimension q. In [25, p. 436] Kodaira and Spencer proved that the manifold $X = (\mathbb{C}^q/\Gamma) \times \mathbb{P}^n$ has obstructed deformations for every $q \geqslant 2$ and every $n \geqslant 1$. This was the first example of obstructed manifold; the same example is also discussed, with a different approach, in [7]. Here, we refine this result with the following theorem.

Theorem 3.1. *For every $q \geqslant 2$ and every $n \geqslant 1$, the manifold $X = (\mathbb{C}^q/\Gamma) \times \mathbb{P}^n$ does not have a universal deformation. The base space of its Kuranishi family is singular, it is analytically isomorphic to the germ at 0 of $\mathbb{C}^{q^2} \times C(q, \mathfrak{sl}(n+1))$ and it is reducible for every $n \geqslant 3$ and $q \geqslant 3 + 8/(n-2)$.*

Proof. Denote by $B^i \subset A^{0,i}_{\mathbb{C}^q/\Gamma}$ the subspace of invariant forms of type $(0, i)$ on the torus. If $z_1, \ldots, z_q$ are linear coordinates on $\mathbb{C}^q$, then B^1 is the vector space generated by $d\overline{z_1}, \ldots, d\overline{z_q}$, $B^i = \bigwedge^i B^1$ and the inclusion $B^i \subset A^{0,i}_{\mathbb{C}^q/\Gamma}$ induces an isomorphism $B^i = H^i(\mathcal{O}_{\mathbb{C}^q/\Gamma})$ [14, p. 301]. The natural inclusion

$$B = \bigoplus_{i\geqslant 0} B^i = \bigwedge^* B^1 \xrightarrow{\imath} A^{0,*}_{\mathbb{C}^q/\Gamma}$$

is a quasi-isomorphism of differential graded algebras and therefore, since the tangent vector bundle of a complex torus is trivial, the inclusion

$$B \otimes H^0(T_{\mathbb{C}^q/\Gamma}) \xrightarrow{\imath\otimes \mathrm{Id}} A^{0,*}_{\mathbb{C}^q/\Gamma}(T_{\mathbb{C}^q/\Gamma})$$

is a quasi-isomorphism of DGLA. According to the Künneth formula, the inclusions

$$Q_1 := B \otimes H^0(T_{\mathbb{C}^q/\Gamma}) \xrightarrow{\imath\otimes p^*} A^{0,*}_X(p^*T_{\mathbb{C}^q/\Gamma}),$$

$$Q_2 := B \otimes H^0(T_{\mathbb{P}^n}) \xrightarrow{\imath\otimes q^*} A^{0,*}_X(q^*T_{\mathbb{P}^n})$$

yield a quasi-isomorphism of differential graded Lie algebras

$$Q := Q_1 \oplus Q_2 = (B \otimes H^0(T_{\mathbb{C}^q/\Gamma})) \oplus (B \otimes H^0(T_{\mathbb{P}^n})) \xrightarrow{\imath\otimes p^* + \imath\otimes q^*} A_X^{0,*}(T_X).$$

Therefore, the functor

$$\mathrm{Def}_Q = \mathrm{Def}_{Q_1} \times \mathrm{Def}_{Q_2}$$

is isomorphic to the functor of infinitesimal deformations of X. Notice that since Q_1 is abelian, the gauge action is trivial and

$$\mathrm{Def}_{Q_1}(A) = H^1(Q_1) \otimes \mathfrak{m}_A = \{\mathrm{Spec}(A) \to (H^1(Q_1), 0)\},$$

so that Def_{Q_1} is prorepresented by $\mathbb{C}^{q^2}$. The subspace

$$Q_2^+ = \bigoplus_{i>0} Q_2^i = \bigoplus_{i>0} B^i \otimes H^0(T_{\mathbb{P}^n})$$

is a differential graded Lie subalgebra of Q_2 and the inclusion $Q_2^+ \to Q_2$ induces an isomorphism of Maurer-Cartan functors: $\mathrm{MC}_{Q_2^+} = \mathrm{MC}_{Q_2}$. Since the gauge action associated with Q_2^+ is trivial, we have $\mathrm{MC}_{Q_2^+} = \mathrm{Def}_{Q_2^+}$. Fixing a basis $b_1, \dots, b_q$ of B^1, we have

$$\frac{1}{2}\left[\sum b_i \otimes \eta_i, \sum b_i \otimes \eta_i\right] = \sum_{i<j} b_i \wedge b_j [\eta_i, \eta_j], \qquad \eta_i \in H^0(T_{\mathbb{P}^n}),$$

and therefore, $\mathrm{Def}_{Q_2^+} = \mathrm{MC}_{Q_2^+}$ is prorepresented by the germ at 0 of the commuting variety

$$C(q, H^0(T_{\mathbb{P}^n})) = C(q, \mathfrak{sl}(n+1)).$$

Finally, since $H^0(T_{\mathbb{P}^n})$ is not abelian, the gauge action on MC_{Q_2} is non-trivial and so $\mathrm{Def}_{Q_2^+} \to \mathrm{Def}_{Q_2}$ is a hull but not an isomorphism. The assertion about reducibility follows immediately from the following Lemma 3.2. $\square$

Lemma 3.2. *If $n \geqslant 4$ and the commuting variety $C(q, \mathfrak{sl}(n, \mathbb{C}))$ is irreducible, then*

$$q < 3 + \frac{8n-12}{(n-2)^2} \quad \textit{for } n \textit{ even},$$
$$q < 3 + \frac{8}{n-3} \quad \textit{for } n \textit{ odd}.$$

Proof. This proof is based on the ideas of [18]. Assume $C(q, \mathfrak{sl}(n))$ is irreducible and consider the projection to the first factor

$$\pi \colon C(q, \mathfrak{sl}(n, \mathbb{C})) \to C(1, \mathfrak{sl}(n, \mathbb{C})) = \mathfrak{sl}(n, \mathbb{C}).$$

Let $D \in \mathfrak{sl}(n, \mathbb{C})$ be a diagonal matrix with distinct eigenvalues, then every matrix commuting with D must be diagonal. Therefore, the fiber $\pi^{-1}(D)$ is irreducible of dimension $(n-1)(q-1)$ and the dimension of $C(q, \mathfrak{sl}(n, \mathbb{C}))$ is less than or equal to

$$n^2 - 1 + (n-1)(q-1).$$

On the other hand, let r be the integral part of $n/2$ and let $N \subset \mathfrak{sl}(n, \mathbb{C})$ be the closed subset of matrices A such that $A^2 = 0$. It is easy to see that N is irreducible of dimension $2r(n-r)$ and therefore, for the generic $A \in N$, we have

$$2r(n-r) + \dim \pi^{-1}(A) < n^2 - 1 + (n-1)(q-1).$$

After a possible change of basis, every $A \in N$ belongs to the space

$$H = \{(h_{ij}) \in \mathfrak{sl}(n, \mathbb{C}) \mid h_{ij} \neq 0 \text{ only if } i > r,\ j \leqslant r\}.$$

Then H is an abelian subalgebra of $\mathfrak{sl}(n,\mathbb{C})$ and therefore,

$$\{A\}\times H^{\oplus q-1}\subset \pi^{-1}(A).$$

In particular,

$$r(n-r)(q-1)=\dim H^{\oplus q-1}<n^2-1+(n-1)(q-1)-2r(n-r).$$

A straightforward computation gives the inequalities of the lemma. □

Remark 3.3. Lemma 3.2 also gives a partial answer to the question of [11, p. 342]. In the same paper, Gerstenhaber proved that $C(q,\mathfrak{sl}(n,\mathbb{C}))$ is irreducible for $n\leqslant 3$ and not irreducible for $q>n\geqslant 4$.

4 Deformations of pairs $(\mathbb{C}^q/\Gamma\times\mathbb{P}^n, L)$

We recall that the Appell-Humbert data on a complex torus $\mathbb{C}^q/\Gamma$ are a pair (α,H), where H is a hermitian form on $\mathbb{C}^n$ such that its imaginary part E is integral on $\Gamma\times\Gamma$ and α is a semicharacter for H:

$$\alpha\colon\Gamma\to\mathrm{U}(1),\qquad \alpha(\gamma_1+\gamma_2)=\alpha(\gamma_1)\alpha(\gamma_2)(-1)^{E(\gamma_1,\gamma_2)}.$$

Denote by $L(\alpha,H)$ the line bundle on $\mathbb{C}^q/\Gamma$ with the factor of automorphy [22, p. 4]

$$A_\gamma(z)=\alpha(\gamma)e^{\pi(H(z,\gamma)+H(\gamma,\gamma)/2)},\qquad \gamma\in\Gamma,\ z\in\mathbb{C}^q.$$

According to Appell-Humbert's theorem, every line bundle on $\mathbb{C}^q/\Gamma$ is isomorphic to $L(\alpha,H)$ for a unique Appell-Humbert data (α,H); moreover, the first Chern class of $L(\alpha,H)$ is equal to the invariant form of type $(1,1)$ corresponding to E [22, Lemma 3.5]. In particular, two line bundles $L(\alpha_1,H_1)$, $L(\alpha_2,H_2)$ have the same Chern class if and only if $H_1=H_2$.

The same proof of the Appell-Humbert theorem given in [22], with minor and straightforward modifications, shows that every line bundle on $\mathbb{C}^q/\Gamma\times\mathbb{P}^n$ is isomorphic to

$$L(\alpha,H,d):=L(\alpha,H)\boxtimes\mathcal{O}(d)$$

for some Appell-Humbert data (α,H) and some integer d.

Denote for simplicity $X=\mathbb{C}^q/\Gamma\times\mathbb{P}^n$ and consider the exact sequence

$$0\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{D}(L(\alpha,H,d))\xrightarrow{\sigma}T_X\longrightarrow 0\,.$$

As observed in Section 2, the induced morphism in cohomology

$$H^1(T_X)\xrightarrow{\delta}H^2(\mathcal{O}_X)$$

is equal to a scalar multiple of the contraction with the first Chern class of the line bundle $L(\alpha,H,d)$.

Lemma 4.1. *In the above notation, if* $\det(H)\neq 0$*, then* δ *is surjective and its kernel contains* $H^1(\mathcal{O}_{\mathbb{C}^q/\Gamma})\otimes H^0(T_{\mathbb{P}^n})$*. In particular, the map*

$$H^2(\mathcal{D}(L(\alpha,H,d)))\longrightarrow H^2(T_X)$$

is injective.

Proof. Since

$$H^1(T_X)=H^1(T_{\mathbb{C}^q/\Gamma})\otimes H^0(\mathcal{O}_{\mathbb{P}^n})\oplus H^1(\mathcal{O}_{\mathbb{C}^q/\Gamma})\otimes H^0(T_{\mathbb{P}^n}),\quad H^2(\mathcal{O}_X)=H^2(\mathcal{O}_{\mathbb{C}^q/\Gamma}),$$

and

$$c_1(L(\alpha,H,d))=p^*c_1(L(\alpha,H))+q^*c_1(\mathcal{O}(d)),$$

it is sufficient to prove that the map

$$H^1(T_{\mathbb{C}^q/\Gamma})\xrightarrow{\lrcorner c_1(L(\alpha,H))}H^2(\mathcal{O}_{\mathbb{C}^q/\Gamma})$$

is surjective.

The elements of $H^2(\mathcal{O}_{\mathbb{C}^q/\Gamma})$ are represented by invariant $(0,2)$-forms: if $z_1,\dots,z_q$ are linear coordinates on $\mathbb{C}^q$, then a basis of $H^2(\mathcal{O}_{\mathbb{C}^q/\Gamma})$ is given by the forms $d\overline{z}_i\wedge d\overline{z}_j$, for $i<j$. Similarly, a basis of $H^1(T_{\mathbb{C}^q/\Gamma})$ is given by the invariant tensors

$$d\overline{z}_i\otimes\frac{\partial}{\partial z_j},\quad \text{for } i,j=1,\dots,q.$$

The first Chern class of $L(\alpha,H)$ is given by the invariant form $\sum h_{rs}dz_r\wedge d\overline{z}_s$, where (h_{rs}) is a scalar multiple of H, and therefore,

$$\left(d\overline{z}_i\otimes\frac{\partial}{\partial z_j}\right)\lrcorner c_1(L(\alpha,H))=d\overline{z}_i\otimes\frac{\partial}{\partial z_j}\lrcorner\sum_{r,s}h_{rs}dz_r\wedge d\overline{z}_s=\sum_s h_{js}d\overline{z}_i\wedge d\overline{z}_s.$$

Therefore, since the matrix (h_{rs}) is invertible, the contraction map is surjective. □

Theorem 4.2. *Let L be an ample line bundle on $X=\mathbb{C}^q/\Gamma\times\mathbb{P}^n$. Then the base space of the semiuniversal deformation of the pair (X,L) has the same singularity type as $C(q,\mathfrak{sl}(n+1,\mathbb{C}))$.*

Proof. We have $L=L(\alpha,H,d)$ for some integer $d>0$ and a positive definite hermitian form H. Denoting by P the fiber product of

$$A_X^{0,*}(\mathcal{D}(L))\xrightarrow{\sigma}A_X^{0,*}(T_X)$$

and the injective quasi-isomorphism

$$Q=B\otimes(H^0(T_{\mathbb{C}^q/\Gamma})\oplus H^0(T_{\mathbb{P}^n}))\to A_X^{0,*}(T_X)$$

introduced in the proof of Theorem 3.1, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_X^{0,*} & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & 0\\
 & & \| & & \big\downarrow\phi & & \big\downarrow & & \\
0 & \longrightarrow & A_X^{0,*} & \longrightarrow & A_X^{0,*}(\mathcal{D}(L)) & \xrightarrow{\sigma} & A_X^{0,*}(T_X) & \longrightarrow & 0,
\end{array}$$

where the rows are exact and the columns are quasi-isomorphisms. In particular, according to Theorem 2.4, the functor Def_P is isomorphic to the functor of deformations of the pair (X,L).

We have a direct sum decomposition

$$Q=(B^0\otimes H^0(T_{\mathbb{C}^q/\Gamma}))\oplus(B^1\otimes H^0(T_{\mathbb{C}^q/\Gamma}))\oplus R,$$

where

$$R=\bigoplus_{i>1}B^i\otimes H^0(T_{\mathbb{C}^q/\Gamma})\oplus B\otimes H^0(T_{\mathbb{P}^n}).$$

Since $R^1=B^1\otimes H^0(T_{\mathbb{P}^n})$, the functor Def_R is prorepresented by the germ at 0 of the commuting variety $C(q,\mathfrak{sl}(n+1,\mathbb{C}))$; therefore, in order to prove the theorem, it is sufficient to prove that the composition $P\to Q\to R$ is surjective on H^1 and injective on H^2. Equivalently, it is sufficient to prove that

$$H^2(\mathcal{D}(L(\alpha,H,d)))\longrightarrow H^2(T_X)$$

is injective and the image of

$$H^1(\mathcal{D}(L(\alpha,H,d)))\longrightarrow H^1(T_X)$$

contains $B^1\otimes H^0(T_{\mathbb{P}^n})$, and this follows from Lemma 4.1. □

5 Deformations of smooth ample divisors of $\mathbb{C}^q/\Gamma \times \mathbb{P}^n$

The aim of this section is to apply the previous results and Horikawa's costability theorems in order to compute the Kuranishi family of a smooth ample divisor of $X = \mathbb{C}^q/\Gamma \times \mathbb{P}^n$. We first point out that the canonical divisor of X is $K_X = L(0, 0, -n-1)$ and

$$L(\alpha_1, H_1, d_1) \otimes L(\alpha_2, H_2, d_2) = L(\alpha_1\alpha_2, H_1 + H_2, d_1 + d_2).$$

Lemma 5.1. *On the manifold $X = \mathbb{C}^q/\Gamma \times \mathbb{P}^n$ we have*

(1) *If $\alpha \neq 1$, then $H^i(L(\alpha, 0, d)) = 0$ for every $i, d \in \mathbb{Z}$.*

(2) *If H is positive definite and $d \geqslant -n$, then*

$$H^i(L(\alpha, H, d)) = 0$$

for every $i > 0$.

(3) *If H is negative definite and $d \leqslant -2$, then*

$$H^i(L(\alpha, H, d)) = H^i(T_X \otimes L(\alpha, H, d)) = 0$$

for every $i \leqslant q + n - 2$.

(4) *If H is negative definite and $d \leqslant -n-2$, then*

$$H^{q+n-1}(T_X \otimes L(\alpha, H, d)) = 0.$$

Proof. The determination of the cohomology of line bundles on $\mathbb{C}^q/\Gamma$ [22, Theorem 3.9] gives that

(i) If $\alpha \neq 1$, then $H^i(L(\alpha, 0)) = 0$ for every i.

(ii) If H is negative definite, then $H^i(L(\alpha, H)) = 0$ for every $i < q$.

By the Künneth formula we get (1). Assume now that H is negative definite; then by the Künneth formula we get $H^i(L(\alpha, H, d)) = 0$ for every α, every $d < 0$ and every $i < q + n$. By the Serre duality, we get (2).

The bundle $T_X \otimes L(\alpha, H, d)$ is the direct sum of $L(\alpha, H) \boxtimes T_{\mathbb{P}^n}(d)$ and q copies of $L(\alpha, H, d)$. Since $H^i(T_{\mathbb{P}^n}(d)) = 0$ for every $d \leqslant -2$ and every $i \leqslant n-2$, we get (3).

If $d \leqslant -n-2$ then $H^{n-1}(T_{\mathbb{P}^n}(d)) = 0$, and this implies (4). □

Remark 5.2. For simplicity of exposition, in Lemma 5.1 we stated only the vanishing theorems we need for our application: as pointed out by the referee, the same proof gives a slightly stronger result.

Theorem 5.3. *Let $L = L(\alpha, H, d)$ be an ample line bundle on the variety $X = \mathbb{C}^q/\Gamma \times \mathbb{P}^n$, and let S be a smooth divisor which is the zero locus of a section of L. Assume $q, d \geqslant 2$, $n \geqslant 1$ and $q + n + d \geqslant 6$. Then every deformation of S is projective and the base space of the Kuranishi family of S has the same singularity type as the commuting variety $C(q, \mathfrak{sl}(n+1, \mathbb{C}))$.*

Proof. Let us denote by $\mathrm{Def}_{(X,L)}$ the functor of deformations of the pair (X, L), by $\mathrm{Def}_{(X,S)}$ the functor of deformations of the pair (X, S) and by Def_S the functor of deformations of S.

According to Theorem 4.2, it is sufficient to prove that the natural morphisms

$$\mathrm{Def}_{(X,S)} \to \mathrm{Def}_S, \qquad \mathrm{Def}_{(X,S)} \to \mathrm{Def}_{(X,L)}$$

are smooth. By Lemma 5.1 we have $H^1(L) = 0$ and this implies that $\mathrm{Def}_{(X,S)} \to \mathrm{Def}_{(X,L)}$ is smooth. The ampleness of S also implies that every deformation of the pair (X, S) is projective.

On the other hand, by Lemma 5.1, $H^2(T_X(-S)) = 0$ and then by Horikawa's costability theorem [17, Thmeorem 8.3], the morphism $\mathrm{Def}_{(X,S)} \to \mathrm{Def}_S$ is smooth. □

Example 5.4. Let $A = \mathbb{C}^2/\Gamma$ be an abelian surface and S a smooth irreducible surface contained in $X = A \times \mathbb{P}^1$. Then S is of general type if and only if $\mathcal{O}_X(S) = L(\alpha, H, d)$ for some H positive definite and $d \geqslant 3$. In fact, since $H^0(\mathcal{O}_X(S)) \neq 0$, we get by Künneth's formula that $d \geqslant 0$, H is positive semidefinite and $\alpha \equiv 1$ on $\ker H \cap \Gamma$. Assume that S is of general type and denote by r the dimension of the kernel of H; if $r = 2$ then S is isomorphic to A, while if $r = 1$ then S is an elliptic fibration. Therefore, H is positive definite and the degree d of the projection $S \to A$ must be greater than or equal to 2. Thus, if S is of general type, we have $\mathcal{O}_X(K_X + S) = L(\alpha, H, d-2)$ and then, by Lemma 5.1 we have $H^1(X, m(K_X + S)) = 0$ for every $m > 0$. Therefore, $H^0(X, m(K_X + S)) \to H^0(S, mK_S)$ is surjective for every $m > 0$ and if $d = 2$, then the image of the pluricanonical map of S is an abelian surface. Conversely, if $d > 2$ and H is positive definite, then K_S is ample.

Therefore, if S is of general type, according to Theorem 5.3, the base space of the universal deformation of S has the same singularity type as $C(2, \mathfrak{sl}(2, \mathbb{C}))$.

Remark 5.5. The same ideas can be used to understand the deformation type of sufficiently ample and generic complete intersections in $\mathbb{C}^q/\Gamma \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

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