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Real Paley-Wiener theorems for the Clifford Fourier transform

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Abstract Associated with the Dirac operator and partial derivatives, this paper establishes some real Paley-Wiener type theorems to characterize the Clifford-valued functions whose Clifford Fourier transform (CFT) has compact support. Based on the Riemann-Lebesgue theorem for the CFT, the Boas theorem is provided to describe the CFT of Clifford-valued functions that vanish on a neighborhood of the origin.

Keywords Clifford Fourier transform, Dirac operator, Paley-Wiener theorem, Boas theorem

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1 Introduction

The classical Paley-Wiener theorem describes the Fourier transform (FT) of L^2-functions on the real line with compact support in a symmetric interval as entire functions of exponential type whose restriction to the real line is L^2-functions. Higher dimensional extensions of the Paley-Wiener theorem have been studied. In [12], Kou and Qian generalized the classical Paley-Wiener theorem for Clifford-valued functions in the Clifford algebra setting owing to the imbedding of $\mathbb{R}^n$ into the real Clifford algebra. On the other hand, recently there has also been a great interest to real Paley-Wiener theorems discovered by Bang [3], and Tuan and Zayed [20], in which the adjective "real" expresses that information about the support of the FT comes from growth rates associated to f on $\mathbb{R}^n$, rather than on $\mathbb{C}^n$ as in the classical "complex Paley-Wiener theorem". The set-up is as follows. For any $f \in L^2(\mathbb{R}^n)$, the FT $\mathcal{F}f$ of f has a compact support if and only if $f \in C^\infty(\mathbb{R}^n)$, $\triangle^k f \in L^2(\mathbb{R}^n)$ and

$$\lim_{k\to\infty} \|\triangle^k f\|_{L^2(\mathbb{R}^n)}^{\frac{1}{2k}} = \sup\{|\omega| : \omega \in \mathcal{F}f(\omega) \neq 0\} < \infty,$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\triangle$ denotes the Laplace operator in $\mathbb{R}^n$ defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{1.1}$$

This result is due to [17]. A wide number of papers have been devoted to the extension of the theory on higher dimensions and many other integral transforms [1, 2, 6, 7, 14, 15, 18, 19]. A comprehensive overview of the literature on the real Paley-Wiener theory was included in [2].

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The Clifford Fourier transform is a generalization of the FT to higher dimensions in the framework of the Clifford algebra, motivated by applications to higher dimensional signal processing. The Dirac operator may be looked upon as the square root of the Laplace operator in $\mathbb{R}^n$ defined by (1.1) and this factorization is one of the most fundamental features in the Clifford analysis. Many efforts have been devoted to some important properties and applications of the CFT [4, 5, 8–11, 13].

In this paper, inspired by the treatment of the Dirac operator in Clifford analysis, we will derive some real Paley-Wiener theorems to characterize the Clifford-valued functions whose CFT has compact support and also a Boas theorem to describe the CFT of these functions that vanish on a neighborhood of the origin.

The paper is organized as follows: Section 2 is devoted to reviewing some definitions and basic properties of the real Clifford algebra and Clifford analysis. In Section 3, based on the definition of the CFT, we recall some properties of the CFT, such as the Plancherel theorem, the vector derivative and so on. The Riemann-Lebesgue theorem is shown for the CFT. In Section 4, we proceed with the study of real Paley-Wiener theorems for the CFT associated with the Dirac operator and the partial derivative. In Section 5, we establish a Boas theorem for the CFT.

2 Preliminaries

Let $\mathbb{R}_n$ be the 2^n-dimensional universal real Clifford algebra over $\mathbb{R}^n$ constructed from the basis $\{e_1, e_2, \dots, e_n\}$ under the usual relations

$$e_k e_l + e_l e_k = 2\delta_{kl}, \quad 1 \leqslant k, l \leqslant n,$$

where δ_{kl} is the Kronecker delta function. An element $f \in \mathbb{R}_n$ can be represented as $f = \sum_A f_A e_A$, $f_A \in \mathbb{R}$, where $e_A = e_{j_1 j_2 \cdots j_k} = e_{j_1} e_{j_2} \cdots e_{j_k}$, $A = \{j_1, j_2, \dots, j_k\}$ with $1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n$, and $e_0 = e_\emptyset = 1$ is the identity element of $\mathbb{R}_n$. The elements of the algebra $\mathbb{R}_n$ for which $|A| = k$ are called k-vectors. We denote the space of all k-vectors by $\mathbb{R}_n^k := \operatorname{span}_{\mathbb{R}}\{e_A : |A| = k\}$. It is clear that the spaces $\mathbb{R}$ and $\mathbb{R}^n$ can be identified with $\mathbb{R}_n^0$ and $\mathbb{R}_n^1$, respectively.

Of interest for this work is the (unit oriented) pseudoscalar element $i_n = e_1 e_2 \cdots e_n$. Observe that $i_n^2 = -1$ and $i_n^{-1} = -i_n$ for $n = 2, 3 \,(\mathrm{mod}\ 4)$. For the sake of simplicity, if not otherwise stated, n is always assumed to be $n = 2, 3 \,(\mathrm{mod}\ 4)$ for the remaining of this paper.

We define the anti-automorphism reversion $\tilde{}: \mathbb{R}_n \to \mathbb{R}_n$ by its action on the basis elements $\tilde{e_A} = (-1)^{\frac{k(k-1)}{2}} e_A$, for $|A| = k$, and its reversion property $\tilde{fg} = \tilde{g}\tilde{f}$ for every $f, g \in \mathbb{R}_n$. In particular, we remark that $\tilde{i_n} = -i_n$.

The elliptic, rotation-invariant, vector differential operator of first order

$$\partial_{\boldsymbol{x}} := \sum_{j=1}^{n} e_j \partial_{x_j} \tag{2.1}$$

called Dirac operator, may be looked upon as the square root of the Laplace operator in $\mathbb{R}^n$ defined by (1.1) satisfying $\triangle = \partial_{\boldsymbol{x}}^2$, where ∂_{x_j} is a short notation for the partial operator $\frac{\partial}{\partial x_j}$.

In what follows, we will require two types of scalar products. First, we introduce the (real valued) scalar product of $f, g \in \mathbb{R}_n$ as the scalar part of their geometric product

$$f \cdot g := [f\tilde{g}]_0 = \sum_A f_A g_A. \tag{2.2}$$

As usual, when we set $f = g$ we obtain the square of the modulus (or magnitude) of the multivector $f \in \mathbb{R}_n$,

$$|f|^2 = [f\tilde{f}]_0 = \sum_A f_A^2. \tag{2.3}$$

Second, we require an inner product in the function space under consideration. We denote by $L^p(\mathbb{R}^n;\mathbb{R}_n)$ the left module of all Clifford-valued functions $f:\mathbb{R}^n\to\mathbb{R}_n$ with finite norm

$$\|f\|_p=\begin{cases}\left(\int_{\mathbb{R}^n}|f(\boldsymbol{x})|^p d^n\boldsymbol{x}\right)^{\frac{1}{p}}, & 1\leqslant p<\infty,\\ \operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^n}|f(\boldsymbol{x})|, & p=\infty,\end{cases} \tag{2.4}$$

where $d^n\boldsymbol{x}=dx_1dx_2\cdots dx_n$ represents the usual Lebesgue measure in $\mathbb{R}^n$. In the particular case of $p=2$, we shall denote this norm by $\|f\|$. Given two functions $f,g\in L^2(\mathbb{R}^n,\mathbb{R}_n)$, we define the Clifford-valued inner product by

$$(f,g):=\int_{\mathbb{R}^n}f(\boldsymbol{x})\tilde{g}(\boldsymbol{x})d^n\boldsymbol{x}, \tag{2.5}$$

from which we can induce the L^2-norm in $L^2(\mathbb{R}^n,\mathbb{R}_n)$ by taking the scalar part and $f=g$.

3 Clifford Fourier transform

First of all, we recall the definition of the CFT as follows, originally introduced by Felsberg [9].

Definition 3.1. Let $f\in L^1(\mathbb{R}^n,\mathbb{R}_n)$. The CFT of f at the point $\omega\in\mathbb{R}^n$ is defined as the $\mathbb{R}_n$-valued (Lebesgue) integral

$$\mathcal{F}f(\omega)=\int_{\mathbb{R}^n}f(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}. \tag{3.1}$$

The function $\mathcal{F}f(\omega)$ is called the CFT of f.

Concerned with the behavior of the CFT at infinity, we show that the Riemann-Lebesgue theorem holds true for it, which plays a key role in proving the Boas theorem for the CFT.

Theorem 3.2. *If $f\in L^1(\mathbb{R}^n,\mathbb{R}_n)$, then it holds*

$$\mathcal{F}f(\omega)\to 0,\quad \text{as } |\omega|\to\infty.$$

Proof. It is easy to verify that $f\in L^1(\mathbb{R}^n,\mathbb{R}_n)$ if and only if $f_A\in L^1(\mathbb{R}^n;\mathbb{R})$, where $f=\sum_A f_Ae_A$, $f_A\in\mathbb{R}$. A direct calculation leads to $\mathcal{F}f(\omega)=\sum_A\mathcal{F}\{f_A\}(\omega)e_A$, which tells us that the computation of the CFT of $f\in L^1(\mathbb{R}^n,\mathbb{R}_n)$ can be reduced to that of the real-valued functions $f_A\in L^1(\mathbb{R}^n;\mathbb{R})$. Therefore, we simply need to prove the result to each of the real component function $f_A\in L^1(\mathbb{R}^n;\mathbb{R})$.

We do it in three steps. First, assume that $g(\boldsymbol{x})$ is piecewise constant on a compact domain $D=[a,b]^n\subset\mathbb{R}^n$ which means that D is subdivided into a finite number of subdomains $D_\alpha=D_{j_1j_2\cdots j_n}=\prod_{k=1}^n[a_{j_k-1},a_{j_k}]$, $j_k=1,2,\ldots,N$, $k=1,2,\ldots,n$ $(a_0=a, a_N=b)$ and that $g(\boldsymbol{x})$ has a certain constant value c_α for $\boldsymbol{x}\in D_\alpha$. This means that we can write $g(\boldsymbol{x})=\sum_\alpha c_\alpha\chi_\alpha(\boldsymbol{x})$, where $\chi_\alpha(\boldsymbol{x})=1$ on D_α and $\chi_\alpha(\boldsymbol{x})=0$ outside of D_α. Then, we get

$$\begin{aligned}\int_D g(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}&=\sum_\alpha c_\alpha\int_{D_\alpha}\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\\&=\frac{i_n^n}{\prod_{j=1}^n\omega_j}\sum_\alpha c_\alpha\prod_{k=1}^n(\mathrm{e}^{-i_na_{j_k}\omega_j}-\mathrm{e}^{-i_na_{j_k-1}\omega_j}),\end{aligned}$$

which leads to

$$\left|\int_D g(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\right|\leqslant\frac{C}{\prod_{j=1}^n|\omega_j|}\to 0,\quad \text{as } |\omega|\to\infty \tag{3.2}$$

with the positive constant $C=2^n\sum_\alpha|c_\alpha|$ independent of ω. Second, let f_A be an arbitrary function satisfying $f_A\in L^1(D;\mathbb{R})$. Then for all $\varepsilon>0$ there exists a piecewise constant function $g(\boldsymbol{x})$ such that

$$\int_D|f_A(\boldsymbol{x})-g(\boldsymbol{x})|d^n\boldsymbol{x}<\frac{\varepsilon}{2}.$$

Thus, it follows that

$$\begin{aligned}\left|\int_D f_A(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\right| &= \left|\int_D (f_A(\boldsymbol{x})-g(\boldsymbol{x}))\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}+\int_D g(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\right|\\ &\leqslant \int_D |f_A(\boldsymbol{x})-g(\boldsymbol{x})|d^n\boldsymbol{x}+\left|\int_D g(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\right|\\ &< \frac{\varepsilon}{2}+\left|\int_D g(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\right|,\end{aligned}$$

where the last integral tends to zero as $|\omega|\to\infty$ by the preceding case, which leads to

$$\left|\int_D f_A(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\right|\to 0,\quad \text{as } |\omega|\to\infty.$$

Finally, since $f_A\in L^1(\mathbb{R}^n;\mathbb{R})$, there exists a compact domain $D\subset\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n-D}|f_A(\boldsymbol{x})|d^n\boldsymbol{x}<\frac{\varepsilon}{2}.$$

Thus, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f_A(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\right| &\leqslant \left|\int_D f_A(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\right|+\int_{\mathbb{R}^n-D}|f_A(\boldsymbol{x})|d^n\boldsymbol{x}\\ &< \left|\int_D f_A(\boldsymbol{x})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{x}}d^n\boldsymbol{x}\right|+\frac{\varepsilon}{2},\end{aligned}$$

where the first term tends to zero as $|\omega|\to\infty$ by the preceding case. This completes the proof. □

Based on Theorem 3.2, we can infer that if $F(\omega)\in L^1(\mathbb{R}^n;\mathbb{R}_n)$ then

$$\lim_{|N|\to\infty}\int_{\mathbb{R}^n}F(\omega)\mathrm{e}^{i_nN\cdot\omega}d^n\omega\to 0, \tag{3.3}$$

where $N=\sum_{k=1}^n N_ke_k$. Similarly, following the same lines as the proof of Theorem 3.2, we can conclude that if $F(\omega)\in L^1(\mathbb{R}^n;\mathbb{R}_n)$ then for a fixed point $\boldsymbol{x}\in\mathbb{R}^n$ we can get a series of integrals whose limits are equal to zero when $|N|$ tends to infinity, such as

$$\lim_{|N|\to\infty}\int_{\mathbb{R}^n}F(\omega)\mathrm{e}^{i_nN_1\omega_1}\mathrm{e}^{i_nx_2\omega_2}\cdots\mathrm{e}^{i_nx_n\omega_n}d^n\omega=0$$

and

$$\lim_{|N|\to\infty}\int_{\mathbb{R}^n}F(\omega)\mathrm{e}^{i_nN_1\omega_1}\mathrm{e}^{i_nN_2\omega_2}\cdots\mathrm{e}^{i_nx_n\omega_n}d^n\omega=0,$$

and so on. Associating this fact with (3.3), it follows that if $F(\omega)\in L^1(\mathbb{R}^n;\mathbb{R}_n)$ then for a fixed point $\boldsymbol{x}\in\mathbb{R}^n$,

$$\begin{aligned}&\lim_{|N|\to\infty}\int_{\mathbb{R}^n}F(\omega)\prod_{k=1}^n(\mathrm{e}^{i_nN_k\omega_k}-\mathrm{e}^{i_nx_k\omega_k})d^n\omega\\ &\quad=\int_{\mathbb{R}^n}F(\omega)(-1)^n\mathrm{e}^{i_nx_1\omega_1}\mathrm{e}^{i_nx_2\omega_2}\cdots\mathrm{e}^{i_nx_n\omega_n}d^n\omega=(-1)^n\int_{\mathbb{R}^n}F(\omega)\mathrm{e}^{i_n\omega\cdot\boldsymbol{x}}d^n\omega,\end{aligned}$$

which will play a key role in the proof of the Boas theorem for CFT.

Moreover, we recall some properties of the CFT as follows. For more details, see [10].

Lemma 3.3 (CFT Plancherel). *For two Clifford module functions $f,g\in L^1(\mathbb{R}^n,\mathbb{R}_n)\cap L^2(\mathbb{R}^n,\mathbb{R}_n)$, one has*

$$(f,g)=\frac{1}{(2\pi)^n}(\mathcal{F}f,\mathcal{F}g),$$

where the Clifford-valued inner product $(\cdot,\cdot)$ is defined by (2.5).

For $g = f$ the Plancherel theorem above has a CFT Parseval theorem as a direct corollary.

Lemma 3.4 (CFT Parseval). *If $f \in L^1(\mathbb{R}^n, \mathbb{R}_n) \cap L^2(\mathbb{R}^n, \mathbb{R}_n)$, then*

$$\|f\|^2 = \frac{1}{(2\pi)^n} \|\mathcal{F}f\|^2.$$

Lemma 3.4 asserts that the CFT is a bounded linear operator on $L^1(\mathbb{R}^n, \mathbb{R}_n) \cap L^2(\mathbb{R}^n, \mathbb{R}_n)$. Hence, standard density arguments allow us to extend the CFT in a unique way to the whole of $L^2(\mathbb{R}^n, \mathbb{R}_n)$. In what follows we always consider the properties of the CFT as an operator from $L^2(\mathbb{R}^n, \mathbb{R}_n)$ into $L^2(\mathbb{R}^n, \mathbb{R}_n)$.

It is easy to check that the CFT of $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ can be inverted by means of

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}f(\omega) \mathrm{e}^{i_n \omega \cdot \boldsymbol{x}} d^n\omega \tag{3.4}$$

with $d^n\omega = d\omega_1 d\omega_2 \cdots d\omega_n$.

Lemma 3.5 (CFT partial derivative). *The CFT of $\partial_{x_k} f(\boldsymbol{x}) \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ is given by*

$$\mathcal{F}\{\partial_{x_k} f(\boldsymbol{x})\}(\omega) = \omega_k \mathcal{F}f(\omega) i_n.$$

In general, for $m \in \mathbb{Z}_+$ we have

$$\mathcal{F}\{\partial_{x_k}^m f(\boldsymbol{x})\}(\omega) = \omega_k^m \mathcal{F}f(\omega) i_n^m.$$

Lemma 3.6 (CFT vector differential). *Let $a \in \mathbb{R}^n$ be an arbitrary vector. Then the CFT of $(a \cdot \partial_{\boldsymbol{x}})^m f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ is given by*

$$\mathcal{F}\{(a \cdot \partial_{\boldsymbol{x}})^m f(\boldsymbol{x})\}(\omega) = (a \cdot \omega)^m \mathcal{F}f(\omega) i_n^m, \quad m \in \mathbb{Z}_+,$$

where $\partial_{\boldsymbol{x}}$ is the Dirac operator defined by (2.1).

Lemma 3.7 (CFT left vector derivative). *The CFT of $\partial_{\boldsymbol{x}}^m f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ (applied from the left-hand side of f) is*

$$\mathcal{F}\{\partial_{\boldsymbol{x}}^m f(\boldsymbol{x})\}(\omega) = \omega^m \mathcal{F}f(\omega) i_n^m, \quad m \in \mathbb{Z}_+.$$

Lemma 3.8 (CFT right vector derivative). *The CFT of $f\partial_{\boldsymbol{x}}^m \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ (applied from the right-hand side of f) is for $n = 3$ (mod 4),*

$$\mathcal{F}\{f\partial_{\boldsymbol{x}}^m\}(\omega) = i_n^m \mathcal{F}f(\omega)\omega^m, \quad m \in \mathbb{Z}_+,$$

and for $n = 2$ (mod 4),

$$\mathcal{F}\{f\partial_{\boldsymbol{x}}^m\}(\omega) = \mathcal{F}f((-1)^m\omega)\omega^m i_n^m, \quad m \in \mathbb{Z}_+.$$

4 Paley-Wiener theorem

Let us recall some basic facts on the modulus in the case of Clifford number. As we all know, the modulus of the Clifford numbers is not multiplicative, i.e., for any two elements $a, b \in \mathbb{R}_n$ we have $|ab| \leqslant 2^{\frac{n}{2}}|a||b|$. However, inside a Clifford algebra there is the possibility, in some special cases, that the modulus is multiplicative. These cases are described in the following lemma.

Lemma 4.1. *Let $b \in \mathbb{R}_n$ be such that $b\tilde{b} = |b|^2$. Then*

$$|ab| = |ba| = |a||b|, \quad \forall\, a \in \mathbb{R}_n.$$

Proof. Consider $|ab|^2$. By (2.3) we have

$$|ab|^2 = [ab\widetilde{ab}]_0 = [ab\tilde{b}\tilde{a}]_0 = [a|b|^2\tilde{a}]_0 = [a\tilde{a}]_0|b|^2 = |a|^2|b|^2.$$

Similarly, we can prove that $|ba|^2 = |a|^2|b|^2$. □

Since it holds $i_n^m \tilde{i_n^m} = 1 = |i_n^m|^2$, for any Clifford-valued functions $F(\cdot): \mathbb{R}^n \to \mathbb{R}_n$, Lemma 4.1 gives

$$|i_n^m F(\omega)| = |F(\omega) i_n^m| = |F(\omega)||i_n^m| = |F(\omega)|. \tag{4.1}$$

Furthermore, note that the squares of vectors $\omega \in \mathbb{R}^n$ are positive definite scalars and so are all the even powers of the vectors

$$\omega^2 = |\omega|^2 \geqslant 0, \quad \omega^m = |\omega|^{m'} \geqslant 0 \quad \text{for } m = 2m', m' \in \mathbb{Z}_+,$$

which leads to

$$\omega\tilde{\omega} = \omega^2 = |\omega|^2, \quad \omega^m \tilde{\omega^m} = |\omega|^{2m} \quad \text{for } m = 2m', m' \in \mathbb{Z}_+.$$

Moreover, for all the odd powers of the vectors $\omega \in \mathbb{R}^n$, a simple calculation leads to

$$\omega^m \tilde{\omega^m} = |\omega|^{2m} \quad \text{for } m = 2m'+1, m' \in \mathbb{Z}_+.$$

Thus, in general, for any $m \in \mathbb{Z}_+$ it holds

$$\omega^m \tilde{\omega^m} = |\omega|^{2m}, \quad |\omega^m| = |\omega|^m.$$

Associating with Lemma 4.1, we get the following corollary which will play a key role in proving the Paley-Wiener theorems for the CFT.

Corollary 4.2. *Let $\omega \in \mathbb{R}^n$. Then for any Clifford-valued functions $F(\cdot): \mathbb{R}^n \to \mathbb{R}_n$ it holds*

$$|\omega^m F(\omega)| = |F(\omega)\omega^m| = |F(\omega)||\omega^m| = |F(\omega)||\omega|^m.$$

Let $C^\infty(\mathbb{R}^n, \mathbb{R}_n)$ denote the module of Clifford-valued infinitely differentiable functions. Let $\mathrm{B}(0,\sigma)$ be the open ball in $\mathbb{R}^n$ centered at 0 with radius σ, i.e., $\mathrm{B}(0,\sigma) = \{\omega \in \mathbb{R}^n : |\omega| < \sigma\}$. Denote $\mathrm{supp}\mathcal{F}f(\omega)$ to be the support of $\mathcal{F}f(\omega)$ (or spectrum of f) describing the smallest closed set in $\mathbb{R}^n$ outside which $\mathcal{F}f(\omega)$ vanishes almost everywhere.

Theorem 4.3. *Let $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$. Then $\mathrm{supp}\mathcal{F}f(\omega) \subset \mathrm{B}(0,\sigma)$ if and only if $f \in C^\infty(\mathbb{R}^n, \mathbb{R}_n)$, $\partial_{\boldsymbol{x}}^k f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ and*

$$\lim_{k\to\infty} \|\partial_{\boldsymbol{x}}^k f\|^{\frac{1}{k}} = \sigma < \infty, \tag{4.2}$$

where $\sigma = \sup\{|\omega| : \omega \in \mathrm{supp}\mathcal{F}f(\omega)\}$ and the Dirac operator $\partial_{\boldsymbol{x}}$ is defined by (2.1).

Proof. Necessity. We can assume that $\|f\| > 0$, otherwise $\sigma = 0$ and (4.2) is trivial. Suppose that $\mathrm{supp}\mathcal{F}f(\omega) \subset \mathrm{B}(0,\sigma)$. The compactness of the support of $\mathcal{F}f(\omega)$ and the square integrability of f imply that $\omega^k \mathcal{F}f(\omega) \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ and thus $\partial_{\boldsymbol{x}}^k f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ for any $k = 0, 1, 2, \ldots$ Moreover, by Lemma 3.7 we have

$$\mathcal{F}\{\partial_{\boldsymbol{x}}^k f\}(\omega) = \omega^k \mathcal{F}f(\omega) i_n^k.$$

Applying (4.1), Lemma 3.4 and Corollary 4.2, we get

$$\begin{aligned}
\|\partial_{\boldsymbol{x}}^k f\|^2 &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\omega^k \mathcal{F}f(\omega) i_n^k|^2 d^n\omega = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\omega^k|^2 |\mathcal{F}f(\omega) i_n^k|^2 d^n\omega \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\omega|^{2k} |\mathcal{F}f(\omega)|^2 d^n\omega = \frac{1}{(2\pi)^n} \int_{\mathrm{supp}\mathcal{F}f} |\omega|^{2k} |\mathcal{F}f(\omega)|^2 d^n\omega.
\end{aligned}$$

Consequently,

$$\|\partial_{\boldsymbol{x}}^k f\|^{\frac{1}{k}} = \|f\|^{\frac{1}{k}} \left\{ \frac{1}{(2\pi)^n} \frac{1}{\|f\|^2} \int_{\mathrm{supp}\mathcal{F}f} |\omega|^{2k} |\mathcal{F}f(\omega)|^2 d^n\omega \right\}^{\frac{1}{2k}}. \tag{4.3}$$

Notice that $|\omega|^{2k}, |\mathcal{F}f(\omega)|^2$ are real-valued functions, so we can use the well-known result in classical Lebesgue space $L^p(\Omega, d\mu)$, which tells us that if μ is a Lebesgue measure on a set Ω with $\mu(\Omega) = 1$ then it holds

$$\lim_{p\to\infty} \|\varphi\|_{L^p(\Omega,d\mu)} = \|\varphi\|_{L^\infty(\Omega,d\mu)}. \tag{4.4}$$

Let $\Omega = \mathrm{supp}\mathcal{F}f, \varphi = |\omega|, p = 2k$, and $d\mu = \frac{1}{(2\pi)^n}\frac{1}{\|f\|^2}|\mathcal{F}f(\omega)|^2 d^n\omega$. It is easy to check that $\mu(\mathrm{supp}\mathcal{F}f) = 1$. Hence, formula (4.4) leads to

$$\lim_{k\to\infty}\left\{\frac{1}{(2\pi)^n}\frac{1}{\|f\|^2}\int_{\mathrm{supp}\mathcal{F}f}|\omega|^{2k}|\mathcal{F}f(\omega)|^2 d^n\omega\right\}^{\frac{1}{2k}} = \sup_{\omega\in\mathrm{supp}\mathcal{F}f}|\omega| = \sigma.$$

Since $\lim_{k\to\infty}\|f\|^{\frac{1}{k}} = 1$, by (4.3) we obtain

$$\lim_{k\to\infty}\|\partial_{\boldsymbol{x}}^k f\|^{\frac{1}{k}} = \sigma.$$

Sufficiency. Suppose that $\partial_{\boldsymbol{x}}^k f \in L^2(\mathbb{R}^n, \mathbb{R}_n), k = 0, 1, 2, \ldots$ and

$$\lim_{k\to\infty}\|\partial_{\boldsymbol{x}}^k f\|^{\frac{1}{k}} = d < \infty.$$

We want to prove that $\sigma = \sup\{|\omega| : \mathcal{F}f(\omega) \neq 0\} < \infty$. We attack the proof by the reduction to absurdity. Suppose that $\mathcal{F}f(\omega) \neq 0$, a.e. $\omega \in \mathbb{R}^n$, thus for any $M > 0$ we have

$$\int_{\mathbb{R}^n}|\omega|^{2k}|\mathcal{F}f(\omega)|^2 d^n\omega \geqslant \int_U |\omega|^{2k}|\mathcal{F}f(\omega)|^2 d^n\omega \geqslant CM^{2k},$$

where $U := \{\omega : |\omega| \geqslant M\}$ and C is some positive constant independent of k. The above inequality implies

$$\lim_{k\to\infty}\|\partial_{\boldsymbol{x}}^k f\|^{\frac{1}{k}} = \infty,$$

which contradicts the assumption of the convergence of the sequence $\|\partial_{\boldsymbol{x}}^k f\|^{\frac{1}{k}}$. Thus, $\mathcal{F}f$ is compactly supported. Finally, the same technique as the part of necessity yields that $d = \sigma$. We have finished the proof. □

We remark that the Dirac operator $\partial_{\boldsymbol{x}}$ is involved in applying to the left-hand side of the functions $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ in Theorem 4.3. Due to the noncommutative property of the Clifford algebra, the Dirac operator can also be used to describe the results from the right-hand side of the functions $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$. According to Lemma 3.8, applying the similar technique as that of Theorem 4.3 leads to the following result. For simplicity, we omit the proof here.

Theorem 4.4. *Let $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$. Then $\mathrm{supp}\mathcal{F}f(\omega) \subset \mathrm{B}(0, \sigma)$ if and only if $f \in C^\infty(\mathbb{R}^n, \mathbb{R}_n)$, $f\partial_{\boldsymbol{x}}^k \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ and*

$$\lim_{k\to\infty}\|f\partial_{\boldsymbol{x}}^k\|^{\frac{1}{k}} = \sigma < \infty,$$

where $\sigma = \sup\{|\omega| : \omega \in \mathrm{supp}\mathcal{F}f(\omega)\}$ and the Dirac operator $\partial_{\boldsymbol{x}}$ is defined by (2.1).

Furthermore, associated with the Dirac operator, we will show another real Paley-Wiener-type theorem for functions $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ with the support given by the symmetric body. According to [16], let $K = \{\omega : \omega^2 \leqslant 1, \forall\, \omega \in \mathbb{R}^n\}$ be a convex, compact and symmetric set in $\mathbb{R}^n$ with nonempty interior which is called a symmetric body (symmetric means $-\omega \in K$ if $\omega \in K$). Then the set $K^* = \{a \in \mathbb{R}^n : a \cdot \omega \leqslant 1, \forall\, \omega \in K\}$ is called the polar set of K, where $a \cdot \omega = \sum_{k=1}^n a_k\omega_k$ with $a = \sum_{k=1}^n a_k e_k$ and $\omega = \sum_{k=1}^n \omega_k e_k$. We can see that K^* is also a symmetric body and $(K^*)^* = K$.

Theorem 4.5. *Let $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$. Then the CFT $\mathcal{F}f(\omega)$ vanishes outside a symmetric body K if and only if $f \in C^\infty(\mathbb{R}^n; \mathbb{R}_n)$, $\partial_{\boldsymbol{x}}^k f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ and*

$$\sup_{a\in K^*}\|(a\cdot\partial_{\boldsymbol{x}})^k f\| \leqslant M, \quad k = 1, 2, \ldots, \tag{4.5}$$

where the Dirac operator $\partial_{\boldsymbol{x}}$ is defined by (2.1) and M is a positive constant independent of k.

Proof. Necessity. Suppose that the CFT $\mathcal{F}f(\omega)$ of $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ vanishes outside the symmetric body K. Then $f \in C^\infty(\mathbb{R}^n, \mathbb{R}_n)$ and $\partial_{\boldsymbol{x}}^k f$ belongs to $L^2(\mathbb{R}^n, \mathbb{R}_n)$ together with all its partial derivatives. By Lemma 3.6 it follows

$$\mathcal{F}\{(a \cdot \partial_{\boldsymbol{x}})^k f\}(\omega) = (a \cdot \omega)^k \mathcal{F}f(\omega) i_n^k.$$

Applying Lemma 3.4, we obtain

$$\|(a \cdot \partial_{\boldsymbol{x}})^k f\|^2 = \frac{1}{(2\pi)^n} \|(a \cdot \omega)^k \mathcal{F}f(\omega) i_n^k\|^2 = \frac{1}{(2\pi)^n} \|(a \cdot \omega)^k \mathcal{F}f(\omega)\|^2.$$

Since K is a symmetric body in $\mathbb{R}^n$, it holds $|a \cdot \omega| \leqslant 1$ for all $\omega \in K$ and $a \in K^*$. Hence, we have

$$\begin{aligned}\|(a \cdot \omega)^k \mathcal{F}f(\omega)\|^2 &= \int_{\mathbb{R}^n} |a \cdot \omega|^{2k} |\mathcal{F}f(\omega)|^2 d^n\omega = \int_K |a \cdot \omega|^{2k} |\mathcal{F}f(\omega)|^2 d^n\omega \\ &\leqslant \int_K |\mathcal{F}f(\omega)|^2 d^n\omega = \|\mathcal{F}f(\omega)\|^2 = (2\pi)^n \|f\|^2\end{aligned}$$

which means $\sup_{a \in K^*} \|(a \cdot \partial_{\boldsymbol{x}})^k f\| \leqslant \|f\| = M$. Conversely, suppose (4.5) is valid for all k. Since $f \in C^\infty(\mathbb{R}^n; \mathbb{R}_n)$ and $\partial_{\boldsymbol{x}}^k f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$, the CFT of $(a \cdot \partial_{\boldsymbol{x}})^k f$ exists. Based on (4.1), Lemmas 3.4 and 3.6, it follows

$$\sup_{a \in K^*} \|(a \cdot \omega)^k \mathcal{F}f(\omega)\|^2 = (2\pi)^n \sup_{a \in K^*} \|(a \cdot \partial_{\boldsymbol{x}})^k f\|^2 \leqslant M, \quad k = 1, 2, \ldots$$

Sufficiency. Suppose that ω_0 does not belong to the symmetric body K, then there exists $a_0 \in K^*$ such that $a_0 \cdot \omega_0 > 1$. Thus, there is a neighborhood U_{ω_0} of ω_0 with the property $a_0 \cdot \omega > \frac{a_0 \cdot \omega_0 + 1}{2} > 1$ for $\omega \in U_{\omega_0}$. We have

$$\begin{aligned}M &\geqslant \sup_{a \in K^*} \|(a \cdot \omega)^k \mathcal{F}f(\omega)\|^2 \geqslant \|(a_0 \cdot \omega)^k \mathcal{F}f(\omega)\|^2 \\ &\geqslant \int_{U_{\omega_0}} (a_0 \cdot \omega)^{2k} |\mathcal{F}f(\omega)|^2 d^n\omega \geqslant \left|\frac{a_0 \cdot \omega_0 + 1}{2}\right|^k \int_{U_{\omega_0}} |\mathcal{F}f(\omega)|^2 d^n\omega.\end{aligned}$$

Note that $|\frac{a_0 \cdot \omega_0 + 1}{2}|^k$ approaches ∞ as $k \to \infty$. Therefore, the inequalities above hold only if $\int_{U_{\omega_0}} |\mathcal{F}f(\omega)|^2 d^n\omega = 0$, which means that ω_0 does not belong to the support of $\mathcal{F}f(\omega)$, which leads to $\mathrm{supp}\mathcal{F}f \subset K$. □

Actually, apart from the Dirac operator $\partial_{\boldsymbol{x}}$, we can further establish the real Paley-Wiener theorem for the CFT of $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ by partial derivatives. Given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we write as usual $|\alpha| = \sum_{j=1}^n \alpha_j$, $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ for partial derivatives.

Theorem 4.6. *Let $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$. Then the CFT $\mathcal{F}f(\omega)$ is compactly supported in $[-\sigma, \sigma]^n$ if and only if all partial derivatives $D^\alpha f(\boldsymbol{x}) \in L^2(\mathbb{R}^n; \mathbb{R}_n)$, $\prod_{j=1}^n \omega_j^{\alpha_j} \mathcal{F}f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ for all $\alpha \in \mathbb{Z}_+^n$ and*

$$\lim_{|\alpha| \to \infty} \|D^\alpha f\|^{1/|\alpha|} = \sigma,$$

where $\sigma = \sup\{|\omega_j|, 1 \leqslant j \leqslant n : \mathcal{F}f(\omega) \neq 0, \omega = \sum_{j=1}^n \omega_j e_j \in \mathbb{R}^n\}$.

Proof. Necessity. Suppose that $\mathrm{supp}\mathcal{F}f(\omega) = [-\sigma, \sigma]^n$. The compactness of the support of $\mathcal{F}f$ and $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ imply that $\prod_{j=1}^n \omega_j^{\alpha_j} \mathcal{F}f$ belongs to $L^1(\mathbb{R}^n, \mathbb{R}_n) \cap L^2(\mathbb{R}^n, \mathbb{R}_n)$, thus partial derivatives $D^\alpha f(\boldsymbol{x})$ exist and belong to $L^2(\mathbb{R}^n, \mathbb{R}_n)$ for all $\alpha \in \mathbb{Z}_+^n$. Moreover, by Lemma 3.5 we have

$$\mathcal{F}\{D^\alpha f\}(\omega) = \prod_{j=1}^n \omega_j^{\alpha_j} \mathcal{F}f(\omega) i_n^{|\alpha|}.$$

Applying Lemma 3.4, it follows

$$\|D^\alpha f\|^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left|\prod_{j=1}^n \omega_j^{\alpha_j} \mathcal{F}f(\omega) i_n^{|\alpha|}\right|^2 d^2\omega,$$

i.e., by (4.1) we have

$$\|D^\alpha f\|^2 = \frac{1}{(2\pi)^n}\int_{[-\sigma,\sigma]^n}\prod_{j=1}^n |\omega_j|^{2\alpha_j}|\mathcal{F}f(\omega)|^2 d^n\omega. \tag{4.6}$$

Thus, we obtain

$$\|D^\alpha f\|^2 \leqslant \frac{1}{(2\pi)^n}\sigma^{2|\alpha|}\|\mathcal{F}f\|^2 = \sigma^{2|\alpha|}\|f\|^2,$$

which leads to

$$\|D^\alpha f\|^{1/|\alpha|} \leqslant C^{1/|\alpha|}\sigma$$

with the constant $C = \|f\|$ independent of $|\alpha|$. Then, we have

$$\limsup_{|\alpha|\to\infty}\|D^\alpha f\|^{1/|\alpha|} \leqslant \sigma.$$

On the other hand, using (4.6) again, for $\epsilon \in (0, \sigma/2)$, there hold

$$\begin{aligned}\|D^\alpha f\|^2 &\geqslant \frac{1}{(2\pi)^n}\int_{[\sigma-2\epsilon,\sigma-\epsilon]^n}\prod_{j=1}^n |\omega_j|^{2\alpha_j}|\mathcal{F}f(\omega)|^2 d^n\omega \\ &\geqslant (\sigma-2\epsilon)^{2|\alpha|}\frac{1}{(2\pi)^n}\int_{[\sigma-2\epsilon,\sigma-\epsilon]^n}|\mathcal{F}f(\omega)|^2 d^n\omega,\end{aligned}$$

which lead to

$$\liminf_{|\alpha|\to\infty}\|D^\alpha f\|^{1/|\alpha|} \geqslant \sigma - 2\epsilon.$$

The arbitrariness of ϵ implies

$$\liminf_{|\alpha|\to\infty}\|D^\alpha f\|^{1/|\alpha|} \geqslant \sigma.$$

Therefore, we can conclude that $\lim_{|\alpha|\to\infty}\|D^\alpha f\|^{1/|\alpha|} = \sigma$.

Sufficiency. Suppose that all partial derivatives $D^\alpha f(\boldsymbol{x}) \in L^2(\mathbb{R}^n, \mathbb{R}_n)$, $\prod_{j=1}^n \omega_j^{\alpha_j}\mathcal{F}f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ for all $\alpha \in \mathbb{Z}_+^n$ and

$$\lim_{|\alpha|\to\infty}\|D^\alpha f\|^{1/|\alpha|} = d < \infty. \tag{4.7}$$

We need to prove that $\sigma := \sup\{|\omega_j|, 1 \leqslant j \leqslant n : \mathcal{F}f(\omega) \neq 0\} < \infty$. We will prove it by reduction to absurdity. Suppose $\mathcal{F}f(\omega) \neq 0$, a.e. $\omega \in \mathbb{R}^n$, thus associating with (4.6) we obtain that for arbitrary $M > 0$ there hold

$$\begin{aligned}\|D^\alpha f\|^2 &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\prod_{j=1}^n |\omega_j|^{2\alpha_j}|\mathcal{F}f(\omega)|^2 d^n\omega \\ &\geqslant \frac{1}{(2\pi)^n}\int_E\prod_{j=1}^n |\omega_j|^{2\alpha_j}|\mathcal{F}f(\omega)|^2 d^n\omega \geqslant CM^{2|\alpha|},\end{aligned}$$

where $E = \{\omega \in \mathbb{R}^n : |\omega_j| \geqslant M, 1 \leqslant j \leqslant n\}$ and C is some positive constant independent of $|\alpha|$. The above inequalities imply

$$\lim_{|\alpha|\to\infty}\|D^\alpha f\|^{1/|\alpha|} = \infty,$$

which contradicts the assumption (4.7). Thus, we have

$$\sigma := \sup\{|\omega_j|, 1 \leqslant j \leqslant n : \mathcal{F}f(\omega) \neq 0\} < \infty,$$

which means $\mathcal{F}f(\omega)$ is compactly supported in $[-\sigma, \sigma]^n$. Finally, the same technique as the part of the proof for the necessity yields that $d = \sigma$. The proof is complete. □

5 Boas theorem

Real Paley-Wiener theorems describe the compactness of the support of $\mathcal{F}f$ on $\mathbb{R}^n$. The following theorem, called Boas theorem, provides another description of high frequency signals in the CFT domain by an integral operator on $\mathbb{R}^n$. To this end, for $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$, define an integral operator $\mathcal{I}$ by

$$\mathcal{I}f(\boldsymbol{x}) := \int_{x_1}^{\infty}\int_{x_2}^{\infty}\cdots\int_{x_n}^{\infty} f(\boldsymbol{y})d^n\boldsymbol{y}$$

with $\boldsymbol{x} = \sum_{k=1}^{n} x_k e_k$, $\boldsymbol{y} = \sum_{k=1}^{n} y_k e_k$, and $d^n\boldsymbol{y} = dy_1 dy_2 \cdots dy_n$.

Theorem 5.1. *Let $f \in L^2(\mathbb{R}^n, \mathbb{R}_n)$. Then the CFT $\mathcal{F}f(\omega)$ vanishes in a neighborhood of the origin if and only if $\mathcal{I}^m f$ is well defined and belongs to $L^2(\mathbb{R}^n, \mathbb{R}_n)$ for all $m \in \mathbb{Z}_+$, and*

$$\lim_{m\to\infty} \|\mathcal{I}^m f\|^{\frac{1}{m}} = d < \infty.$$

Moreover, $d = \gamma^{-n}$ with $\gamma = \inf\{|\omega_j|, 1 \leqslant j \leqslant n : \omega \in \mathrm{supp}\mathcal{F}f(\omega)\}$.

Proof. Necessity. Suppose that the CFT $\mathcal{F}f(\omega)$ vanishes on $(-\gamma, \gamma)^n$ with $\gamma > 0$. Set

$$\mathcal{I}_N f(\boldsymbol{x}) := \int_{x_1}^{N_1}\int_{x_2}^{N_2}\cdots\int_{x_n}^{N_n} f(\boldsymbol{y})d^n\boldsymbol{y},$$

with $N = \sum_{k=1}^{n} N_k e_k$. Obviously, if $|N| \to \infty$, the limit of the operator $\mathcal{I}_N f$ is $\mathcal{I}f$. Based on the Clifford-valued inner product $(\cdot,\cdot)$ defined by (2.5), Lemma 3.3 leads to

$$\mathcal{I}_N f(\boldsymbol{x}) = (f, \chi_N) = \frac{1}{(2\pi)^n}(\mathcal{F}f, \mathcal{F}\chi_N),$$

where $\chi_N(\boldsymbol{y}) = 1$ on $[x_1, N_1] \times [x_2, N_2] \times \cdots \times [x_n, N_n]$ and $\chi_N(\boldsymbol{y}) = 0$ outside it. A simple calculation leads to the CFT of the indicator function $\chi_N(\boldsymbol{y})$ of the domain $[x_1, N_1] \times [x_2, N_2] \times \cdots \times [x_n, N_n]$,

$$\mathcal{F}\chi_N(\omega) = \int_{\mathbb{R}^n} \chi_N(\boldsymbol{y})\mathrm{e}^{-i_n\omega\cdot\boldsymbol{y}}d^n\boldsymbol{y} = \frac{i_n^n}{\prod_{k=1}^n \omega_k}\prod_{k=1}^{n}(\mathrm{e}^{-i_n N_k\omega_k} - \mathrm{e}^{-i_n x_k\omega_k}).$$

Thus, we have

$$\mathcal{I}_N f(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\mathcal{F}f(\omega)\tilde{i_n^n}}{\prod_{k=1}^n \omega_k}\prod_{k=1}^{n}(\mathrm{e}^{i_n N_k\omega_k} - \mathrm{e}^{i_n x_k\omega_k})d^n\omega.$$

Note that $\frac{\mathcal{F}f(\omega)\tilde{i_n^n}}{\prod_{k=1}^n \omega_k} \in L^1(\mathbb{R}^n, \mathbb{R}_n) \cap L^2(\mathbb{R}^n, \mathbb{R}_n)$ and vanishes on $(-\gamma, \gamma)^n$ since the CFT $\mathcal{F}f(\omega)$ vanishes on $(-\gamma, \gamma)^n$. Therefore, letting $|N| \to \infty$ and using (3.4), we have

$$\mathcal{I}f(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\mathcal{F}f(\omega)\tilde{i_n^n}}{\prod_{k=1}^n \omega_k}\mathrm{e}^{i_n\omega\cdot\boldsymbol{x}}d^n\omega.$$

Recognizing the inverse CFT given by (3.4), it follows $\mathcal{F}\{\mathcal{I}f(\boldsymbol{x})\} = \frac{\mathcal{F}f(\omega)\tilde{i_n^n}}{\prod_{k=1}^n \omega_k}$. Since $\frac{\mathcal{F}f(\omega)\tilde{i_n^n}}{\prod_{k=1}^n \omega_k} \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ and it vanishes on $(-\gamma, \gamma)^n$, $\mathcal{I}f(\boldsymbol{x})$ is also a finite-energy high frequency signal. By induction one can show that $\mathcal{I}^m f(\boldsymbol{x})$ is a finite-energy high frequency signal for any $m \in \mathbb{Z}_+$. Hence, $\mathcal{I}^m f(\boldsymbol{x}) \in L^2(\mathbb{R}^n, \mathbb{R}_n)$ for any $m \in \mathbb{Z}_+$, and moreover, it holds

$$\mathcal{I}^m f(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\mathcal{F}f(\omega)(\tilde{i_n^n})^m}{\prod_{k=1}^n \omega_k^m}\mathrm{e}^{i_n\omega\cdot\boldsymbol{x}}d^n\omega.$$

Applying Lemma 3.4 yields

$$\|\mathcal{I}^m f\|^2 = \frac{1}{(2\pi)^n}\left\|\frac{\mathcal{F}f(\omega)(\tilde{i_n^n})^m}{\prod_{k=1}^n \omega_k^m}\right\|^2,$$

which leads to

$$\|\mathcal{I}^m f\|^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{|\mathcal{F}f(\omega)|^2}{\prod_{k=1}^n \omega_k^{2m}} d^n\omega.$$

Since $\mathcal{F}f(\omega)$ vanishes on $(-\gamma,\gamma)^n$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\mathcal{F}f(\omega)|^2}{\prod_{k=1}^n \omega_k^{2m}} d^n\omega &= \int_E \frac{|\mathcal{F}f(\omega)|^2}{\prod_{k=1}^n \omega_k^{2m}} d^n\omega \leqslant \frac{1}{\gamma^{2mn}}\int_E |\mathcal{F}f(\omega)|^2 d^n\omega \\ &= \gamma^{-2mn}\int_{\mathbb{R}^n} |\mathcal{F}f(\omega)|^2 d^n\omega = \gamma^{-2mn}(2\pi)^n\|f\|^2,\end{aligned}$$

where $E = \{\omega \in \mathbb{R}^n : \gamma < |\omega_k| < \infty, 1 \leqslant k \leqslant n\}$. Consequently,

$$\limsup_{m\to\infty} \|\mathcal{I}^m f\|^{\frac{1}{m}} \leqslant \gamma^{-n}. \tag{5.1}$$

On the other hand, because γ is the infimum of $|\omega_k|, 1 \leqslant k \leqslant n$, where ω belongs to the support of $\mathcal{F}f(\omega)$, for any positive ϵ we have

$$\int_G |\mathcal{F}f(\omega)|^2 d^n\omega > 0,$$

where $G = \{\omega \in \mathbb{R}^n : \gamma < |\omega_k| < \gamma+\epsilon, 1 \leqslant k \leqslant n\}$. Hence, it follows

$$\int_{\mathbb{R}^n} \frac{|\mathcal{F}f(\omega)|^2}{\prod_{k=1}^n \omega_k^{2m}} d^n\omega \geqslant \int_G \frac{|\mathcal{F}f(\omega)|^2}{\prod_{k=1}^n \omega_k^{2m}} d^n\omega \geqslant \frac{1}{(\gamma+\epsilon)^{2mn}}\int_G |\mathcal{F}f(\omega)|^2 d^n\omega,$$

and therefore, $\liminf_{n\to\infty} \|\mathcal{I}^n f\|^{\frac{1}{2n}} \geqslant (\gamma+\epsilon)^{-1}$. Because $\epsilon > 0$ is arbitrary, we can infer that

$$\liminf_{m\to\infty} \|\mathcal{I}^m f\|^{\frac{1}{m}} \geqslant \gamma^{-n}. \tag{5.2}$$

Combining (5.1) and (5.2) leads to

$$\lim_{m\to\infty} \|\mathcal{I}^m f\|^{\frac{1}{m}} = \gamma^{-n}.$$

Sufficiency. Suppose that $\mathcal{I}^m f$ is well defined, belongs to $L^2(\mathbb{R}^n, \mathbb{R}_n)$ for all $m \in \mathbb{Z}_+$ and

$$\lim_{m\to\infty} \|\mathcal{I}^m f\|^{\frac{1}{m}} = d < \infty.$$

Note that a simple calculation leads to

$$\frac{\partial^{mn}}{\partial x_1^m \partial x_2^m \cdots \partial x_n^m}\mathcal{I}^m f(\boldsymbol{x}) = (-1)^{mn} f(\boldsymbol{x}).$$

According to Lemma 3.5, applying the CFT to the both sides of the formula above, we have

$$\prod_{k=1}^n \omega_k^m \mathcal{F}\{\mathcal{I}^m f\}(\omega) i_n^{mn} = (-1)^{mn}\mathcal{F}f(\omega).$$

Hence

$$\mathcal{F}\{\mathcal{I}^m f\}(\omega) = \frac{(-1)^{mn}\mathcal{F}f(\omega) i_n^{-mn}}{\prod_{k=1}^n \omega_k^m}.$$

Using (4.1) and Lemma 3.4 yields

$$\|\mathcal{I}^m f\|^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{|\mathcal{F}f(\omega)|^2}{\prod_{k=1}^n \omega_k^{2m}} d^n\omega.$$

Now, suppose that $\mathcal{F}f$ does not vanish in any neighborhood of 0, and for arbitrary positive number ϵ, setting $U = \{\omega \in \mathbb{R}^n : |\omega_k| < \epsilon, 1 \leqslant k \leqslant n\}$, we have

$$\int_U |\mathcal{F}f(\omega)|^2 d^n\omega > 0.$$

Therefore

$$\|\mathcal{I}^m f\|^2 \geqslant \frac{1}{(2\pi)^n}\int_U \frac{|\mathcal{F}f(\omega)|^2}{\prod_{k=1}^n \omega_k^{2m}} d^n\omega \geqslant \frac{1}{\epsilon^{2mn}}\frac{1}{(2\pi)^n}\int_U |\mathcal{F}f(\omega)|^2 d^n\omega,$$

which leads to

$$\lim_{m\to\infty} \|\mathcal{I}^m f\|^{\frac{1}{m}} \geqslant \frac{1}{\epsilon^n}.$$

Letting $\epsilon \to 0^+$, we get $\lim_{m\to\infty} \|\mathcal{I}^m f\|^{\frac{1}{m}} = \infty$, which contradicts the assumption of the convergence of the sequence $\lim_{m\to\infty} \|\mathcal{I}^m f\|^{\frac{1}{m}}$. Therefore, we conclude that there exists a positive number γ such that $\mathcal{F}f$ vanishes in $(-\gamma, \gamma)^n$. Finally, the same technique as in the proof of the necessity part concludes that d equals γ^{-n}. The proof of this theorem is completed. □

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References

1 Andersen N B. Real Paley-Wiener theorems for the Hankel transform. J Fourier Anal Appl, 2006, 12: 17–25

2 Andersen N B, De Jeu M. Real Paley-Wiener theorems and local spectral radius formulas. Trans Amer Math Soc, 2010, 362: 3613–3640

3 Bang H H. A property of infinitely differentiable functions. Proc Amer Math Soc, 1990, 108: 73–76

4 Brackx F, De Schepper N, Sommen F. The Clifford Fourier transform. J Fourier Anal Appl, 2005, 11: 669–681

5 Brackx F, De Schepper N, Sommen F. The two-dimensional Clifford Fourier transform. J Math Imag Vision, 2006, 26: 5–18

6 Chen Q H, Li L Q, Ren G B. Generalized Paley-Wiener theorems. Int J Wavelets Multiresolut Inf Process, 2012, 10: 1250020 (7 pages)

7 Chettaoui C, Othmani Y, Trimèche K. On the range of the Dunkl transform on $\mathbb{R}$. Math Sci Res J, 2004, 8: 85–103

8 Ebling J, Scheuermann G. Clifford Fourier transform on vector fields. IEEE Trans Vis Comput Graph, 2005, 11: 469–479

9 Felsberg M. Low-level image processing with the structure multivector. PhD Thesis. Christian-Albrechts-Universität, Institut für Informatik und Praktische Mathematik, Kiel, 2002

10 Hitzer E S M, Bahri M. Clifford Fourier transform on multivector fields and uncertainty principle for dimensions $n = 2\,(\mathrm{mod}\ 4)$ and $n = 3\,(\mathrm{mod}\ 4)$. Adv Appl Clifford Algebr, 2008, 18: 715–736

11 Hitzer E S M, Mawardi B. Uncertainty principle for Clifford geometric algebras $Cl_{n,0}, n = 3\,(\mathrm{mod}\ 4)$ based on Clifford Fourier transform. In: Qian T, Vai M I, Xu Y S, eds. Wavelet Analysis and Applications, Springer Book Series Applied and Numerical Harmonic Analysis. Switzerland: Birkhäuser, 2007, 47–56

12 Kou K I, Qian T. The Paley-Wiener theorem in $\mathbb{R}^n$ with the Clifford analysis setting. J Funct Anal, 2002, 189: 227–241

13 Mawardi B, Hitzer E M S. Clifford Fourier transformation and uncertainty principle for the Clifford geometric algebra $Cl_{3,0}$. Adv Appl Clifford Algebr, 2006, 16: 41–61

14 Mejjaoli H. Spectrum of functions and distributions for the Jacobi-Dunkl transform on $\mathbb{R}$. Mediterr J Math, 2013, 10: 753–778

15 Mejjaoli H, Trimèche K. Spectrum of functions for the Dunkl transform on $\mathbb{R}^d$. Fract Calc Appl Anal, 2007, 10: 19–38

16 Stein E M, Weiss G. Introduction to Fourier Analysis on Euclidean Spaces. Princeton: Princeton University Press, 1971

17 Tuan V K. Paley-Wiener-type theorems. Fract Calc Appl Anal, 1999, 2: 135–143

18 Tuan V K. Spectrum of signals. J Fourier Anal Appl, 2001, 7: 319–323

19 Tuan V K. A real-variable Paley-Wiener theorem for the Dunkl transform. In: Abstr Appl Anal. River Edge, NJ: World Sci Publ, 2004, 365–371

20 Tuan V K, Zayed A I. Paley-Wiener-Type theorems for a class of integral transforms. J Math Anal Appl, 2002, 266: 200–226