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Some infinite dimensional representations of reductive groups with Frobenius maps

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Abstract In this paper, we construct certain irreducible infinite dimensional representations of algebraic groups with Frobenius maps. In particular, a few classical results of Steinberg and Deligne & Lusztig on complex representations of finite groups of Lie type are extended to reductive algebraic groups with Frobenius maps.

Keywords infinite dimensional representation, reductive group, induced module

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0 Introduction

The construction of induced representations of Frobenius for finite groups has various generalizations for infinite groups. It seems that for infinite groups of Lie type the original form of construction of Frobenius was not used much. In this paper, we try to study abstract representations of algebraic groups by using the original construction of Frobenius directly. We are mainly interested in reductive groups with Frobenius maps. A few classical results of Steinberg and Deligne & Lusztig on complex representations of finite groups of Lie type are extended to reductive algebraic groups with Frobenius maps, see Propositions 2.3 and 2.4, Theorems 3.2 and 3.4, etc..

The paper is organized as follows. In Section 1 we give some trivial extensions for several results in representation theory of finite groups and introduce the concept of quasi-finite groups (see Subsection 1.8). A few general results on irreducibility of a representation for a quasi-finite group are established, if the representation is a “limit” of the irreducible representations (Lemmas 1.5 and 1.6). A partial generalization of Mackeys criterion on irreducibility of induced modules for quasi-finite groups is given (see Subsection 1.9).

In Section 2 we consider algebraic groups with split BN-pairs. The main objects of this section are induced representations of certain one-dimensional representations of a Borel subgroup of an algebraic group with split BN-pairs. In particular, the Steinberg module of the algebraic group is constructed (Proposition 2.3(b)). In Section 3 we consider reductive groups with Frobenius maps. The main results are Theorems 3.2 and 3.4. The first one says that the Steinberg module of a reductive group over an algebraically closed field of positive characteristic is irreducible when the base field of the Steinberg module is the field of complex numbers or the ground field of the reductive group, the second one says that the induced representations of certain one-dimensional complex representations of a Borel subgroup are irreducible.

Gelfand-Graev modules of reductive groups are defined in Section 4, which are similar to those for finite groups of Lie type. In Section 5 a few questions are raised. In Section 6 we discuss type A_1. Section 7 is devoted to discussing representations of some infinite Coxeter groups and infinite dimensional groups of Lie type.

This work was partially motivated by trying to find an algebraic counterpart for Lusztig's theory of character sheaves, the author is grateful to Professor G. Lusztig for his series of lectures on character sheaves delivered at the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, in 2012.

1 General setting

1.1. In this section, we give some trivial extensions for several results in representation theory of finite groups. There are many good references, say [10] and [4].

Let H be a subgroup of a group G and k a field. In this section all modules are assumed to be over k. For an H-module M, we can consider the naive induced module of M: $\mathrm{Ind}_H^G M = kG \otimes_{kH} M$, where kG and kH are the group algebras of G and H over the field k respectively.

As the case of finite groups, we can define the induced module in another way. Let $\mathscr{M}$ be the set of all functions $f : G \to M$ satisfying $f(gh) = h^{-1}f(g)$ for any $g \in G$ and $h \in H$. For $f \in \mathscr{M}$ and $x \in G$, set $xf(g) = f(x^{-1}g)$. This defines a kG-module structure on $\mathscr{M}$.

Let G/H be the set of left cosets of H in G. A function $f : G \to M$ is called to have finite support on G/H if all $f(gH) = 0$ except for finitely many left cosets of H in G. Let $\mathscr{M}_0$ be the subset of $\mathscr{M}$ consisting of all functions f in $\mathscr{M}$ with finite support on G/H. It is clear that $\mathscr{M}_0$ is a kG-submodule of $\mathscr{M}$. The following result is known for finite groups.

Lemma 1.2. *The kG-module $\mathscr{M}_0$ is isomorphic to the induced module $kG \otimes_{kH} M$.*

Proof. Let $\{g_i\}_{i\in I}$ be a set of representatives of left cosets of H in G. Then the map $f \to \sum_i g_i \otimes f(g_i)$ defines an isomorphism of kG-module from $\mathscr{M}_0$ to $kG \otimes_{kH} M$. □

The induced modules above are extremely important in representation theory of finite groups and Lie algebras, but seem not studied much for infinite groups of Lie type. We have some trivial properties for these induced modules, such as Frobenius's reciprocity, etc..

Lemma 1.3. (a) *Let M be an H-module and N be a G-module. Then we have*

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G M, N) \simeq \mathrm{Hom}_H(M, \mathrm{Res}_H N) \quad \textit{and} \quad \mathrm{Ind}_H^G(M \otimes \mathrm{Res}_H N) \simeq \mathrm{Ind}_H^G M \otimes N,$$

where Res_H denotes the restriction functor from G-modules to H-modules.

(b) *Let $H \subset K$ be subgroups of G and M an H-module. Then $\mathrm{Ind}_K^G(\mathrm{Ind}_H^K M)$ is isomorphic to $\mathrm{Ind}_H^G M$.*

The following result should be known.

Lemma 1.4. *Assume that G is commutative and each element of G has finite order. If k is algebraically closed, then any irreducible representation of G over k is one-dimensional.*

Proof. Let M be an irreducible kG-module. By Schur lemma, $\mathrm{End}_G M$ is a division algebra over k. Since G is commutative, for any $x \in G$, the map $\varphi_x : M \to M$, $a \to xa$ is a G-homomorphism. Since x has finite order, φ_x is algebraic over k. Now k is algebraically closed, so φ_x must be in k, i.e., x acts on M by multiplication of a scalar in k. Thus M must be one-dimensional since M is irreducible. The lemma is proved. □

Remark. The author is grateful to Binyong Sun for pointing out that the lemma above cannot be extended to arbitrary commutative groups, say, the field $\mathbb{C}$ is an irreducible representation of $\mathbb{C}^*$, but it is of infinite dimension over $\bar{\mathbb{Q}}$.

Lemma 1.5. *Let $(I,\preceq)$ be a directed set, $\{A_i,\ f_{ij}\}$ be a direct system of algebras over k (resp. groups) and $\{M_i,\ \varphi_{ij}\}$ be a direct system of vector spaces over k. Assume that M_i is A_i-module for each i and for any $i,j\in I$ with $i\preceq j$, the homomorphism $\varphi_{ij}:M_i\to M_j$ is compatible with the homomorphism $f_{ij}:A_i\to A_j$, i.e., $\varphi_{ij}(ux)=f_{ij}(u)\varphi_{ij}(x)$ for any $u\in A_i$ and $x\in M_i$. Then $M=\varinjlim M_i$ is naturally a module of $A=\varinjlim A_i$. Moreover, if M_i is irreducible A_i-module for each $i\in I$, then M is irreducible A-module.*

Proof. The proof is easy. For convenience, we give the details. Let $\bigsqcup_i M_i$ be the disjoint union of all M_i. By definition, $M=\bigsqcup_i M_i/\sim$, here for $x'\in M_i$ and $y'\in M_j$, $x'\sim y'$ if and only if there exists some $r\in I$ such that $\varphi_{ir}(x')=\varphi_{jr}(y')$.

Let $x\in M$ (resp. $u\in A$). Choose $x'\in M_i$ (resp. $u'\in A_j$) such that x' (resp. u') is in the equivalence class x (resp. u). Choose $r\in I$ such that $i\preceq r$ and $j\preceq r$. Then set ux to be the class containing $f_{jr}(u')\varphi_{ir}(x')$. One can verify that this defines an A-module structure on M.

Assume that M_i is irreducible A_i-module for each i. To prove that M is irreducible A-module it suffices to prove that $M=Ax$ for any nonzero element x in M. Assume that x and y are two nonzero elements in M. Let $x'\in M_i$ (resp. $y'\in M_j$) be an element in the equivalence class x (resp. y). Choose $r\in I$ such that $i\preceq r$ and $j\preceq r$. Then $x''=\varphi_{ir}(x')$ (resp. $y''=\varphi_{jr}(y')$) is in the class x (resp. y). Since M_r is irreducible A_r-module, there exists $u'\in A_r$ such that $x''=u'y''$. Let u be the equivalence class containing u'. Then u is element of A and $ux=y$. The lemma is proved. □

The following two simple lemmas will be used frequently.

Lemma 1.6. (a) *Let A be an algebra over k and M be an A-module. Assume that A has a sequence of subalgebras $A_1,A_2,\ldots,A_n,\ldots$ and that M has a sequence of k-subspaces $M_1,M_2,\ldots,M_n,\ldots$ such that M is the union of all M_i and for any positive integers i,j there exists a positive integer r such that M_i and M_j are contained in M_r. If M_i is an irreducible A_i-submodule of M for any i, then M is an irreducible A-module.*

(b) *Let G be a group and M be a G-module. Assume that G has a sequence of subgroups $G_1,G_2,\ldots,G_n,\ldots$ and that M has a sequence of k-subspaces $M_1,M_2,\ldots,M_n,\ldots$ such that M is the union of all M_i and for any positive integers i,j there exists an integer r such that M_i and M_j are contained in M_r. If M_i is an irreducible G_i-submodule of M for any i, then M is an irreducible G-module.*

Proof. (a) We only need to prove that $M=Ax$ for any nonzero element x in M. Assume that x and y are two nonzero elements in M. Then we can find some positive integer i such that both x and y are contained in M_i. Since M_i is irreducible A_i-submodule of M, there exists u in A_i such that $ux=y$. Therefore $M=Ax$.

(b) Applying (a) to $A=kG$ and $A_i=k[G_i]$, we see that (b) is a special case of (a).

The lemma is proved. □

Lemma 1.7. *Let H be a subgroup of G and M a kH-module. Assume that G has a sequence $G_1,G_2,\ldots,G_n,\ldots$ of subgroups such that G is the union of all G_i and for any positive integers i,j there exists an integer r such that G_i and G_j are contained in G_r. Then the following results hold*:

(a) *As G_i-modules, $kG_i\otimes_{k(G_i\cap H)}M$ is isomorphic to the G_i-submodule Y_i of $kG\otimes_{kH}M$ generated by all $x\otimes m$, where $x\in kG_i$ and $m\in M$.*

(b) *$kG\otimes_{kH}M$ is the union of all Y_i.*

(c) *$kG\otimes_{kH}M$ is irreducible if each Y_i is irreducible G_i-module.*

(d) *Let M_i be an $H_i=H\cap G_i$-submodule of M. Then we have a natural homomorphism of G_i-module $\varphi_i:kG_i\otimes_{kH_i}M_i\to kG\otimes_{kH}M$. If M is the union of all M_i and M_i is a subspace of M_j whenever G_i is a subgroup of G_j, then $kG\otimes_{kH}M$ is the union of all the images $Im\varphi_i$.*

Proof. (a), (b) and (d) are clear, (c) follows from (b) and Lemma 1.6(b). □

1.8. Let A be an algebra over a field k. Assume that A has a sequence of subalgebras $A_1,A_2,\ldots,A_n,\ldots$ such that A is the union of all A_i and for any positive integers i,j there exists an integer r such that A_i and A_j are contained in A_r. We can consider a category $\mathscr{F}$ of A-modules whose objects are those A-modules M with a finite dimensional A_i-submodule M_i for each i such that M is the union of all M_i and for any positive integers i, j, M_i and M_j are contained in M_r whenever both A_i and A_j are contained in A_r. Let (M, M_i) and (N, N_i) be two objects in $\mathscr{F}$. The morphisms from (M, M_i) to (N, N_i) are just those homomorphisms of A-module from M to N such that $f(M_i) \subset N_i$ for all i. Clearly $\mathscr{F}$ is an abelian category.

Let A be as above. We say that A is quasi-finite if all A_i are finite dimensional over k. Similarly we say that a group G is quasi-finite if G has a sequence $G_1, G_2, \ldots, G_n, \ldots$ of finite subgroups such that G is the union of all G_i and for any positive integers i, j there exists an integer r such that G_i and G_j are contained in G_r. The sequence $G_1, G_2, G_3, \ldots$ is called a quasi-finite sequence of G. A subgroup of a quasi-finite group is clearly quasi-finite. Clearly if a group G is quasi-finite then the group algebra kG is a quasi-finite algebra over k.

Example. (1) Let W_n be a Weyl group of one type A_n (resp. B_n $(n \geqslant 2)$, D_n $(n \geqslant 4)$). Then we have a canonical imbedding $W_n \to W_{n+1}$. Let $W_\infty = \bigcup_n W_n$. Then W_∞ is a quasi-finite group and is also a Coxeter group.

(2) Let $\mathbb{F}_q$ be a finite field of q elements and $\bar{\mathbb{F}}_q$ be its algebraic closure. The additive group of $\bar{\mathbb{F}}_q$ is quasi-finite and is the union of all $\mathbb{F}_{q^a}$, $a = 1, 2, \ldots$ Also the multiplication group $\bar{\mathbb{F}}_q^*$ is quasi-finite and is the union of all $\mathbb{F}_{q^a}^*$, $a = 1, 2, \ldots$

(3) Let G be an algebraic group defined over $\mathbb{F}_q$. By (2) we see that the $\bar{\mathbb{F}}_q$-points $G(\bar{\mathbb{F}}_q)$ of G is quasi-finite and is the union of all $G(q^a)$, $a \geqslant 1$, where $G(q^a)$ is the $\mathbb{F}_{q^a}$-points of G.

(4) Let G_n be $GL_n(k)$ (resp. $SL_n(k)$, $SO_{2n}(k)$, $SO_{2n+1}(k)$, $Sp_{2n}(k)$). Then G_n is naturally embedded into G_{n+1}. Let G_∞ be the union of all G_n. If k is finite then G_∞ is quasi-finite.

More generally, direct union of quasi-finite groups is also quasi-finite, in particular, G_∞ is quasi-finite if $k = \bar{\mathbb{F}}_q$. (The author is grateful to a referee for pointing out this fact.)

In the rest of this section we assume that all groups are quasi-finite unless other specifications are given. For a quasi-finite group G, we fix a quasi-finite sequence $G_1, G_2, G_3, \ldots$ For a subgroup H of G, the quasi-finite sequence of H is chosen to be $H \cap G_1, H \cap G_2, H \cap G_3, \ldots$, called the quasi-finite sequence of H induced from the given quasi-finite sequence of G.

Assume that N is a finitely generated G-module, say, generated by $x_1, \ldots, x_n$. For each positive integer i, let N_i be the G_i-submodule of M generated by $x_1, \ldots, x_n$. Then N_i is a subspace of N_j if G_i is a subgroup of G_j, and N is the union of all N_i.

We shall say that an irreducible module (or representation) N of G is *quasi-finite* (with respect to the quasi-finite sequence $G_1, G_2, G_3, \ldots$) if it has a sequence of subspaces $N_1, N_2, N_3, \ldots$ of N such that (1) each N_i is an irreducible G_i-submodule of N, (2) if G_i is a subgroup of G_j, then N_i is a subspace of N_j, and (3) N is the union of all N_i. The sequence $N_1, N_2, N_3, \ldots$ will be called a quasi-finite sequence of N. If the intersection $\bigcap_i N_i$ of all N_i is nonzero, then a nonzero element in the intersection $\bigcap_i N_i$ will be called primitive since such an element generates an irreducible G_i-submodule of N for any i. It is often that G_1 is a subgroup of all G_i, in which case N_1 is the intersection of all N_i and any nonzero element in N_1 is primitive.

Question 1. Is every irreducible G-module quasi-finite (with respect to a certain quasi-finite sequence of G)?

When the irreducible module N is finite dimensional, the answer is affirmative, since the map $kG = \bigcup_i kG_i \to \mathrm{End}_k N$ is surjective and $\mathrm{End}_k N$ is finite dimensional. A weak version of the above question is the following.

Question 2. Assume that N is an irreducible G-module. Does there exist an irreducible G_i-submodule N_i of N for each i such that N is the union of all N_i.

In the rest of this section k has characteristic 0.

1.9. For quasi-finite groups, a partial generalization of Mackey's criterion on irreducibility is stated as follows.

Let G be a quasi-finite group and H be a subgroup of G. Let M be a kH-module. Then $\mathrm{Ind}_H^G M$ is irreducible G-module if the following two conditions are satisfied:

(1) M is quasi-finitely irreducible (with respect to the quasi-finite sequence of H induced from the given quasi-finite sequence of G).

(2) Let $M_1, M_2, M_3, \ldots$ be a quasi-finite sequence of M. For any positive integer i and $s \in G_i - H \cap G_i$, the two representations $M_{i,s}$ and M_i of $H_{s,i} = sHs^{-1} \cap H \cap G_i$ have no common composition factors, where M_i is regarded as $H_{s,i}$-module by restriction and $M_{i,s}$ is the $H_{s,i}$-module with M_i as base space and the action of $g \in H_{s,i}$ on M_s is the same action on M of $s^{-1}gs$.

Proof. Assume the conditions (1) and (2) are satisfied. By Mackey's criterion, we know that $kG_i \otimes_{kH_i} M_i$ is irreducible G_i-module. By Lemma 1.7(d) and Lemma 1.6(b) we see that $\mathrm{Ind}_H^G M$ is irreducible. □

1.10. Let A be a normal subgroups of a group G. Then for any representation $\rho : A \to GL(V)$ and $s \in G$, we can define a new representation ${}^s\rho : A \to GL(V)$ by setting ${}^s\rho(g) = \rho(s^{-1}gs)$ for any $g \in A$. In this way we get an action of G on the set of representations of A.

Now assume that (1) A is commutative and each element of A has finite order, and (2) $G = H \ltimes A$ for some subgroup H of G. By Lemma 1.4, any irreducible representation of A is one-dimensional. Note that the set $X = \mathrm{Hom}(A, k^*)$ is a group. We have seen that H acts on X. Denote by X/H the set of H-orbits in X. Let $(\chi_\alpha)_{\alpha \in X/H}$ be a complete set of representatives of the H-orbits. For each $\alpha \in X/H$, let H_α be the subgroup of H consisting of $h \in H$ with ${}^h\chi_\alpha = \chi_\alpha$ and let $G_\alpha = AH_\alpha$. Define $\chi_\alpha(gh) = \chi_\alpha(g)$ for any $g \in A$ and $h \in H_\alpha$. In this way the representation χ_α is extended to a representation of G_α, denoted again by χ_α.

Let ρ be an irreducible representation of H_α. Through the homomorphism $G_\alpha \to H_\alpha$ we get an irreducible representation $\tilde{\rho}$ of G_α. The tensor product $\tilde{\rho} \otimes \chi_\alpha$ then is an irreducible representation of G_α. Let $\theta_{\alpha,\rho} = \mathrm{Ind}_{G_\alpha}^G(\tilde{\rho} \otimes \chi_\alpha)$.

Proposition 1.11. *Assume that G is quasi-finite. Keep the notation above. If ρ is quasi-finite (with respect to the quasi-finite sequence of H_α induced from the given quasi-finite sequence of G), then*

(a) *$\theta_{\alpha,\rho}$ is irreducible.*

(b) *If $\theta_{\alpha,\rho}$ is isomorphic to $\theta_{\alpha',\rho'}$, then $\alpha = \alpha'$, ρ and ρ' are isomorphic.*

Proof. The argument is similar to that for [10, Proposition 25(a), (b)]. Let $G_1, G_2, G_3, \ldots$ be the quasi-finite sequence of G. Then every G_i is finite, G is the union of all G_i and for any pair i, j there exists an integer r such that both G_i and G_j are contained in G_r. Set $H_{\alpha,i} = H_\alpha \cap G_i$. Let M be the kH_α-module affording the representation ρ of H_α and $M_1, M_2, M_3, \ldots$ be a quasi-finite sequence of M (with respect to the sequence $H_{\alpha,1}, H_{\alpha,2}, H_{\alpha,3}, \ldots$). Let V be the one-dimensional kG_α-module affording representation χ_α. Let A act on each M_i trivially. Then M_i becomes an irreducible $G_{\alpha,i} = AH_{\alpha,i}$-module. Regarding V as a $G_{\alpha,i}$-module by restriction, then $M_i \otimes V$ is an irreducible $G_{\alpha,i}$-module.

We claim that $\mathrm{Ind}_{G_{\alpha,i}}^{G_iA}(M_i \otimes V)$ is irreducible G_iA-module. For any t in $G_iA - H_{\alpha,i}A$, there exists an s in $G_i \cap H - G_i \cap H_\alpha$ such that ${}^t\chi_\alpha = {}^s\chi_\alpha$. For s in $G_i \cap H - G_i \cap H_\alpha$, we have ${}^s\chi_\alpha \neq \chi_\alpha$. This implies that there exists some $a_s \in A$ such that $\chi_\alpha(a_s) \neq \chi_\alpha(s^{-1}a_s s)$. Since G is the union of all G_j and for any pair j, j' there exists an integer r such that both $G_j, G_{j'}$ are contained in G_r, we can find an r such that a_s is in G_r for any s in $G_i \cap H - G_i \cap H_\alpha$, thanks to all G_j being finite. Thus a_s is in $A_r = A \cap G_r$ for all s in $G_i \cap H - G_i \cap H_\alpha$. For s in $G_i \cap H - G_i \cap H_\alpha$, set $K_s = H_{\alpha,i}A_r \cap sH_{\alpha,i}A_r s^{-1}$. Note that the restriction to $H_{\alpha,i}A_r$ of the $H_{\alpha,i}A$-module $M_i \otimes V$ is irreducible. Through the two injections $K_s \to H_{\alpha,i}A_r$, $x \to x$ and $x \to s^{-1}xs$ we get two K_s-module structures on the vector space $M_i \otimes V$. The restriction of the first K_s-module structure on $M_i \otimes V$ to A_r is the direct sum of some copies of $\mathrm{Res}_{A_r}\chi_\alpha$, and the restriction of the second K_s-module structure on $M_i \otimes V$ to A_r is the direct sum of some copies of $\mathrm{Res}_{A_r}{}^s\chi_\alpha$. Since a_s is in A_r and $\chi_\alpha(a) \neq \chi_\alpha(s^{-1}a_s s)$, the restrictions of the two K_s-modules to A_r are not isomorphic, hence the two K_s-modules are not isomorphic. By Mackey's criterion on irreducibility, we see that $\mathrm{Ind}_{H_{\alpha,i}A_r}^{G_iA_r}(M_i \otimes V)$ is irreducible G_iA_r-module. The natural map $\mathrm{Ind}_{H_{\alpha,i}A_r}^{G_iA_r}(M_i \otimes V) \to \mathrm{Ind}_{G_{\alpha,i}}^{G_iA}(M_i \otimes V)$ is homomorphism of G_iA_r-module, hence $\mathrm{Ind}_{G_{\alpha,i}}^{G_iA}(M_i \otimes V)$ is irreducible G_iA-module. Using Lemma 1.7(d) and Lemma 1.6(b) we see that $\theta_{\alpha,\rho}$ is irreducible.

(2) The restriction of $\theta_{\alpha,\rho}$ to A is completely reducible and involves only characters in the orbit $H\chi_\alpha$ of χ_α, which shows that $\theta_{\alpha,\rho}$ determines α. Let N be the subspace of $\mathrm{Ind}_{G_\alpha}^G(M \otimes V)$ consisting of all $x \in \mathrm{Ind}_{G_\alpha}^G(M \otimes V)$ such that $\theta_{\alpha,\rho}(a)x = \chi_\alpha(a)x$ for all $a \in A$. The subspace N is stable under H_α and

one checks easily that the representation of H_α in N is isomorphic to ρ, hence $\theta_{\alpha,\rho}$ determines ρ.

The proposition is proved. □

Remark. (1) The above proposition and argument are valid even if A is not commutative. The author is grateful to a referee for this observation.

(2) It is not clear whether any irreducible representation of G is isomorphic to a certain $\theta_{\alpha,\rho}$.

1.12. Let G be a quasi-finite group. Assume that there exists a sequence $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that all G_i's are normal subgroups of G and G_i/G_{i-1} are abelian. Examples of such groups include Borel subgroups of a reductive group over $\bar{\mathbb{F}}_q$.

Question. Is each irreducible representation of G isomorphic to the induced representation of a one-dimensional representation of a subgroup of G?

2 Algebraic groups with split BN-pairs

2.1. In this section we assume that G is an algebraic group with a split BN-pair. By definition (see, for example, [3, p. 50]), G has closed subgroups B and N with the following properties:

(i) The set $B \cup N$ generates G, while $T = B \cap N$ is a normal subgroup of N and all elements of T are semisimple.

(ii) The group $W = N/T$ is generated by a set S of elements s_i, $i \in I$, of order 2.

(iii) If $n_i \in N$ maps to $s_i \in S$ under the natural homomorphism $N \to W$, then $n_iBn_i \neq B$.

(iv) For each $n \in N$ and each n_i we have $n_iBn \subseteq Bn_inB \cup BnB$.

(v) B has a closed normal unipotent subgroup U such that $B = T \ltimes U$.

(vi) $\bigcap_{n\in N} nBn^{-1} = T$.

It is known that W is a Weyl group. Let R be the root system of W and α_i, $i \in I$ be simple roots. For any $w \in W$, U has two subgroups U_w and U'_w such that $U = U'_wU_w$ and $wU'_xw^{-1} \subseteq U$. If $w = s_i$ for some i, we simply write U_i and U'_i for U_w and U'_w, respectively. For each $w \in W$ we choose an element $n_w \in N$ such that its natural image in W is w and let n_i stand for n_{s_i}. The Bruhat decomposition says that G is a disjoint union of the double cosets Bn_wB, $w \in W$. Note that $G_i = B \cup Bn_iB$ is a subgroup G.

Any representation of T can be regarded naturally as a representation of B through the homomorphism $B \to T$. Let k be a field. In this section all representations are assumed over k. Let θ be a one-dimensional representation of T, we use the same letter when it is regraded as a representation of B. Let k_θ denote the corresponding B-module. We are interested in the induced module $M(\theta) = kG \otimes_{kB} k_\theta$.

Let $P \supseteq B$ be a parabolic subgroup of G and L be a Levi subgroup of P containing T. Let U_P be the unipotent radical of P. Then $P = L \ltimes U_P$. Moreover, $B_L = B \cap L$ is a Borel subgroup and $(B_L, N \cap L)$ forms a BN-pair of L. By abusing notation, we also use k_θ for its restriction to B_L. Set $M_L(\theta) = kL \otimes_{B_L} k_\theta$. Let U_P act on $M_L(\theta)$ trivially. Then $M_L(\theta)$ becomes a P-module. The following result is easy to check.

Lemma 2.2. *$M(\theta)$ is isomorphic to $kG \otimes_{kP} M_L(\theta)$.*

If θ is trivial we shall use $M(tr)$ for $M(\theta)$ and k_{tr} for k_θ, respectively. Let 1_{tr} be a nonzero element in k_{tr}. For x in kG we simply denote the element $x \otimes 1_{tr}$ in $M(tr)$ by $x1_{tr}$. For any element $t \in T$ and $n \in N$ we have $nt1_{tr} = n1_{tr}$, so for $w = nT \in W$, the notation $w1_{tr} = n1_{tr}$ is well defined.

For any subset J of S, we shall denote by W_J the subgroup of W generated by J and let w_J be the longest element of W_J. Set $\eta_J = \sum_{w\in W_J}(-1)^{l(w)}w1_{tr}$, where $l(w)$ is the length of w. The following result is a natural extension of [13, Theorem 1, p. 348] and part (a) seems new even for finite groups.

Proposition 2.3. *Keep the notation above. Let J be a subset of S. Then*

(a) *The space $kUW\eta_J$ is a submodule of $M(tr)$ and is denoted by $M(tr)_J$.*

(b) *In particular, $kU\eta_S = kU\sum_{w\in W}(-1)^{l(w)}w1_{tr}$ is a submodule of $M(tr)$. This submodule will be called a Steinberg module of G and is denoted by* St.

Proof. The argument for [13, Theorem 1] works well here. Clearly $kUW\eta_J$ is stable under the action of B. Since G is generated by B and N, it remains to check that $kUW\eta_J$ is stable under the action of N.

But N is generated by all n_i and T, so we only need to check that $n_ikUW\eta_J \subseteq kUW\eta_J$. We need to show that $n_iuh\eta_J \in kUW\eta_J$ for any $u \in U$ and $h \in W$. Let $u = u'_iu_i$, where $u_i \in U_i$ and $u'_i \in U'_i$. Then $n_iuh\eta_J = n_iu'_in_i^{-1}n_iu_ih\eta_J$. Since $n_iu'_in_i^{-1} \in U$, it suffices to check that $n_iu_ih\eta_J \in kUW\eta_J$. When $u_i = 1$, this is clear. Now assume that $u_i \neq 1$. Since $s\eta_J = -\eta_J$ for any $s \in J$, it is no harm to assume that $l(hw_J) = l(h) + l(w_J)$.

If $hw_J \leqslant s_ihw_J$, then $hw \leqslant s_ihw$ for all $w \in W_J$. In this case, we have $n_iu_ih\eta_J = n_ih\eta_J \in kUW\eta_J$.

If $s_ih \leqslant h$, then $n_iu_ih\eta_J = n_iu_in_i(s_ih)\eta_J$. Note that $n_i^2 \in T$. Since $G_i = B \cup Bn_iB$ is a subgroup G, if $u_i \neq 1$, we have $n_iu_in_i = n_iu_in_i^{-1}n_i^2 = xn_ity$ for some $x, y \in U_i$ and $t \in T$. Thus $n_iu_ih\eta_J = xn_iy(s_ih)\eta_J = xn_i(s_ih)\eta_J = xh\eta_J$ since $(s_ih)w_J \leqslant hw_J$.

Now assume that $h \leqslant s_ih$ but $s_ihw_J \leqslant hw_J$. Then we must have $s_ih = hs_j$ for some $s_j \in J$. If $w \in W_J$ and $w^{-1}(\alpha_j)$ is a positive root, then we have $hw \leqslant s_ihw$, hence

(i) $n_iu_ihw1_{tr} = n_ihw1_{tr} = s_ihw1_{tr} = hs_jw1_{tr}$,

(ii) $hw1_{tr} = xhw1_{tr}$,

(iii) $n_iu_ihs_jw1_{tr} = n_iu_in_i^{-1}hw1_{tr} = xs_ihw1_{tr} = xhs_jw1_{tr}$.

Multiplying (i), (ii) and (iii) by $(-1)^{l(w)}$, $(-1)^{l(w)}$ and $(-1)^{l(s_jw)}$, respectively, adding them, then summing on all $w \in W_J$ satisfying $l(s_jw) = l(w)+1$, we get $(1-x+n_iu_i)h\eta_J = 0$. Thus $n_iuh\eta_J = n_iu'_iu_ih\eta_J = n_iu'_in_i^{-1}n_iu_ih\eta_J = n_iu'_in_i^{-1}(x-1)\eta_J \in kUW\eta_J$. The proposition is proved. □

An analogue of [5, Proposition 7.3] is the following result.

Proposition 2.4. *Let θ be a one-dimensional representation of T. Then $M(\theta) \otimes \mathrm{St}$ is isomorphic to $\mathrm{Ind}_T^G k_\theta$.*

Proof. Let 1_θ be a nonzero element in k_θ and $\eta = \sum_{w\in W}(-1)^{l(w)}w1_{tr}$. Then it is easy to check that the map $g1_\theta \to g(1_\theta \otimes \eta)$ defines an isomorphism of G-module between $\mathrm{Ind}_T^G k_\theta$ and $M(\theta) \otimes \mathrm{St}$. The proposition is proved. □

Lemma 2.5. *For each $n \in N = N_G(T)$, $kn1_\theta$ is T-stable. If each U_i is infinite, then any T-stable one-dimensional subspace of $M(\theta)$ is contained in $\sum_{n\in N} kn1_\theta$, which is of dimension $|W|$.*

Proof. It is clear. □

2.6. Let J be a subset of S and let $M(tr)'_J$ be the sum of all $M(tr)_K$ (see Proposition 2.3(a) for definition) with $J \subsetneq K$. Then $M(tr)'_J$ is a proper submodule of $M(tr)_J$. Let $E_J = M(tr)_J/M(tr)'_J$.

Proposition 2.7. *Assume that each U_i is infinite. If J and K are different subsets of S, then E_J and E_K are not isomorphic.*

Proof. For any $w \in W$, let $c_w = \sum_{y\leqslant w}(-1)^{l(y)}P_{y,w}(1)y1_{tr}$, where $P_{y,w}$ are Kazhdan-Lusztig polynomials. Note that $c_w = \eta_J$ if $w = w_J$ for some subset J of S.

We claim that $M(tr)_J$ is the sum of all kUc_w, $w \in W$ with $l(ww_J) = l(w) - l(w_J)$. Since $M(tr)_J = kUW\eta_J = kUWc_{w_J}$, we only need to show that kWc_{w_J} is spanned by all c_w, $w \in W$ with $l(ww_J) = l(w) - l(w_J)$. But this follows from and [6, (2.3.a), (2.3.c) and Proposition 2.4].

Let $\bar{c}_w$ be the image of c_w in E_J. Let A_J be the subset of W consisting of all $w \in W$ such that $w \leqslant ws$ for all $s \in S - J$ and $ws \leqslant w$ for all $s \in J$. Then $\bar{c}_w$ is nonzero if and only if $w \in A_J$ and E_J is the sum of all $kU\bar{c}_w$. Since U_i is infinite for each i, any T-stable line in E_J is contained in $\sum_{w\in A_J} k\bar{c}_w = E_J^T$. If there exists a G-isomorphism $\phi : E_J \to E_K$, then we must have $\phi(E_J^T) = E_K^T$. Thus $\phi(\bar{c}_{w_J}) = \sum_{w\in A_K} a_w\bar{c}_w$, $a_w \in k$. But $\bar{c}_{w_J} \neq 0$ is uniquely determined by the following two conditions: (1) $n_i\bar{c}_{w_J} = -\bar{c}_{w_J}$ if and only if $s_i \in J$, and (2) $U_i\bar{c}_{w_J} = \bar{c}_{w_J}$ if and only if $s_i \notin J$. Therefore, $J \neq K$ implies that any nonzero element in E_K^T does not satisfy the conditions for $\bar{c}_{w_J}$, hence ϕ does not exist. The proposition is proved. □

2.8. Let P be a parabolic subgroup of G with unipotent radical U_P. Assume that L is a Levi subgroup of P. Any kL-module E is naturally a kP-module through the homomorphism $P \to L$. Then we can define the induced module $\mathrm{Ind}_P^G E = kG \otimes_{kP} E$. If P contains B and E is one-dimensional P-module, then $\mathrm{Ind}_P^G E = kG \otimes_{kP} E$ is a quotient module of some $M(\theta)$.

Let P_J $(J \subset S)$ be a standard parabolic subgroup of G. Let P_J act on k trivially. Define $1_{P_J}^G = kG \otimes_{P_J} k$.

Clearly $1_{P_J}^G$ is a quotient module of $M(tr)$. Assume that each U_i is infinite. By the discussion above we see that $\mathrm{Hom}_G(1_{P_J}^G, 1_{P_K}^G)$ is nonzero if and only if J is a subset of K.

3 Reductive groups with Frobenius maps

3.1. In this section we assume that G is a connected reductive group defined over a finite field $\mathbb{F}_q$ of q elements, where q is a power of a prime p. Lang's theorem implies that G has a Borel subgroup B defined over $\mathbb{F}_q$ and B contains a maximal torus T defined over $\mathbb{F}_q$. For any power q^a of q, we denote by G_{q^a} the $\mathbb{F}_{q^a}$-points of G and shall identify G with its $\bar{\mathbb{F}}_q$-points, where $\bar{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$. Then we have $G = \bigcup_{a=1}^{\infty} G_{q^a}$. Similarly we define B_{q^a} and T_{q^a}.

Let N be the normalizer of T in G. Then B and N form a BN-pair of G. Let k be a field. For any one-dimensional representation θ of T over k. As in Section 2 we can define the kG-module $M(\theta) = kG \otimes_{kB} k_\theta$. When θ is trivial representation of T over k, as in Section 2 we write $M(tr)$ for $M(\theta)$ and let 1_{tr} be a nonzero element in k_θ. We shall also write $x1_{tr}$ instead of $x \otimes 1_{tr}$ for $x \in kG$. Let U be the unipotent radical of B.

Recall that for $w \in W = N/T$, the element $w1_{tr}$ is defined to be $n_w 1_{tr}$, where n_w is a representative in N of w (cf. the paragraph below Lemma 2.2).

Theorem 3.2. (a) *Assume that $k = \mathbb{C}$ is the field of complex numbers. Then $kU \sum_{w\in W} (-1)^{l(w)} w1_{tr}$ is an irreducible G-module.*

(b) *Assume that $k = \bar{\mathbb{F}}_q$. Then $kU \sum_{w\in W} (-1)^{l(w)} w1_{tr}$ is an irreducible G-module.*

Proof. (a) Let U_{q^a} be the $\mathbb{F}_{q^a}$-points of U. Then $U = \bigcup_{a=1}^{\infty} U_{q^a}$. Let $\eta = \sum_{w\in W} (-1)^{l(w)} w1_{tr}$. Then $\mathbb{C}[U_{q^a}]\eta$ is isomorphic to the Steinberg module of G_{q^a}, so it is an irreducible G_{q^a}-module. We have $\mathbb{C}U\eta = \bigcup_{a=1}^{\infty} \mathbb{C}[U_{q^a}]\eta$ and $\mathbb{C}[U_{q^a}]\eta \subset \mathbb{C}[U_{q^{ab}}]\eta$ for any integer $b \geqslant 1$. By Lemma 1.6(b), $\mathbb{C}U\eta$ is an irreducible G-module.

The argument for (b) is similar. The theorem is proved. □

3.3. According to [13, Theorems 2 and 3], the G_{q^a}-module $k[U_{q^a}] \sum_{w\in W} (-1)^{l(w)} w1_{tr}$ is irreducible if and only if $\mathrm{char} k$ does not divide $\sum_{w\in W} q^{al(w)}$. Therefore $kU \sum_{w\in W} (-1)^{l(w)} w1_{tr}$ is irreducible G-module if $\mathrm{char} k$ does not divide $\sum_{w\in W} q^{al(w)}$ for all positive integers a. Unfortunately, it is by no means easy to determine the prime factors of $\sum_{w\in W} q^{al(w)}$ even for type A_1 (in this case W has only two elements). So it seems we need to find other ways to see whether $kU \sum_{w\in W} (-1)^{l(w)} w1_{tr}$ is irreducible G-module if $\mathrm{char} k$ is different from 0 and from $\mathrm{char}\bar{\mathbb{F}}_q = p$.

Let θ be a group homomorphism from T to k^*. For any $w \in W$, define ${}^w\theta : T \to k,\ t \to \theta(w^{-1}tw)$.

Theorem 3.4. *Assume that $k = \mathbb{C}$. Then $M(\theta)$ has at most $|W_\theta|$ composition factors, where $W_\theta = \{w \in W \mid {}^w\theta = \theta\}$. In particular, if ${}^w\theta \neq \theta$ for any $1 \neq w \in W$ (i.e., there exists $t \in T$ such that $\theta(w^{-1}tw) \neq \theta(t)$), then $M(\theta)$ is an irreducible G-module.*

Proof. We can find an integer a such that for any $b \geqslant a$ we have $W_\theta = \{w \in W \mid {}^w\theta_{T_{q^b}} = \theta_{T_{q^b}}\}$, where $\theta_{T_{q^b}}$ denotes the restriction of θ to T_{q^b}. Assume that $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_h = M(\theta)$ is a filtration of submodules of $M(\theta)$. Then there exist $x_i \in M_i - M_{i-1}$ for $i = 1, 2, \ldots, h$. Clearly there exists $c \geqslant a$ such that all x_i are in $\mathbb{C}G_{q^c}(1 \otimes 1_\theta)$, where 1_θ is nonzero element in k_θ. But it is known that $\mathbb{C}G_{q^c}(1 \otimes 1_\theta)$ has at most W_θ composition factors. The theorem is proved. □

Proposition 3.5. *Let $\theta : T \to \mathbb{C}^*$ be a group homomorphism. Assume that W_θ is a parabolic subgroup W_J of W. Then the element $\sum_{w\in W_J} (-1)^{l(w)} w1_\theta$ generates an irreducible submodule of $M(\theta)$ and the elements $(s - e)1_\theta$, $s \in W_J$ being simple reflections, generate a maximal submodule of $M(\theta)$, where e is the neutral element of W.*

Proof. It is known that the kG_{q^a}-submodule of $M(\theta)$ generated by $\sum_{w\in W_J} (-1)^{l(w)} w1_\theta$ is irreducible for all positive integers a and the kG_{q^a}-submodule of $kG1_\theta$ generated by all $(s - e)1_\theta$, $s \in W_J$ being simple reflections, is a maximal submodule of $kG_{q^a}1_\theta$. The proposition then follows Lemma 1.6(b). □

3.6. Assume that $k = \mathbb{C}$. It is an interesting question to determine the composition factors of $M(\theta)$. Assume that P is a parabolic subgroup containing B. Let P act trivially on $\mathbb{C}$. Then $\mathrm{Ind}_P^G\mathbb{C} = kG \otimes_{kP} \mathbb{C}$

is a quotient module of $M(tr)$, so it has finitely many composition factors. If P is a maximal parabolic subgroup, then $\mathrm{Ind}_P^G\mathbb{C}$ has much less composition factors than $M(tr)$.

Let G be a connected reductive group over $\mathbb{F}_q$ such that its derived group is of type A_n. Let P be a maximal parabolic subgroup of G containing the F-stable Borel subgroup B and assume that the derived subgroup of a Levi subgroup of P has type A_{n-1}. Using Lemma 1.6(b) and representation theory for G_{q^a}, it is easy to see that $\mathrm{Ind}_P^G\mathbb{C}$ has a unique irreducible quotient module which is trivial and a unique irreducible submodule.

It is known that there is a bijection between the composition factors of G_{q^a}-submodule $\mathbb{C}G_{q^a}1_{tr}$ of $M(tr)$ and the composition factors of the regular module $\mathbb{C}W$ of W, which preserves multiplicities. But this result cannot be extended to $M(tr)$ since by the proof for Proposition 2.7 it is easy to see that $M(tr)$ has at least $2^{|S|}$ composition factors which are pairwise non-isomorphic.

3.7. Assume that $k=\bar{\mathbb{F}}_q$. Then for each dominant weight $\lambda: T\to k^*$, we have Weyl module $V(\lambda)$ and its irreducible quotient $L(\lambda)$. Clearly $V(\lambda)$ is a quotient module of $M(\lambda)$. Also it is clear that the tensor product $M(\theta)\otimes V(\lambda)$ has a filtration of submodules such that the quotient modules of the filtration are some $M(\theta+\mu)$, where μ are weights of $V(\lambda)$. It is not clear whether some $M(\theta)$ have infinite composition factors. It might be interesting to study $\mathrm{St}\otimes L(\lambda)$.

It is easy to see that the trivial module of $\bar{\mathbb{F}}_qU$ is the unique irreducible $\bar{\mathbb{F}}_qU$-module. A question comes naturally: is every irreducible $\bar{\mathbb{F}}_qB$-module one-dimensional?

If $\mathrm{char}k$ is different from 0 and from $\mathrm{char}\bar{\mathbb{F}}_q=p$, the structure of the modules $M(\theta)$ are more complicated.

4 Gelfand-Graev modules

4.1. Keep the notation in Subsection 3.1. Thus G is a connected reductive group defined over $\mathbb{F}_q$, B a maximal Borel subgroup of G defined over $\mathbb{F}_q$ and T a maximal torus in B defined over $\mathbb{F}_q$. Let U be the unipotent radical of B. The group G and its subgroups are identified with their $\bar{\mathbb{F}}_q$-points, so $G=G(\bar{\mathbb{F}}_q),\ B=B(\bar{\mathbb{F}}_q),\ T=T(\bar{\mathbb{F}}_q)$, etc..

Let R be the root system of G and $\Delta=\{\alpha_1,\dots,\alpha_l\}$ be the set of simple roots corresponding to B. Denote by R^+ the set of positive roots. For each positive root $\alpha\in R$, let U_α be the corresponding root subgroup in G. We choose an isomorphism $\varepsilon_\alpha:\bar{\mathbb{F}}_q\to U_\alpha$ so that $t\varepsilon_\alpha(a)t^{-1}=\varepsilon_\alpha(\alpha(t)a)$ for any $a\in\bar{\mathbb{F}}_q$ and $t\in T$. It is known that the subgroup U' of B generated by all U_α, $\alpha\in R^+-\Delta$, is a normal subgroup of B and the quotient group U/U' is isomorphic to the direct product $U_{\alpha_1}\times U_{\alpha_2}\times\cdots\times U_{\alpha_l}$ and B/U' is isomorphic to the semidirect product $T\ltimes U/U'$.

Each irreducible representation of B/U' gives rise naturally an irreducible representation of B. In general it is hard to get a classification of irreducible representations for groups B and U.

4.2. Clearly, there is a bijection between one-dimensional representations of U with U' in the kernel and the sets (σ_j), where σ_j is a one-dimensional representation of U_{α_j}. A one-dimensional representation of U with U' in its kernel is called non-degenerate if all σ_j are non trivial. The group T acts naturally on the set of irreducible representations of U/U'.

It is known that all non-degenerate one-dimensional complex representations of U_q form a T_q-orbit if the center of G is connected. But the set Φ of non-degenerate one-dimensional complex representations of U is uncountable. This implies that the T-orbits in Φ is uncountable. For a non-degenerate one-dimensional complex representation σ of U, we may consider the induced representation $\mathrm{Ind}_U^G\sigma$ and call it a Gelfand-Graev representation of G. It seems not easy to decompose these Gelfand-Graev representations (cf. [5, Section 10]).

5 Some questions

There are some natural questions.

1. Develop a theory of kG-modules for infinite quasi-finite groups. A particular question is to classify irreducible kG-modules for some interesting quasi-finite groups, say, reductive groups with Frobenius

maps and their Borel subgroups, the infinite Coxeter group W_∞ (see 1.8 Example (1)) and the group G_∞ (see 1.8 Example (4)), etc..

According to [2, Theorem 10.3 and Corollary 10.4], we know that except the trivial representation, all other irreducible representations of kG are infinite dimensional if G is a semisimple algebraic group over $\bar{\mathbb{F}}_q$ and k is infinite with characteristic different from char$\bar{\mathbb{F}}_q$.

2. Let G be a connected reductive group over $\mathbb{F}_q$. Then G has a Frobenius map $F : G \to G$. So for any representation ρ of G we can define a new representation ${}^F\rho$ by setting ${}^F\rho(g) = \rho(F(g))$. We say that ρ is F-stable if ρ and ${}^F\rho$ are isomorphic.

Question. Are there any good relations between the set of isomorphism classes of irreducible complex representations of G which are F-stable and the set of isomorphism classes of irreducible complex representations of G^F?

Replacing G by $GL_n(\mathbb{F}_{q^a})$ or $SL_n(\mathbb{F}_{q^a})$, the above question is answered positively by the theory of Shintani decent (see, for example, [1, 11, 12]). For character sheaves, there is a similar result (see [9]).

6 Type A_1

In this section G will denote $GL_2(\bar{\mathbb{F}}_q)$ or $SL_2(\bar{\mathbb{F}}_q)$. Let T be the torus of G consisting of diagonal matrices and B be the Borel subgroup consisting of upper triangle matrices. Let U be the unipotent radical of B. In this section we consider complex representations of these groups.

6.1. We first consider representations of B over $\mathbb{C}$. We have $B = T \ltimes U$. Let $X = \mathrm{Hom}(U, \mathbb{C}^*)$. For $t \in T$, $\chi \in X$, define ${}^t\chi : U \to \mathbb{C}^*$, $u \to \chi(t^{-1}ut)$. Then we get an action of T on X. Note that U is isomorphic to the additive group $\bar{\mathbb{F}}_q$, so as abelian group, U is a direct sum of countable copies of $\mathbb{F}_p$, where p is the characteristic of $\bar{\mathbb{F}}_q$. Therefore, the set X of homomorphism $U \to \mathbb{C}^*$ is uncountable. This implies the set of T-orbits in X is uncountable.

Denote by X/T the set of T-orbits in X and let $(\chi_\alpha)_{\alpha\in X/T}$ be a complete set of representatives of the T-orbits. For each $\alpha \in X/T$, let T_α be the subgroup of T consisting of $t \in T$ with ${}^t\chi_\alpha = \chi_\alpha$ and let $B_\alpha = T_\alpha U$. Define $\chi_\alpha(tu) = \chi_\alpha(t)$ for any $t \in T$ and $u \in U$. In this way the representation χ_α is extended to a representation of B_α, denoted again by χ_α. Note that T_α is the center of B if χ_α is non-trivial, is the whole T if χ_α is trivial.

Let ρ be an irreducible complex representation of T_α, which is one-dimensional since T_α is commutative. Through the homomorphism $B_\alpha \to T_\alpha$ we get an irreducible representation $\tilde{\rho}$ of B_α. The tensor product $\tilde{\rho} \otimes \chi_\alpha$ then is an irreducible representation of G_α. Let $\theta_{\alpha,\rho}$ be the corresponding induced representation of B. According to Proposition 1.11 we have the following result.

Lemma 6.2. *The complex representation $\theta_{\alpha,\rho}$ of B is irreducible. Moreover, $\theta_{\alpha,\rho}$ is isomorphic to $\theta_{\alpha',\rho'}$ if and only if $\alpha = \alpha'$ and $\rho = \rho'$.*

We can further induce $\theta_{\alpha,\rho}$ to G. By the lemma above, if χ_α is trivial, then $B_\alpha = B$ and $\theta_{\alpha,\rho}$ is just $\tilde{\rho}$. Since the commutator group $[B, B]$ of B is U, any homomorphism $\theta : B \to \mathbb{C}^*$ has the form $\tilde{\rho}$. According to Theorems 3.2 and 3.4, we have the following result.

Proposition 6.3. *Let $\theta : B \to \mathbb{C}^*$ be a group homomorphism. Then*

(a) *$M(\theta) = \mathbb{C}G \otimes_{\mathbb{C}B} \mathbb{C}_\theta$ is irreducible G-module if θ is not trivial.*

(b) *If θ is trivial, then $M(\theta)$ has a unique nonzero proper submodule and unique quotient module. The nonzero proper submodule is the Steinberg module. The quotient module is the trivial module of G.*

6.4. Let B_q, T_q and U_q be the F_q-points of B, T and U, respectively. Keep the notation in Subsection 6.1. Assume that χ_α is nontrivial. If the restriction of χ_α to U_q is not trivial, we can consider the induced representation $\theta_{\alpha,\rho,q}$ of B_q from the restriction of $\tilde{\rho} \otimes \chi_\alpha$ to $G_\alpha \cap B_q$, which is irreducible. It is known that the action of B_q on $\theta_{\alpha,\rho,q}$ can be extended to actions of G_q and in this way one can get all cuspidal representations of G_q. But the author has not been able to see how to extend the B action on $\theta_{\alpha,\rho}$ to an action of G.

7 Miscellany

In this section we give some discussion to representations of the groups listed in 1.8 Example (1) and Example (4).

7.1. Let $W = W_\infty$ be the group defined in 1.8 Example (1). Since W is a Coxeter group, we can use Kazhdan-Lusztig cells to construct representations of W and its Hecke algebras. Let $s_1, \ldots, s_n$ be the simple reflections of W_n and let S be the set of all simple reflections.

(1) Assume that W is of type A. Let C_w, $w \in W$, be the Kazhdan-Lusztig basis of $\mathbb{C}W$ (cf. [6, Theorem 1.1]). For each left cell σ of W, let I_σ be the subspace of $\mathbb{C}W$ spanned by all C_w, $w \in \sigma$. Denote by $I_{<\sigma}$ the subspace of $\mathbb{C}W$ spanned by all C_w, $w \leqslant_L u$ for some $u \in \sigma$ but $w \notin \sigma$. Then $\mathbb{C}W$ is the direct sum of all I_σ, and both $I_\sigma + I_{<\sigma}$, $I_{<\sigma}$ are left ideals of $\mathbb{C}W$. According to [6, Theorem 1.4] and Lemma 1.6(b), $L_\sigma = (I_\sigma + I_{<\sigma})/I_{<\sigma}$ is an irreducible $\mathbb{C}W$-module. When two left cells σ and τ are in the same two-sided cell, the right star actions leads to an isomorphism between L_σ and L_τ. Moreover, L_σ and L_τ are isomorphic $\mathbb{C}W$-modules if and only if σ and τ are in the same two-sided cell of W. Similar results hold for Hecke algebra of W over $\mathbb{C}(q^{\frac{1}{2}})$ with parameter q (here q is an indeterminate).

According to the proof of [6, Theorem 1.4], any two-sided cell of W contains some w_P, where P is a finite subset of S and w_P is the longest element of the subgroup of W generated by P. Let σ_P be the left cell of W containing w_P. For subsets of P and Q of S, w_P and w_Q are in the same two-sided cell of W if and only if w_P and w_Q are in the same two-sided cell of W_n whenever both w_P and w_Q are contained in W_n.

Unlike the group W_n, some irreducible $\mathbb{C}W$-modules are not isomorphic to any L_σ, for example, the sign representation of W is not isomorphic to any L_σ. It is also less easy to discuss irreducible $\bar{\mathbb{F}}_qW$-modules.

(2) Assume that W is of type B. Let H be the Hecke algebra of W defined over $\mathscr{A} = \mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ (q an indeterminate) with $\mathscr{A}$-basis T_w, $w \in W$, and multiplication relations $(T_{s_i} - q_i)(T_{s_i} + 1) = 0$ and $T_wT_u = T_{wu}$ if $l(wu) = l(w) + l(u)$, where $q_1 = q$, $q_i = q^2$ for all $i \geqslant 2$. Let C_w, $w \in W$, be the Kazhdan-Lusztig basis of H defined in [7, Proposition 2]. The corresponding cells are called generalized cells (φ-cells in [7]). Regarding $\mathbb{C}$ as an $\mathscr{A}$-module by specifying q to 1, then we have $\mathbb{C}W = H \otimes_{\mathscr{A}} \mathbb{C}$. By abuse notation we use also notation C_w for its image in $\mathbb{C}W$. For each generalized left cell σ of W, let I_σ be the subspace of $\mathbb{C}W$ spanned by all C_w, $w \in \sigma$. Denote by $I_{<\sigma}$ the subspace of $\mathbb{C}W$ spanned by all C_w, $w \leqslant_L u$ for some $u \in \sigma$ but $w \notin \sigma$. Then $\mathbb{C}W$ is the direct sum of all I_σ, and both $I_\sigma + I_{<\sigma}$, $I_{<\sigma}$ are left ideals of $\mathbb{C}W$. According to [7, Theorem 11] and Lemma 1.6(b), $L_\sigma = (I_\sigma + I_{<\sigma})/I_{<\sigma}$ is an irreducible $\mathbb{C}W$-module. However, it seems not clear whether L_σ and L_τ are isomorphic $\mathbb{C}W$-modules when the generalized left cells σ and τ are in the same generalized two-sided cell of W. Similar results hold for the Hecke algebra $\bar{H} = H \otimes_{\mathscr{A}} \mathbb{C}(q^{\frac{1}{2}})$. According to [7, Section 10], if one chooses $q_1 = q^3$, $q_i = q^2$ for all $i \geqslant 2$, the above results remain valid.

(3) Assume that W is of type D. Let C_w, $w \in W$, be the Kazhdan-Lusztig basis of $\mathbb{C}W$ (cf. [6, Theorem 1.1]). For each left cell σ of W, as in (1) we can define the subspaces I_σ and $I_{<\sigma}$ of $\mathbb{C}W$ and the $\mathbb{C}W$-module L_σ. Unlike the case of types A or B, L_σ may not be irreducible. In [8, Chapter 12], Lusztig has proved that the $\mathbb{C}W_n$-module afforded by a left cell of W_n is multiplicity free and the number of irreducible components in the $\mathbb{C}W_n$-module is a power of 2. So it is likely that the $\mathbb{C}W$-module L_σ is semisimple (i.e., a direct sum of some irreducible submodules).

7.2. In the rest of this section G_n and G_∞ are as in 1.8 Example (4). Let T be the subgroup of G consisting of diagonal matrices in G and N be the normalizer of T in G. We can choose naturally a subgroup B of G so that B and N form a BN-pair for G. For example, B can be chosen to be the subgroup of G consisting of upper triangular matrices in G if $G = GL_\infty$ or SL_∞. Let U be the kernel of the natural homomorphism $B \to T$. It is easy to see that $W = N/T$ is just a group in 1.8 Example (1). Let $S = \{s_1, s_2, s_3, \ldots\}$ be the set of simple reflections of W. For each s_i we choose a representative $n_i \in N$ of s_i. For each i, there exist subgroups U_i and U_i' of U such that $U = U_i'U_i$ and $n_iU_i'n_i^{-1} \in U$. Note that if $u_i \in U_i$, we have $n_iu_in_i^{-1} = xn_ity$ for some $x, y \in U_i$ and $t \in T$.

Set $T_n = T \cap G_n$ and $B_n = B \cap G$. Let $N_n = N \cap G_n$ and $W_n = N_n/T_n$. Note that W_n can be regarded as a subgroup of W in a natural way.

Assume that $k = \bar{k}$, by Lemma 1.6(b) the natural representation V of G is irreducible. We may consider to decompose the tensor product of m copies of V. Many classical results for G_n can be extended to G_∞.

Let $\lambda : T \to k^*$ be a character of T. Assume that the restriction λ_n of λ to T_n is a dominant weight of G_n for each n. Then we have an irreducible rational G_n-module V_n with highest weight λ_n. Clearly, we have a natural embedding $V_n \hookrightarrow V_{n+1}$ for each n. Moreover, the embedding is a G_n-homomorphism. Let $V(\lambda)$ be the union of all V_n. Then $V(\lambda)$ is naturally a G-module. By Lemma 1.6(b) it is an irreducible G-module.

It is known that a G_n-module M_n is called rational if for any $x \in M$, the G-submodule of G generated by x is a finite dimensional rational G-module. We call a G-module M rational if the restriction of M to G_n is rational. Clearly, $V(\lambda)$ a rational G-module in this sense.

7.3. Keep the notation in Subsection 7.2. For any homomorphism $\theta : T \to K$, where K is a field, let K_θ be the corresponding one-dimensional B-module. As in Section 2 we may consider the induced module $M(\theta) = KG \otimes_{KB} K_\theta$. We can define Steinberg module for KG but which is not a submodule of $M(tr)$.

Let $\mathrm{St} = KU\xi$ be a free KU-module generated by the element ξ. Note that KG (resp. kU) is the union of all KG_n (resp. $K(U \cap G_n)$). By the proof for Proposition 2.3 we get the following result.

Proposition 7.4. *The KU-module structure on* St *can be uniquely extended to a KG-module structure as follows*:

(1) $tu\xi = tut^{-1}\xi$ *for any* $t \in T$ *and* $u \in U$,

(2) $n_i u\xi = -n_i u n_i^{-1}\xi$ *if* $u \in U_i'$,

(3) $n_i u_i' u_i \xi = n_i u_i' n_i^{-1}(x-1)\xi$ *for* $u_i' \in U_i'$ *and* $1 \neq u_i \in U_i$, *where* $x \in U_i$ *is defined uniquely by the formula* $n_i u_i n_i^{-1} = x n_i t y$, $t \in T$, $y \in U_i$.

Naturally, we call the G-module St a Steinberg module of G. Proposition 2.4 also has its counterpart here, i.e., $M(\theta) \otimes \mathrm{St}$ is isomorphic to $\mathrm{Ind}_T^G K_\theta$.

Using [13, Theorem 2], Theorems 3.2 and 3.4 and Lemma 1.6(b), we get the following result.

Theorem 7.5. (a) *Assume that* (1) $k = \mathbb{F}_q$ *or* $\bar{\mathbb{F}}_q$, (2) $K = \mathbb{C}$ *or* $\bar{\mathbb{F}}_q$, *then* St *is irreducible KG-module.*

(b) *Assume that* $K = \mathbb{C}$ *and* $\theta : T \to \mathbb{C}^*$ *is a character of* T. *If* $W_\theta = \{w \in W \mid {}^w\theta = \theta\}$ *has only one element, then* $M(\theta)$ *is irreducible KG-module.*

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