A Route to Turbulence in Delay-Differential System

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Abstract In this paper a singular perturbation approach is developed to study the dynamical behaviour of the delay-differential system at the points of transition from quasi-periodic to more chaotic states. The results reveal that some attractor crisis windows appear at merging points of the chaotic bands. The widths and the critical values of the windows vary with the different linear modes of the system regularly, leading to some harmonic bifurcation as $\omega_0 \to 3\omega_0 \to 5\omega_0 \to \cdots (2k_m+1)\omega_0$ and some hysteresis behaviour. The author argues that this unique harmonic bifurcation and hysteresis motion is an essential route to chaos in the delay-differential system.

Keywords: turbulence, delay differential system.

Among the investigations on bifurcation and chaos phenomena the delay-differential system has occupied an important position[1-10]. There are many important backgrounds present in this kind of system including optical bistability[3-4], acousto-electric hybrid system[5], and neurobiologic science. Also, the system itself has richer and more interesting properties than other kinds of nonlinear differential or difference equations. For example, the delay-differential system itself is an infinite-dimensional system, but often exhibits finite-dimensional attractor[1,2]; the time-evolution behaviour of the system is controlled by both the difference and differential processes and thus usually displays unexpected phenomena[4]. Up to now, great progress has been made on the research of this kind of time delay system. However, the route to chaos in the system remains an open question. The experimental results[6] obtained with optical-electric hybrid apparatus as well as the numerical simulation results[7] on some concrete delay-differential equations showed that harmonic bifurcation process is the important route to chaos in this kind of system. However, this viewpoint was challenged by Berre *et al*[8]. This is because Berre *et al*. did not observe the harmonic motion in their simulation, thus they raised query to the results reported in Refs. [6] and [7]; in addition, it is not convincing enough to draw a general conclusion from the results of special systems. Using a singular perturbation method we study the dynamical behaviour of the delay-differential system at the points of transition from quasi-periodic to chaotic state in this paper. The validity of this method for a time delay system has been demonstrated by our previous research results[9,10] which have predicted some new phenomena, such as discontinuous transition and hysteresis at each bifurcation point, and these predictions have already been proved by several related experiments[11,12]. Therefore, on such evidence we deal with the delay-differential system by means of the method which is similar, but more generalized, to that previously used.

The general equation for this kind of delay-differential system is[13,14].

$$\prod_{i=1}^{N}\tau_i\frac{d^NX(t)}{dt^N}+\cdots+\sum_{i=1}^{N}\sum_{j>i}^{N}\sum_{k>j}^{N}\tau_i\tau_j\tau_k\frac{d^3X(t)}{dt^3}+\sum_{i=1}^{N}\sum_{j>i}^{N}\tau_i\tau_j\frac{d^2X(t)}{dt^2}$$

$$+\sum_{i=1}^{N}\tau_i\frac{dX(t)}{dt}+X(t)=\mu F[X(t-t_d)], \tag{1}$$

where $F[X(t)]$ is a nonlinear function of $X(t)$, μ is the control parameter, t_d is the external delay time, and τ_i the internal characteristic response time. Considering the special case proposed by Vellee *et al.*[13,15], i.e. $\tau_i=\tau/N$, Eq. (1) becomes

$$\sum_{m=0}^{N}(\tau/N)^m \overset{N}{\underset{m}{C}} X^{(m)}(t)=\mu F[X(t-t_d)], \tag{2}$$

in which $X^{(m)}(t)=\frac{d^mX(t)}{dt^m}$.

When $N\to\infty$, based on the following fact

$$\lim_{N\to\infty}\sum_{m=0}^{N}(\tau/N)^m \overset{N}{\underset{m}{C}} X^{(m)}(t)=\exp\left(\tau\frac{d}{dt}\right)X(t), \tag{3}$$

where $\exp\left(\tau\frac{d}{dt}\right)$ is the retarded operator, Eq. (2) then degenerates to

$$X(t)=\mu F[X(t-(t_d+\tau))]. \tag{4}$$

Especially, in the case of long delay time limit, i.e. when $t_d>\tau$, Eq. (4) is a proper approximation of Eq. (2). In fact, no matter whether it is in the long delay time limit or not, the contribution of the higher order differential terms in Eq. (2) is always to force the system to close in Eq. (4), since time delay always means a series of higher order differential terms including the infinite-order one. Therefore, in the case of the single response time, i.e. $\tau_i=\tau/N$, the most typical delay-differential system which can reflect most sufficiently the disparity between the difference equation and the delay-differential system itself is the first-order one, i.e.

$$\tau\frac{dX(t)}{dt}+X(t)=\mu F[X(t-t_d)]. \tag{5}$$

It should be pointed out here that the results given by Vellee *et al.*[14] who have studied the Nth-order delay-differential equation with $\tau_i=\tau/N$ hardly involves any intrinsic new dynamical properties. The conclusions drawn by them are somewhat similar to those reported before[9,16]. In order to study the features of the Nth-order delay-differential system, we must deal with Eq. (1) starting from the very beginning.

Under the transformation

$$t\to t-P(\tau_1,\tau_2,\cdots,\tau_N;\omega),$$

where ω is a parameter yet to be determined, and function $P(\tau_1,\tau_2,\cdots,\tau_N;\omega)$ is given as

$$P(\tau_1,\tau_2,\cdots,\tau_N;\omega)$$
$$=\sum_{n=1}^{n\leqslant N+1}(-1)^{n+1}\left\{\frac{1}{(2n-2)!}\ \frac{1}{(2n-1)!}\right\}\omega^{2n-2}\left(\sum_{i=1}^{N}\tau_i^{2n-1}\right), \tag{6}$$

in which we define $0!=1/2$. Eq. (1) then becomes

$$X(t)-\left\{\sum_{n=1}^{n\leqslant N}\left[\frac{1}{(2n-1)!}-\frac{1}{(2n)!}\right]X^{(2n)}(t)\left(\sum_{i=1}^{N}\tau_i^{2n}\right)\right\}$$
$$+O\left(\left(\sum_{i=1}^{N}\tau_i\right)^{2N+1}\right)=\mu F[X(t-t_d-P(\tau_1,\tau_2,\cdots,\tau_N;\omega))]. \tag{7}$$

Before the system enters into the completely chaotic state, the motion in the chaotic bands tends to be regarded as a certain quasi-periodic motion: the motion is periodic between bands and is chaotic within a given band. This kind of motion also corresponds to the periodic motion with a random fluctuation in its amplitude. Based on this point of view, we might suppose that the circle frequency of the motion of the kth-order mode is just the parameter ω (it is denoted by ω_k thereinafter). In the case of plane wave motion and long delay time limit, i.e.

$$t_d>\max\ \{\tau_1,\tau_2,\cdots,\tau_N\},$$

Eq. (7) thus becomes an equation which has a form similar to that reported before[10]:

$$X(t)=\mu'(\tau_1,\tau_2,\cdots,\tau_N;\omega_k)F[X(t-t_d'(\tau_1,\tau_2,\cdots,\tau_N;\omega_k))], \tag{8}$$

where μ' and t_d' are the effective control parameter and delay time, respectively, with

$$\mu'(\tau_1,\tau_2,\cdots,\tau_N;\omega_k)=\mu/\{1+S(\tau_1,\tau_2,\cdots,\tau_N;\omega_k)\}, \tag{9a}$$
$$t_d'=t_d+P(\tau_1,\tau_2,\cdots,\tau_N;\omega_k), \tag{9b}$$

in which the function $S(\tau_1,\tau_2,\cdots,\tau_N;\omega_k)$ can be expressed as

$$S(\tau_1,\tau_2,\cdots,\tau_N;\omega_k)=\sum_{n=1}^{n\leqslant N}(-1)^{n+1}\left[\frac{1}{(2n-1)!}-\frac{1}{(2n)!}\right]\omega_k^{2n}\left(\sum_{i=1}^{N}\tau_i^{2n}\right). \tag{10}$$

Therefore, within a general long delay time limit, Eq. (1) has become, in the form and only in the form, a one-dimensional map. Eq. (10) is not exactly equal to a one-dimensional map because both the control parameter and the delay time are coupled with the dynamical states of the system, and may result in some intrinsic differences from one-dimensional map. So, the Nth-order delay-differential system can be regarded as, in a sense, a one-dimensional map with self-feedback and self-control, and the information on the typical one-dimensional map can be used in the analyses on the higher order effects of the differential terms. First of all, we must review some dynamical features of the one-dimensional map briefly. For a standard one-dimensional map

$$X(t)=\alpha F[X(t-t_d)], \tag{11}$$

the different quasi-periods of occurrence are

$$T^{(M)} = 2^{M+1}t_d, \ M = 0,1,2,\cdots,$$

where M is the number of the bifurcation time of the chaotic bands. For the quasi-period state denoted by $T^{(M)}$, there are two critical values for the control parameter: one is the critical value $\alpha_+^{(M)}$ through which the state switches to $T^{(M+1)}$ by bifurcation, and the other is denoted by $\alpha_-^{(M)}$ at which it converts into $T^{(M-1)}$ by remerging. This is almost the same as the bifurcation processes of the exactly periodic state[10] except for the reverse order of the critical values. For the bifurcation or merging processes of the chaotic bands (See Fig. 1(a)) the value $\alpha_-^{(M)}$ is larger than $\alpha_+^{(M)}$. Considering the continuity of the state at each bifurcation point, the critical values of the control parameter must satisfy

$$\alpha_+^{(M)} = \alpha_-^{(M+1)}, \alpha_+^{(M-1)} = \alpha_-^{(M)}. \tag{12}$$

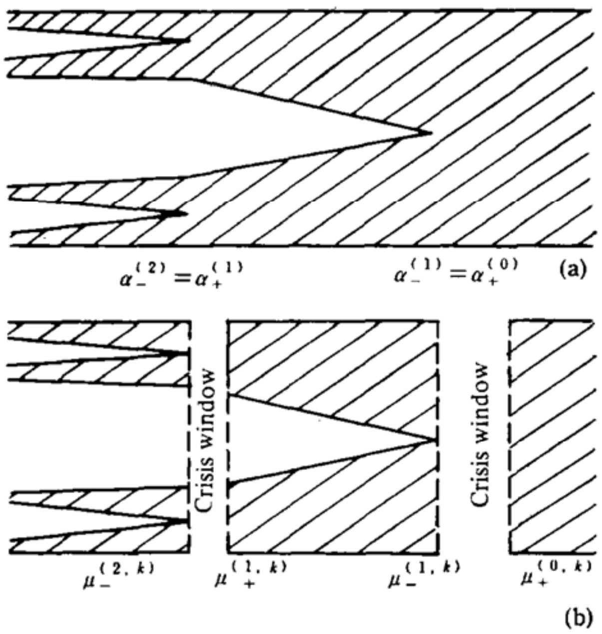

Fig. 1. (a) The combination process in the chaotic band of a one-dimensional map. (b) The attractor crisis windows appearing in the combination process of a chaotic band for a given mode in the differential-delay system.

However, there are some differences for the time delay system. The one-dimensional-map form of Eq. (8) demands μ' to satisfy Eq. (12), that is,

$$\mu'^{(M-1,k)}_+ = \mu'^{(M,k)}_-$$

$$\text{or } \frac{\mu_+^{(M-1,k)}}{1 + S(\tau_1,\tau_2,\cdots,\tau_N;\omega_k^{(M-1)})} = \frac{\mu_-^{(M,k)}}{1 + S(\tau_1,\tau_2,\cdots,\tau_N;\omega_k^{(M)})}, \tag{13}$$

where the subscripts "+" and "−" represent the bifurcation and merging processes, respectively; M and k are the indexes of the chaotic band and the linear mode of the system respectively. From Eq. (13), we can obtain

$$\mu_+^{(M-1,k)} - \mu_-^{(M,k)} \geqslant \mu_-^{(M,k)}\{S(\tau_1,\tau_2,\cdots,\tau_N;\omega_k^{(M-1)}) - S(\tau_1,\tau_2,\cdots,\tau_N;\omega_k^{(M)})\} \geqslant 0, \tag{14}$$

where $\omega_k^{(M)} = \frac{(2k+1)\omega_0}{2^{M+1}}$, $k = 0,1,2,\cdots,k_m$; $M = 0,1,2,\cdots$, and ω_0 is determined by

$$\omega_0\{t_d + P(\tau_1,\tau_2,\cdots,\tau_N;\omega_0)\} = \pi. \tag{15}$$

The Eq. (14) shows that in a process in which the M band combines with the $M-1$ band, a forbidden zone exists for a range of values of μ. When μ satisfies

$$\mu_-^{(M,k)} < \mu < \mu_+^{(M-1,k)},$$

the system has already passed through the critical combination point for the M band: the system must pass through the combination and enters the $M-1$ band state. As far as the $M-1$ band is concerned, it does not satisfy the basic condition of entering the $M-1$ band: $\mu > \mu_+^{(M-1,k)}$. Hence, the state entering the $M-1$ band is excluded from the system for the given mode. This contradictory result hints that the quasi-periodic attractor of the system is completely unstable and consequently some sudden changes may occur in the forbidden zone. This is the so-called attractor crisis which is defined as the occurrence of the sudden qualitative changes of chaotic attractor[17,18]. It will be seen below that the attractor crisis in the present system is different from that in one-dimensional map, although both of them behave as a sudden change in qualities of the attractors. Here we just point out the existence of the region where the attractor crisis may occur, and call the difference determined by Eq. (13) the attractor crisis window for the kth-mode at the M order bifurcation point (as shown in Fig. 1(b)).

Now we analyse the occurrence of the attractor crisis and its development. First of all, we consider the arrangement of the critical values of the crisis windows. According to Eq. (9a), the bifurcation critical parameters of the delay-differential system, $\mu_-^{(M,k)}$, are larger than that of the corresponding one-dimensional map, $\mu_-'^{(M,k)}$, by a factor of $[1 + S(\tau_1,\tau_2,\cdots,\tau_N;\omega_k^{(M)})]$. The dependence of the S function on k means that the different modes will reach their own attractor crisis windows in a certain sequence. Fig. 2 demonstrates the typical arrangement of the sequence. It is very clear that the critical values of the crisis windows increase one by one with the addition of the mode from lower order to higher order. This means that when the attractor with a lower order mode (e. g. the fundamental mode) is completely unstable, the higher order mode is relatively stable, leading to a sudden jump of the dynamical state from the lower mode to the higher one, a typical

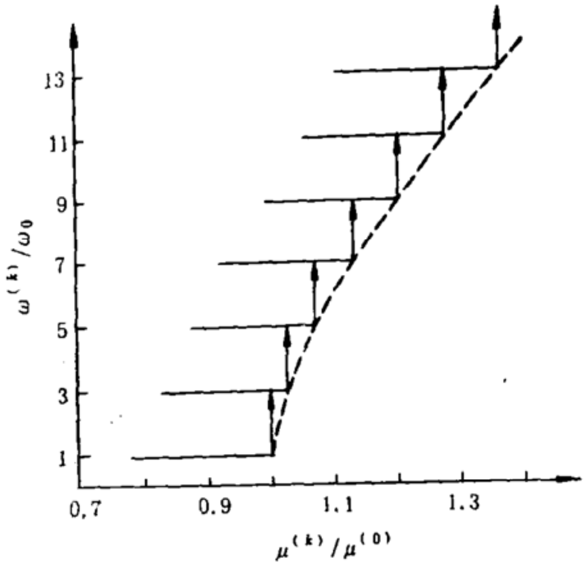

Fig. 2. The arrangement of the critical values of the crisis window of the different modes. The parameters are $N = 2$, $M = 0$, $t_d/\tau_1 = 20$, $\tau_2/\tau_1 = 0.1$.

phenomenon of the attractor crisis. It is worth while to note that in the one-dimensional map as studied by Grebogi *et al.*[17] and Jeffries *et al.*[18], the crises arise from intersection of an unstable orbit with the chaotic attractor, and behave as the sudden jump from one chaotic (or quasi-periodic) state to another chaotic state or even to an exactly periodic state, in which all states are developed from the same mode. However, the crisis in the present system behaves as the transition between the states developed from the different modes. This is the origin of the anomalous mode-locked phenomena observed in the experiments[4-6] or in the numerical simulation[7]. In order to distinguish them from the crises proposed by Grebogi *et al.*[17], we might as well call the crises here anomalous attractor crises.

The arrangement of the critical values as shown in Fig. 2 is not the peculiarity under some specified parameters, but the general character, determined by Eq. (10), of the system. In the general situation, the lower order modes must arrange in a way similar to that in Fig. 2. The development of the motion in the forbidden zone is not only determined by this arrangement, but also controlled by the widths of the crisis window. From Eq. (13) we know that the widths decrease rapidly with increasing M; the maximum appears at $M = 0$. This means that the anomalous attractor crisis occurs most often at the transition point from quasi-periodic to fully chaotic states, which is also one of the reasons why the anomalous mode-locked phenomena are merely observed in this point as performed in the optical experiments[4-6] and numerical simulation[7]. For the lower order modes, the widths of the crisis windows increase with k as shown in Fig. 3.

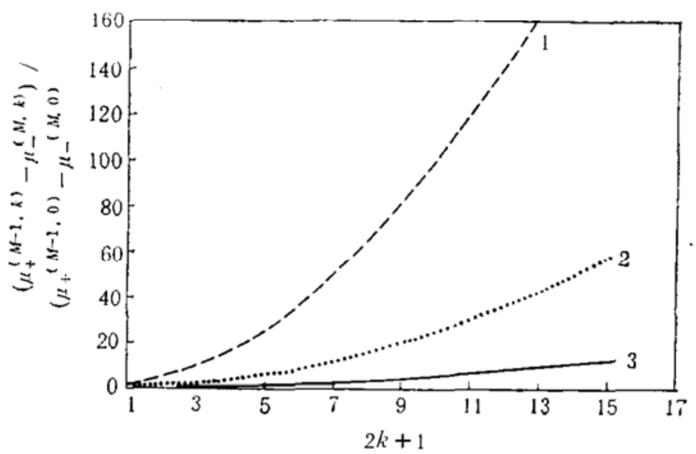

Fig. 3. The widths of the crisis window vs. order number of mode. The parameters are 1, $N = 2$, $M = 0$, $t_d/\tau_1 = 20$, $\tau_2/\tau_1 = 0.1$; 2, $N = 2$, $M = 1$, $t_d/\tau_1 = 20$, $\tau_2/\tau_1 = 0.5$; 3, $N = 2$, $M=2$, $t_d/\tau_1 = 40$, $\tau_2/\tau_1 = 10$.

Based on the analyses above, we can obtain the structure of the crisis window as given in Fig. 4. The unique structure is crucial, as illustrated by the following analyses, to the origin of the harmonic bifurcation process. Eq. (1) has many linear modes with the circle frequencies of $\omega_0, 3\omega_0, 5\omega_0, \cdots, (2k_m + 1)\omega_0$. Generally, the macroscopic behaviour of the system is determined by the excited modes; since the fundamental mode is excited first, and the excited probability of the higher order

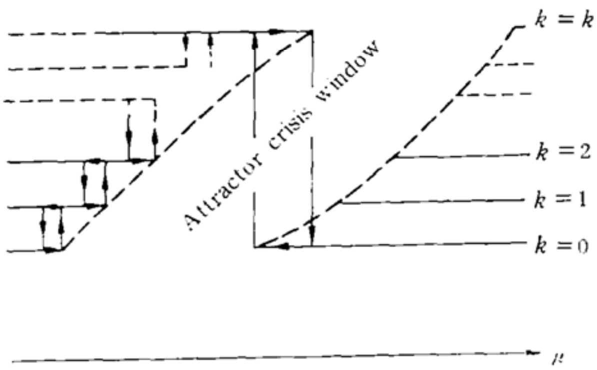

Fig. 4. The transition and hysteresis between the different linear modes in an attractor crisis window located between the M band and the $M-1$ band.

modes decreases successively with increasing k by a factor $t_d/\bar{\tau}\left(\bar{\tau}=\sum_{i=1}^{N}\tau_i/N,\right)$, the state of the system which can be usually observed is the motion of the fundamental mode including its bifurcation and chaos. The higher order modes can appear only when the fundamental mode reaches its attractor crisis window. A series of higher order modes appear one by one as soon as a lower mode enters its own attractor crisis window. Not until the highest order mode is excited and reaches its window does the harmonic bifurcation process cease. Once the highest mode is unstable, the system switches back to the fundamental mode which has already passed over its attractor crisis window and entered into a more chaotic but more stable state. If letting the system move in the reverse process, a hysteresis and transition may appear between the fundamental mode and the highest order mode. In addition, some hysteresis and transition with smaller scale may also emerge between the higher order modes due to the "super-cooling" effect. This is quite similar to that observed by Hopf and Gibbs *et al.*[4,6] and simulated numerically by Ikeda *et al.*[7]

As regards the k_m, generally the relation between k_m and other dynamical parameters of the system is not simple, but in the case of the first-order delay-differential system, it can be expressed approximately as

$$k_m=[(t_d\mu/2\pi\tau)|\delta F[x]/\delta x|_{x=x_s}|-1/4], \qquad (15)$$

in which x_s is the value of x at steady-state; [ ] represents the integer function. It can be seen that the maximum number of the modes which can be excited is approximately proportional to the delay time (it is also true for the Nth-order delay-differential system). So, for a given delayed time and specified crisis window, the number of harmonic mode is finite. In addition, k_m is also related with the gradient of function $F[x]$ at $x=x_s$.

As illustrated above, the occurrence of the harmonic bifurcation at the transition point from quasi-periodic to more chaotic states and hysteresis are the intrinsic nature of the delay-differential system, which is resulting from the anomalous attractor crisis as well as the non-Markov property of the system. All these along with the anomalies in the exactly periodic state as shown in Ref. [10] are the typical characteristics distinguishing the delay differential system from other nonlinear systems. Besides, all results mentioned above are only based on the quasi-periodic motion and this kind of motion exists at any stage of the chaotic scheme structure except for the completely chaotic one. It can be predicted that the harmonic bifurcation process can also appear at any stage in the chaotic attractor as long as the transition from a quasi-periodic motion to a more chaotic one is concerned. This means that the harmonic bifurcation is self-similar in the chaotic attractor and is undoubtedly an intrinsic characteristic of the delay-differential system, and that the topological structure of the chaotic attractor is a new-type one.

In summary, using the singular perturbation approach developed here it turns out that the attractor crisis will appear at the transition point from quasi-periodic to more chaotic state in the delay-differential system, leading to harmonic bifurcation as $\omega_0 \to 3\omega_0 \to 5\omega_0 \cdots \to (2k_m + 1)\omega_0$ and a hysteresis. This harmonic bifurcation is self-similar in the chaotic attractor. Based on the argument above we can draw one important and significant conclusion, i.e. the harmonic bifurcation is an essential route to a full chaos in the Nth-order differential-delay system.

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