Energy velocity and group velocity

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Received June 30, 1994

Abstract A new Lagrangian method for studying the relationship between the energy velocity and the group velocity is described. It is proved that under the usual quasistatic electric field, the energy velocity is identical to the group velocity for acoustic waves in anisotropic piezoelectric (or non-piezoelectric) media.

Keywords: energy velocity, group velocity, Lagrangian, anisotropic.

The study on the relationship between the energy velocity and the group velocity of waves has lasted over a century. With important physical significance, this relationship is related to the problem of wave-particle duality in nature. Generally, it is believed that for non-dispersive waves, the energy velocity and the group velocity are identical[1, 2]. Reyleigh[3], one of the early researchers in this field, gave a proof of the equivalence between the energy velocity and the group velocity for one-dimensional water wave. Following him, many authors[4—6] have shown the equivalence between the two velocities for more general wave systems. However, because of the difficulties in understanding their methods or of the limitation of those methods, up to now this relationship still remains somewhat obscure for some linear homogeneous and conservative wave systems (particularly in anisotropic media). Some papers, e.g. ref. [7] dealt with acoustic waves in piezoelectrics without discussing this relationship. Resitic[8] tried to prove that for acoustic waves in piezoelectrics, the energy velocity and the group velocity are not identical under the usual approximation of quasistatic electric field. However, his proof is not rigorous enough.

In this paper, we shall describe a new method for studying the energy velocity and the group velocity with acoustic waves in piezoelectrics taken as an example, and show that the energy velocity and the group velocity are identical for this system under the approximation of quasistatic electric field. This method has already been used to prove the equivalence between the energy velocity and the group velocity for elastic waves in layered anisotropic media1). Similar to Lighthill's method, it makes use of a Lagrangian density. Though the method applies to a general linear wave system, in order to make the discussion clear in physical terms we shall consider acoustic waves in piezoelectrics. The expression of energy velocity in terms of a Lagrangian density is given by the definition. Lighthill obtained the expression of group velocity using a variation method. We

1) Chen, Y., MPhil Thesis, Nottingham University, England, 1988.

shall obtain the group velocity by implicit differentiation from a generalized dispersion equation. The mathematical logic of the new method is simple and clear and we do not need to consider the concrete expression of a wave field.

1 Lagrangian density of a system

The dynamic characteristics of a linear wave system may be determined by a Lagrangian density of the system. In general, the Lagrangian density is a functional of basic quantities for a system, and of their first derivatives. For acoustic waves in anisotropic piezoelectrics, under the usual approximation of quasistatic electric field, we have the expression of a Lagrangian density

$$L=\frac{1}{2}\rho\dot{U}_i\dot{U}_i-\frac{1}{2}D_j\Phi_{,j}-\frac{1}{2}T_{ij}U_{i,j}, \tag{1}$$

where the dot above the first term on the right-hand side denotes the partial derivative with respect to time, the subscript comma of the second or third term represents the partial derivative with respect to position, and any subscript takes values 1, 2, 3, and the convention of a repeated alphabetic subscript in a term is used for summation. ρ is the mass density of a medium, U_i is a displacement component of a particle, D_i is the component of electric displacement, Φ is quasistatic electric potential, and T_{ij} is the component of the stress tensor. The first term on the right side of (1) corresponds to the kinetic energy density, the second term is associated with the energy density of electric field, and the third term gives the deformation energy density. If using the displacement U_i and the electric potential Φ as the basic quantities of the system, from the constitutive equations of piezoelectricity, we may have the Lagrangian density

$$L=\frac{1}{2}\rho\dot{U}_i\dot{U}_i+\frac{1}{2}\varepsilon^s_{ij}\Phi_{,i}\Phi_{,j}-\frac{1}{2}C^E_{ijkl}U_{i,j}U_{k,l}-e_{kij}U_{i,j}\Phi_{,k}, \tag{2}$$

where C^E_{ijkl}, e_{ijk}, ε^s_{ij} are elastic, piezoelectric and dielectric constants, respectively.

By use of a Lagrangian density, the governing equations of the wave motion can be obtained from Hamilton's variation principle[9, 10]. For Lagrangian density (2), applying the theorem of variation calculation to Hamilton's principle, we have

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial\dot{U}_i}+\frac{\partial}{\partial x_j}\frac{\partial L}{\partial U_{i,j}}=0, \tag{3}$$

$$\frac{\partial}{\partial x_j}\frac{\partial L}{\partial\Phi_{,j}}=0. \tag{4}$$

It is not difficult to show that (3) and (4) are corresponding to the governing equations for acoustic waves under the quasistatic electric field. Eq. (3) is the equation of motion, and (4) represents the zero divergence of the electric displacement vector. This also shows that Lagrangian density (1) is appropriate.

We assume that the solution of the governing equations (3) and (4) is given by

$$\begin{cases} U_i = U_i(\theta), \\ \Phi = \Phi(\theta), \end{cases} \tag{5}$$

where $\theta = \omega t - k_i x_i$ is a phase function. Furthermore, we assume that U_i, Φ and their deivatives all are continuous periodic functions in θ with a period P. In a general wave system, components U_i and Φ are not necessary in phase (e.g. in the case of waves in a waveguide). In order to demonstrate the generality of our method, we consider the solution in the form of (5).

2 Energy velocity

The energy velocity is the ratio of the energy flux (energy flow crossing a unit area per unit time) $\boldsymbol{F}$ averaged over the time-space range, where the wave exists, to the energy density (energy per unit volume) averaged over the same time-space range

$$v_0 = \frac{\langle F \rangle}{\langle E \rangle}, \tag{6}$$

where the symbol $\langle\ \rangle$ denotes the time-space average. When considering a periodic wave, we only need to take the average over a period. For a general system, if a wave field in a certain direction is not periodic (e.g. surface waves in the direction normal to the surface of solid) the average should be taken over the whole space where the wave exists. The energy velocity is a constant which determines a dynamic characteristic of a wave system. Obviously the energy velocity is a vector and it is in the direction of the time-space mean energy flux. If the energy velocity does not use averaged quantities, it is usually a function of time and space (e.g. waves in a waveguide). The energy velocity which varies with time and space has little meaning here.

For piezoelectric acoustic waves, the expression of the energy flux may be obtained from the Poynting vectors of electromagnetic waves and elastic waves under the approximation of quasistatic electric field. The expression may be found in many textbooks (e.g. ref. [7]). In terms of Lagrangian density, a component of the energy flux can be given by

$$F_i = \frac{\partial L}{\partial U_{i,j}} \dot{U}_j - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_{,i}}\right)\Phi. \tag{7}$$

It is not difficult to show that (7) is consistent with that given by ref. [7]. In (7) the first term on the right side is the product of negative stress component and the particle velocity, and the second term is the product of the time derivative of electric displacement and the electric potential.

By use of the energy balance equation, the total energy density of the system may be obtained from the expression of the energy flux. The energy balance equation in a differentiation form can be expressed as

$$\dot{E}=-F_{i,i}. \tag{8}$$

From (7), the divergence of the energy flux is given by

$$F_{i,i}=\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial U_{j,i}}\right)\dot{U}_j+\frac{\partial L}{\partial U_{i,j}}\dot{U}_{j,i}-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_{,i}}\right)\Phi_{,i}, \tag{9}$$

where we have used the result that the divergence of the electric displacement is zero. Applying the equations of motion to the first term on the right side and then using the following relationships

$$\begin{cases}\dfrac{\partial}{\partial t}\left(\dfrac{\partial L}{\partial \dot{U}_j}\right)\dot{U}_j=\dfrac{\partial}{\partial t}\left(\dfrac{\partial L}{\partial \dot{U}_j}\dot{U}_j\right)-\dfrac{\partial L}{\partial \dot{U}_j}\ddot{U}_j,\\[2ex] \dfrac{\partial}{\partial t}\left(\dfrac{\partial L}{\partial \Phi_{,i}}\right)\Phi_{,i}=\dfrac{\partial}{\partial t}\left(\dfrac{\partial L}{\partial \Phi_{,i}}\Phi_{,i}\right)-\dfrac{\partial L}{\partial \Phi_{,i}}\dot{\Phi}_i,\end{cases} \tag{10}$$

we may have

$$F_{i,i}=-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{U}_i}\dot{U}_i+\frac{\partial L}{\partial \Phi_{,i}}\Phi_{,i}-L\right). \tag{11}$$

Comparing (11) with (8) we obtain the total energy density

$$E=\frac{\partial L}{\partial \dot{U}_i}\dot{U}_i+\frac{\partial L}{\partial \Phi_{,i}}\Phi_{,i}-L. \tag{12}$$

It is worth while to mention that the instant total energy density of piezoelectric acoustic waves looks different from those of usual non-piezoelectric acoustic waves, i.e. the second term is not a derivative with respect to time, but we shall later show that its time-space averaged value gives the usual result that the mean total energy density is twice the mean kinetic energy density.

From (7) and (12) the energy velocity can be expressed as

$$\boldsymbol{v}_e=\frac{\langle F\rangle}{\langle E\rangle}\ \frac{\left\langle\dfrac{\partial L}{\partial U_{i,j}}\dot{U}_j-\dfrac{\partial}{\partial t}\left(\dfrac{\partial L}{\partial \Phi_{,i}}\right)\Phi\right\rangle}{\left\langle\dfrac{\partial L}{\partial \dot{U}_j}\dot{U}_j+\dfrac{\partial L}{\partial \Phi_{,j}}\Phi_{,j}-L\right\rangle}\hat{x}_i, \tag{13}$$

where $\hat{x}_i$ is a unit space vector. The energy velocity is in fact independent of the amplitude because the amplitude components all are dependent on one common factor determined by the governing equations of the system, and the common factors on the denominator and on the numerator cancel out each other. On the denominator, we have, from the periodicity of the wave field,

$$\left\langle\frac{\partial L}{\partial \Phi_{,j}}\Phi_{,j}\right\rangle=\left\langle\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial \Phi_{,j}}\Phi\right)-\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial \Phi_{,j}}\right)\Phi\right\rangle=0. \tag{14}$$

3 Generalized dispersion equation

The system in which we are interested is a linear homogeneous and conservative lossless one. For such a system, a wave field is usually of periodicity or is a superposition of periodic waves. A linear periodic dynamic system which obeys Hamilton's principle has an important property: the mean time-space Lagrangian density for the system is zero. This property is obvious for a classical system of harmonic oscillator. However, we have not found a direct proof of such a property in piezoelectric acoustic waves. The vanishing of the mean Lagrangian density is the key to our method so for certainty we shall show that for piezoelectric acoustic waves under the approximation of quasistatic electric field, the mean Lagrangian density is zero. The averaged Lagrangian density over time and space can be expressed as

$$D=\langle L\rangle. \tag{15}$$

First it is easy to show that the first variation δD vanishes if δU_i, $\delta\Phi$ have the same periodicity as U_i, Φ themselves even if δU_i, $\delta\Phi$ do not vanish at the boundary. The first variation δD is given by

$$\delta D=\left\langle \frac{\partial L}{\partial \dot{U}_i}\,\delta\dot{U}_i+\frac{\partial L}{\partial U_{i,j}}\,\delta U_{i,j}+\frac{\partial L}{\partial \Phi_{,i}}\,\delta\Phi_{,i}\right\rangle. \tag{16}$$

As usual this first variation can be rewritten in an alternative form by the commutation property of variation and differentiation. We have

$$\begin{aligned}\delta D=&-\left\langle \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{U}_i}\right)\delta U_i+\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial U_{i,j}}\right)\delta U_i+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \Phi_{,i}}\right)\delta\Phi\right\rangle,\\&+\left\langle \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{U}_i}\,\delta U_i\right)+\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial U_{i,j}}\,\delta U_i\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \Phi_{,i}}\,\delta\Phi\right)\right\rangle.\end{aligned} \tag{17}$$

The first mean integration term vanishes because of the governing equations for piezoelectric acoustic waves, and the second term is zero if δU_i, $\delta\Phi$ have the same periodicity as U_i, Φ. Therefore, the first variation of the mean Lagrangian density is zero.

Second, we use the direct definition of the variation. For simplicity we write the mean Lagrangian density as a functional of U_i and Φ.

$$D=D[U_i,\ \Phi], \tag{18}$$

so that the variation can be given as

$$\delta D=D[\overline{U}_i,\ \overline{\Phi}]-D[U_i,\ \Phi], \tag{19}$$

where $\overline{U}_i=U_i+\delta U_i$, $\overline{\Phi}=\Phi+\delta\Phi$ are arbitrarily varying functions. If we assume that $\delta U_i=\alpha U_i$, $\delta\Phi=\alpha\Phi$ where α is an arbitrary parameter, since D is homogeneous and quadratic in the derivatives of U_i and Φ, the varied mean Lagrangian density can be given by

$$D[\overline{U}_i,\ \overline{\Phi}]=(1+\alpha)^2 D[U_i,\ \Phi],\tag{20}$$

and (19) becomes

$$\delta D=\{(1+\alpha)^2-1\}D[U_i,\ \Phi].\tag{21}$$

Since δU_i, $\delta\Phi$ have the same properties as U_i, Φ themselves, the variation is zero as we have shown. Thus for any non-zero parameter α the mean Lagrangian density $D[U_i,\ \Phi]$ must be zero:

$$D[U_i,\ \Phi]=0.\tag{22}$$

Equation (22) gives a relationship between the frequency and the wavenumber components and is often called dispersion equation. Here we should emphasize that the frequency and the wavenumber components in the phase function have no contribution to the dispersion equation since the average is performed over time and space. In order to make this clear we rewrite eq. (22) as

$$D=\frac{1}{P}\int_0^P L(\theta,\ \omega,\ k_i)\mathrm{d}\theta=D(\omega,\ k_i)=0.\tag{23}$$

4 Group velocity

The group velocity is consistent with the usual concept of the velocity. It is a vector, and its component is determined by the partial derivative of the frequency with respect to the corresponding component of wavenumber. In terms of angular frequency ω and angular wavenumber component k_i, the group velocity can be expressed as

$$v_g=\frac{\partial\omega}{\partial k_i}\ \hat{x}_i.\tag{24}$$

The group velocity is an important physical quantity which describes the essential characteristic of a complete wave system (we should not simply consider it as an approximation of modulated velocity of travelling wave packets). Here, we shall use the Lagrangian density to express the group velocity. We may consider ω as an implicit function of k_i, which is determined by the dispersion equation. According to the rule of implicit differentiation, we have

$$v_g=\frac{\partial\omega}{\partial k_i}\ \hat{x}_i=-\frac{\dfrac{\partial D}{\partial k_i}}{\dfrac{\partial D}{\partial\omega}}\ \hat{x}_i.\tag{25}$$

Let us examine the partial derivative of D with respect to the frequency. Assuming that D is a continuous function of the frequency and the wavenumber, when performing the partial differentiation with respect to the explicit frequency, the limits of the integration are constants with respect to the integral variable θ. So we can interchange the order of differentiation and the integration. The denominator can be rewritten as

$$\frac{\partial D}{\partial\omega}=\left\langle\frac{\partial L}{\partial\dot{U}_i}\ \frac{\partial\dot{U}_i}{\partial\omega}+\frac{\partial L}{\partial U_{i,j}}\ \frac{\partial U_{i,j}}{\partial\omega}+\frac{\partial L}{\partial\Phi_{,i}}\ \frac{\partial\Phi_{,i}}{\partial\omega}\right\rangle.\tag{26}$$

In eq. (26) we should carefully perform the partial differentiation of the particle velocity with respect to the frequency. We may obtain the following equations:

$$\frac{\partial \dot{U}_i}{\partial \omega} = \frac{\partial}{\partial \omega}\left(\omega \frac{\partial U_i}{\partial \theta}\right) = \frac{\partial U_i}{\partial \theta} + \omega \frac{\partial}{\partial \omega}\left(\frac{\partial U_i}{\partial \theta}\right). \tag{27}$$

Applying the operator $\frac{\partial}{\partial \theta} = \frac{\partial}{\omega \partial t}$ to the first term on the right side, and on the second term exchanging the order of derivatives first, then using $\omega \frac{\partial}{\partial \theta} = \frac{\partial}{\partial t}$ to rearrange the term, we obtain

$$\frac{\partial \dot{U}_i}{\partial \omega} = \frac{\dot{U}_i}{\omega} + \frac{\partial}{\partial t} \frac{\partial U_i}{\partial \omega}. \tag{28}$$

In eq. (28) the orders of differentiations on the last term are not commutative. Since we have interchanged the order of differentiation with respect to ω and the integration with respect to θ, when performing the differentiation with respect to ω, we should consider θ as a variable independent of the frequency. Substituting (28) into (26) and carrying out a transformation similar to eq. (10) we get

$$\begin{aligned}\frac{\partial D}{\partial \omega} &= \left\langle \frac{1}{\omega} \frac{\partial L}{\partial \dot{U}_i} \dot{U}_i \right\rangle \\ &+ \left\langle \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{U}_i} \frac{\partial U_i}{\partial \omega}\right) + \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial U_{i,j}} \frac{\partial U_i}{\partial \omega}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \Phi,_i} \frac{\partial \Phi}{\partial \omega}\right)\right\rangle \\ &- \left\langle \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{U}_i}\right)\frac{\partial U_i}{\partial \omega} + \frac{\partial}{\partial x_j}\left(\frac{\partial U_i}{\partial U_{i,j}}\right)\frac{\partial U_i}{\partial \omega} + \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \Phi,_i}\right)\frac{\partial \Phi}{\partial \omega}\right\rangle. \end{aligned} \tag{29}$$

By the periodicity, the second-averaged term on the right side of Eq. (29) is zero. The third averaged term vanishes according to the governing equations of piezoelectric acoustic waves. Thus, we have

$$\frac{\partial D}{\partial \omega} = \left\langle \frac{1}{\omega} \frac{\partial L}{\partial \dot{U}_i} \dot{U}_i \right\rangle. \tag{30}$$

Similarly, by use of the following equation:

$$\left\{\begin{aligned} \frac{\partial U_{i,j}}{\partial k_m} &= \frac{\partial}{\partial k_m}\left(-k_j \frac{\partial U_i}{\partial \theta}\right) = -\delta_{mj}\frac{\dot{U}_i}{\omega} + \frac{\partial}{\partial x_j}\frac{\partial U_i}{\partial k_m}, \\ \frac{\partial \Phi,_j}{\partial k_m} &= \frac{\partial}{\partial k_m}\left(-k_j \frac{\partial \Phi}{\partial \theta}\right) = -\delta_{mj}\frac{\dot{\Phi}}{\omega} + \frac{\partial}{\partial x_j}\frac{\partial \Phi}{\partial k_m}, \end{aligned}\right. \tag{31}$$

where δ_{mj} is Kronecker symbol, and the last two terms on the right side are not commutative. The derivative of D with respect to k_m is given by

$$\frac{\partial D}{\partial k_m}=-\left\langle\frac{1}{\omega}\frac{\partial L}{\partial U_{m,i}}\dot{U}_i+\frac{1}{\omega}\frac{\partial L}{\partial \Phi_{,m}}\dot{\Phi}\right\rangle, \tag{32}$$

where the last term on the right side can be rewritten as

$$\left\langle\frac{1}{\omega}\frac{\partial L}{\partial \Phi_{,m}}\dot{\Phi}\right\rangle=\left\langle\frac{1}{\omega}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_{,m}}\Phi\right)-\frac{1}{\omega}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_{,m}}\right)\Phi\right\rangle$$

$$=-\left\langle\frac{1}{\omega}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_{,m}}\right)\Phi\right\rangle. \tag{33}$$

Using (32) and (30), the group velocity becomes

$$\boldsymbol{v}_g=\frac{\left\langle\frac{\partial L}{\partial U_{i,j}}\dot{U}_j-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Phi_{,i}}\right)\Phi\right\rangle}{\left\langle\frac{\partial L}{\partial \dot{U}_j}\dot{U}_j\right\rangle}\hat{x}_i. \tag{34}$$

Equations (14), (23) and (33) are always true. Comparing (34) with (13) gives the required result. Although the above proof is given for piezoelectric acoustic waves, it is valid for acoustic waves in non-piezoelectric anisotropic media as long as we set the electric potential to be zero. In fact starting from the analytical solutions of a particular case, we may show the equivalence between the energy velocity and the group velocity. But it would be a tedious process to do so (Auld[7] gives a special example: for a cubic non-piezoelectric crystal, the energy velocity is equal to the group velocity). The above proof is relatively simple.

5 Discussion

The study on the energy velocity and the group velocity is not only of theoretical significance but also of important practical meaning. For example, an acoustic method for material characterization or non-destructive evaluation of an anisotropic medium is very often used[11]. In a measurement usually the main signal which we receive is that in the direction of the energy flux vector. As is well known, usually in an anisotropic medium, the group velocity and the wavenumber vector may not be in the same direction. If the energy velocity and the group velocity are identical, we know that the direction of the energy flux does not coincide with that of the wavenumber vector, and the angle between the energy flux and the wave vector may be determined by that of the group velocity and the wave vector, since the group velocity is easier to obtain. Therefore, this should be taken into account when using an acoustic method to evaluate the properties of an anisotropic material. Our proof of the equivalence between the energy velocity and the group velocity for acoustic waves in anisotropic media (including non-piezoelectric media) indirectly gives theoretical evidence that in a measurement the main signal may not be in the direction of the wavenumber vector, and the angle between the wavenumber vector and the energy flux vector may be determined by the angle between the group velocity and the wavenumber vector, since the group velocity is relatively easy to obtain.

This method can be easily applied to other types of linear waves, for example, electromagnetic waves in anisotropic media, surface waves and waves in waveguides. We hope that this method will be useful for further studying the energy velocity and the group velocity.

Acknowledgement The author thanks Profs. G. Z. Zhu, T. Y. Zhou and Z. D. Chen for helpful discussions.

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