The p-Regular Class Length and the p-Rank in a p-Solvable Group

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In Ref. [1], O. Manz investigated the relation between the character degrees and the p-rank in a p-solvable group. In Ref. [2], Y. Q. Wang investigated the relation between the Brauer character degrees and the p-rank in a p-solvable group. Correspondingly, D. Chillag and M. Herzog investigated the relation between the conjugacy class lengths and the p-rank in a solvable group[3]. In this note, we improve the result of D. Chillag and M. Herzog.

In the note, all groups considered are finite. G and p always denote a finite group and a prime number, respectively. G_p is a Sylow p-subgroup of G. $r_p(G)$ and $l_p(G)$ denote the p-rank and the p-length of G, respectively. For a subgroup K of G and an element x of K, we denote the conjugacy class of x in K by $Cl_K(x)$. $\mathrm{Con}(G)$: $=\{C \mid C$ is a conjugacy class of $G\}$. For $C \in \mathrm{Con}(G)$, $|C|$ is called the length of the conjugacy class C. For an element x of G, we say x is p-regular if $p \nmid |x|$. The conjugacy class of a p-regular element in G is called a p-regular class of G.

Let n be a positive integer. If $n = p^a m$ and $p \nmid m$, then we write $\omega_p(n) = a$. For the group G, $\omega c_p(G)$: $= \max\{\omega_p(|C|) C \in \mathrm{Con}(G)\}$[3], we define $\overline{\omega c}_p(G) = \max\{\omega_p(|C|) C \in \mathrm{Con}(G)$ and C is p-regular$\}$. Clearly, we have $\overline{\omega c}_p(G) \leqslant \omega c_p(G)$.

Lemma 1. *Let $N \triangledown G$. Then $\overline{\omega c}_p(N) \leqslant \overline{\omega c}_p(G)$ and $\overline{\omega c}_p(G/N) \leqslant \overline{\omega c}_p(G)$.*

Proof. This is an immediate corollary of Ref. [3].

Lemma 2. *Let P be a p-group and V be a p'-group. Suppose that P acts on V. If every conjugacy class of V is P-invariant, then the action of P on V is trivial.*

Proof. Let C be an arbitrary conjugacy class of V. Since C is P-invariant, the action of P on V induces an action of P on the conjugacy class C. Also notice that $p \nmid |C|$. So, P must fix some element in C, and let $x(C)$ be such an element. Set $H = \langle x(C) \mid C \in \mathrm{Con}(V) \rangle$. Clearly, P acts trivially on H. Since H contains an element from each conjugacy class of V, we conclude that $V = \bigcup_{U \in V} H^v$. Hence $V = H$ and the action of P on V is trivial.

Lemma 3. *Let G be p-nilpotent with a normal p-complement $V=V_1\times\cdots\times V_n$,* where *all V_i's are isomorphic non-Abelian simple groups. If p is odd and $O_p(G)=1$, then $|G_p|\leqslant p^{2\overline{\omega c}_p(G)}$.*

Proof. Put $P=G_p$. Then $G=PV$, $V\triangleleft G$ and $P\cap V=1$. For every element x in P and each $i=1,\cdots,n$, by Ref. [4] we have $xV_ix^{-1}\in\{V_1,\cdots,V_n\}$. Clearly, for every element y in V and each $i=1,\cdots,n$ we have $yV_iy^{-1}=V_i$. Since $C_P(V)\leqslant O_p(G)=1$, $C_p(V)=1$ and so P acts faithfully on V by conjugation. We write $Q=\bigcap_{i=1}^{n}N_p(V_i)$ and $C_i=C_Q(V_i)$, $i=1,\cdots,n$. Since $C_P(V)=1$, $\bigcap_{i=1}^{n}C_i=1$. Also notice that $Q\triangleleft P$ and for each $i=1,\cdots,n$, we have $Q/C_i\leqslant\mathrm{Out}(V_i)$.

We assume that $Q\neq1$. Let P_i be a Sylow p-subgroup of $\mathrm{Aut}(V_i)$, $i=1,\cdots,n$. Since $Q\neq1$ and $\bigcap_{i=1}^{n}C_i=1$, there exists $j\in\{1,\cdots,n\}$ such that $Q/C_j\neq1$. Hence, by Ref. [5] P_j is cyclic. Since all V_i's are isomorphic, for each $i=1\cdots,n$, P_i is cyclic. Also notice that since $\bigcap_{i=1}^{n}C_i=1$, $Q\lesssim Q/C_1\times\cdots\times Q/C_n\lesssim P_1\times\cdots\times P_n$.

For each $i=1,\cdots,n$, P_i acts on the set $\mathrm{Con}(V_i)$ in an obvious way. Suppose that for every $C\in\mathrm{Con}(V_i)$ we have $C_{P_i}(C)>1$. (Here, C is viewed as an element of the set $\mathrm{Con}(V_i)$!) Then, since P_i is cyclic, $\Omega_1(P_i)$ fixes every element of $\mathrm{Con}(V_i)$, and hence we get $\Omega_1(P_i)=1$ by Lemma 2, contradicting $1\neq P_i\leqslant\mathrm{Aut}(V_i)$. So, for each $i=1,\cdots,n$ there exists some $C_i\in\mathrm{Con}(V_i)$ such that $C_{P_i}(C_i)=1$. Set $C=C_1\times\cdots\times C_n$. Then $C\in\mathrm{Con}(V)$ and $C_{P_1\times\cdots\times P_n}(C)=C_{P_1}(C_1)\times\cdots\times C_{P_n}(C_n)=1$. Thus, since $Q\leqslant P_1\times\cdots\times P_n$, we get $C_Q(C)=1$. From C we can get a p-regular class $\tilde{C}$ of $G(=PV)$ such that $|\tilde{C}|=|P:C_P(C)|\,|C|$. Therefore, since $Q\triangleleft P$ and $C_Q(C)=1$, we get $|Q|\,\big|\,|\tilde{C}|$ and so $|Q|\leqslant p^{\overline{\omega c}_p(G)}$. This is also true, of course, when $Q=1$.

Let $\Sigma=\{V_1,\cdots V_n\}$. G acts on the set Σ by conjugation. From the remarks at the beginning of the proof, we conclude that for $x\in G=PV$, x fixes every element of the set Σ if and only if $x\in QV$. $G/QV\simeq P/V$ is therefore a permutation group on the set Σ. Since p is odd, $G/QV\simeq P/Q$ is of odd order. So by Ref. [6] there exists some nonvoid subset W in Σ such that $\mathrm{Stab}_{G/QV}(W)=QV/QV$; that is, $\mathrm{Stab}_G(W)=QV$. We choose the numbering so that $W=\{V_1,\cdots,V_s\}$, $s\leqslant n$. For each $i=1,\cdots,s$, we take a non-identity element x_i in V_i and put $x=x_1\cdots x_s$. Then we have $C_G(x)\leqslant\mathrm{Stab}_G(W)=QV$. From this it follows that $|C_G(x)|_p\leqslant|Q|$, and hence we have $|\mathrm{Cl}_G(x)|_p\geqslant|P:Q|$. Clearly, $\mathrm{Cl}_G(x)$ is a p-regular class of G. Thus we get $|P:Q|\leqslant p^{\overline{\omega c}_p(G)}$ and $|P|\leqslant p^{2\overline{\omega c}_p(G)}$ since $|Q|\leqslant p^{\overline{\omega c}_p(G)}$, thereby completing the proof.

Lemma 4. *Let G have a nilpotent normal p-complement and let $O_p(G)=1$. Then*

(1) $|G_p|\leqslant p^{2\overline{\omega c}_p(G)}$.

(2) *If $|G|$ is odd, then $|G_p|\leqslant p^{\overline{\omega c}_p(G)}$.*

(3) *If* G_p *is Abelian, then* $|G_p| \leqslant p^{\overline{\omega c}_p(G)}$.

Proof. By hypothesis, $G = G_p F(G)$, where $F(G)$ is the fitting subgroup of G, and $F(G)$ is exactly the normal p-complement. Set $\overline{G} = G/\Phi(G)$. Let H be a subgroup of G such that $\Phi(G) \leqslant H$ and $\overline{H} = H/\Phi(G) = O_p(\overline{G})$. Since $\Phi(G) \leqslant F(G)$ and $p \nmid |F(G)|$, $\Phi(G)$ is a p'-group, and hence $H = P \cdot \Phi(G)$, where $P \in \mathrm{Syl}_p(H)$. As $H \lhd G$, we have $G = N_G(P)H = N_G(P) \cdot \Phi(G) = N_G(P)$; that is, $P \leqslant O_p(G) = 1$. So $O_p(\overline{G}) = \overline{H} = 1$. Also notice that $\overline{G} = \overline{G}_p \cdot F(\overline{G})$, $\Phi(\overline{G}) = 1$ and $|G_p| = |\overline{G}_p|$. So by induction on $|G|$ we may assume that $\Phi(G) = 1$. Thus $F(G)$ is Abelian by Ref. [7]. By Ref. [7] we have $C_{G_p}(F(G)) \leqslant G_p \cap F(G) = 1$. This implies that the action of G_p on $F(G)$ by conjugation is faithful. By Passman's results[8], there exists an $x \in F(G)$ such that $|C_{G_p}(x)| \leqslant |G_p|^{1/2}$. If $|G|$ is odd or if G_p is Abelian, then there even exists an $x \in F(G)$ with $|C_{G_p}(x)| = 1$. As $C_G(x) = C_{G_p}(x) \cdot F(G)$, we conclude that $|Cl_G(x)| = |G_p : C_{G_p}(x)| \geqslant |G_p|^{1/2}$, and that $|Cl_G(x)| = |G_p|$ if $|G|$ is odd or if G_p is Abelian. Also notice that $\mathrm{Cl}_G(x)$ is a p-regular class of G. This completes the proof.

Theorem 1. *Let p be an arbitrary prime divisor of $|G|$. If the commutator subgroup G' of G is nilpotent, then* $|G_p : O_p(G)| \leqslant p^{\overline{\omega c}_p(G)}$.

Proof. By induction on $|G|$, we may assume that $O_p(G) = 1$. It is therefore sufficient to show that $|G_p| \leqslant p^{\overline{\omega c}_p(G)}$. Since G' is nilpotent, $G_p F(G) \lhd G$ and so $O_p(G_p F(G)) \leqslant O_p(G) = 1$. Hence, by induction we may assume that $G = G_p F(G)$. Clearly, $G_p \cap F(G) = 1$. It follows that $G_p \simeq G/F(G)$ is Abelian. Hence, by Lemma 4 we get $|G_p| \leqslant p^{\overline{\omega c}_p(G)}$. The proof is now complete.

In Ref. [3], D. Chillag and M. Herzog have proved that if G is solvable with $O_p(G) = 1$, then $r_p(G) \leqslant 2\omega c_p(G)$, and that if $|G|$ is odd and $O_p(G) = 1$, then $r_p(G) \leqslant \omega c_p(G)$. Now let us improve the above-mentioned result.

Theorem 2. *Let $O_p(G) = 1$. Then the following statements hold*:

(1) *If G is p-solvable, then* $r_p(G) \leqslant 2\overline{\omega c}_p(G)$;

(2) *If $|G|$ is odd, then* $r_p(G) \leqslant \overline{\omega c}_p(G)$;

(3) *If G is solvable and G_p is Abelian, then* $r_p(G) \leqslant \overline{\omega c}_p(G)$.

Proof. Since groups of odd order are solvable, G in the above three cases is always p-solvable.

Let V be a minimal normal subgroup of G. As G is p-solvable with $O_p(G) = 1$, V is a p'-group. If $O_p(G/V) = 1$, then since $r_p(G) = r_p(G/V)$ and $\overline{\omega c}_p(G/V) \leqslant \overline{\omega c}_p(G)$, we are done by induction. Hence we assume that $O_p(G/V) \neq 1$. Let $H/V = O_p(G/V)$. We have $H = PV$, where $P \in \mathrm{Syl}_p(H)$ and $P \neq 1$. Clearly, V is a normal p-complement of H and $O_p(H) = 1$.

If V is non-solvable, then p is odd because 2-solvable groups must be solvable. According to Lemma 3, $|P|\leqslant p^{2\overline{\omega c}_p(H)}\leqslant p^{2\overline{\omega c}_p(G)}$. If V is solvable, then V is elementary Abelian. Thus, according to Lemma 4, $|P|\leqslant p^{2\overline{\omega c}_p(H)}\leqslant p^{2\overline{\omega c}_p(G)}$, and $|P|\leqslant p^{\overline{\omega c}_p(H)}\leqslant p^{\overline{\omega c}_p(G)}$ if $|G|$ is odd or if G_p is Abelian. Therefore, all the chief p-factors of G between 1 and H have rank at most $2\overline{\omega c}_p(G)$, and at most $\overline{\omega c}_p(G)$ if $|G|$ is odd or if G is solvable with Abelian G_p.

Clearly, $O_p(G/H)=1$. We therefore can apply induction to G/H and get $r_p(G/H)\leqslant 2\overline{\omega c}_p(G/H)\leqslant 2\overline{\omega c}_p(G)$, and $r_p(G/H)\leqslant\overline{\omega c}_p(G/H)\leqslant\overline{\omega c}_p(G)$ if $|G|$ is odd or if G is solvable with Abelian G_p. It follows that $r_p(G)\leqslant 2\overline{\omega c}_p(G)$ and $r_p(G)\leqslant\overline{\omega c}_p(G)$ if $|G|$ is odd or if G is solvable with Abelian G_p. This completes the proof.

Using Theorem 2 and Huppert's estimates[1], a bound for the p-length of a p-solvable group G, $l_p(G)$, can be obtained in terms of $\overline{\omega c}_p(G)$. Namely,

Theorem 3. *Let G be p-solvable and p be an odd prime. Define $b=1$ if $|G|$ is odd or if G is solvable with Abelian G_p, and $b=2$ if G is an arbitrary p-solvable group. Then*

$$l_p(G/O_p(G))\leqslant\begin{cases}2+\log_p(b\overline{\omega c}_p(G)), & \text{if } p \text{ is not a Fermat prime;}\\ 1+\log_{p-2}(b\overline{\omega c}_p(G)(p-3)+1), & \text{if } p\neq 3, \text{ a Fermat prime;}\\ 2+\log_2(b\overline{\omega c}_p(G)), & \text{if } p=3.\end{cases}$$

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