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Fuzzy tolerance quotient spaces and fuzzy subsets

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Abstract The structure and characteristic of fuzzy subsets are discussed by using the concepts of granulation and hierarchy in quotient space theory. First, the equivalence relation based quotient space theory is extended to the fuzzy tolerance relation. Second, the isomorphism and its discriminant of fuzzy tolerance relations are discussed. Finally, by using the fuzzy tolerance relation to define the fuzzy subset, its properties are addressed. The main results are given below: (1) several equivalent statements of fuzzy tolerance relations; (2) the definition of isomorphism of fuzzy tolerance relations; (3) the isomorphic discriminant of fuzzy tolerance relations; (4) the definition and properties of fuzzy subsets based on the fuzzy tolerance relations; and (5) the necessary and sufficient condition of the isomorphism of fuzzy subsets. These results will help us further comprehend the concepts of fuzzy tolerance relations and fuzzy subsets.

Keywords Fuzzy tolerance relation, quotient space, isomorphism, fuzzy subset

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1 Introduction

1.1 Foreword

Granular computing is one of the hot topics in artificial intelligence recently. There have been three basic theoretical models in granular computing: fuzzy set model [1], rough set model [2] and quotient space model [3].

In the fuzzy set model, the membership function is one of its main concepts. For a given fuzzy subset (or a fuzzy event), different types of membership functions may be chosen by researchers. Therefore, it appears that the results induced by the same fuzzy set theory might be different due to using different membership functions. But in real applications such as fuzzy control, although the control rules and their parameters are obtained from different membership functions, we can almost have the same control performances generally. It shows that fuzzy set methodology has its rationality.

There have been many research results [4–6] on the meaning of fuzzy set membership functions. Most are based on statistics. For example, Liang [4] presented a statistical interpretation of membership functions. Under a certain hypothesis, he proved that the mean of the membership functions defined

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apart will converge to the value of a limit membership function in x (an element in domain) based on the laws of large numbers in statistics. Lucero [5] also regarded a membership function as a random variable so that it will be found via training samples. Mitsuishi [6] introduced some new concepts such as empty fuzzy sets to define the fuzzy set based on membership functions so that the meaning of fuzzy sets will be extended. Lin [7–9] introduced a topological definition of fuzzy sets (topological rough sets) and discussed their structural properties by using the neighborhood system approach. He presented two equivalent definitions of fuzzy membership functions and the necessary and sufficient condition of any two equivalent membership functions. Lin [10] also introduced the concept of granular fuzzy sets. These works help us comprehend the essence of fuzzy sets but cannot explain the conflict between the subjectivity of the choice of fuzzy membership functions and the objectivity of the existence of fuzzy sets.

Both rough set model and quotient space model are based on equivalence relations initially. Since equivalence relations and partitions are equivalent, from the concept of partition, it means that all classes (granules) must mutually disjoin. This requirement is difficult to come by in reality such as clustering in data mining. Therefore, it is needed to extend the equivalence relation based granular computing to more general cases. For rough set model, there have been a lot of works on its extension, e.g., constructing a tolerance approximate space based on tolerance relations. Skowron [11], Doherty [12], Cattaneno [13, 14], and Slowinski [15] discussed fuzzy set theory based on tolerance relations. Shi [16] discussed the concept of tolerance granular spaces and applied it to data mining. Zhu [17] based on rough sets, discussed a tolerance space constructed by coverings and the corresponding upper and lower approximate operations and their properties. Yao [18] introduced the general methodology and art of granulation and granular computing. We [19, 20] discussed the fuzzy quotient space theory and extended the equivalence relation based quotient space theory to the fuzzy tolerance relation based.

Many researchers [19, 21–27] introduced the concept of fuzzy sets to rough set and quotient space theories; and presented the fuzzy rough set and fuzzy quotient space models, respectively. Contrarily, only a few researchers introduced the concepts of granularity and hierarchy in rough set and quotient space theories to the fuzzy set model in order to solve its puzzle. In [21], we presented a primary work on this aspect, i.e., presented a structural definition of fuzzy sets based on the quotient space structural property. In this paper, we first extend the equivalence relation based quotient space theory to the tolerance relation based. Second, we present the structural definition of fuzzy sets by using the fuzzy tolerance relation and discuss its properties. The results can fully explain the relation between the subjectivity of the definition (membership function) of fuzzy sets and the objectivity of the existence of fuzzy sets.

1.2 Fuzzy tolerance relation

First, several equivalent representations of fuzzy tolerance relations are discussed. Zadeh [28] discussed the fuzzy similarity relation that is the same as our fuzzy equivalence relation. The fuzzy ordering that Zadeh defined is the fuzzy relation that satisfies reflexivity, antisymmetry, and transitivity. The two concepts are slightly different from the fuzzy tolerance relation that only satisfies reflexivity and symmetry. So the fuzzy tolerance relation that we will discuss is the extension of the fuzzy similarity relation that Zadeh defined.

For simplicity, the main results that we presented in [19] are shown below.

Definition 1.1. Assume $C_i \subseteq X, i = 1, \ldots$. If $\bigcup_{i=1}^{n} C_i = X$ then $\{C_i | i = 1, \ldots, n\}$ is called a covering of X.

Definition 1.2. Assume $\{C_i | i = 1, \ldots, n\}$ is a covering of X. Construct a function $R : X \times X \to \{0, 1\}$, if $\exists C_i, x, y \in C_i$, let $R(x, y) = 1$; otherwise $R(x, y) = 0$, then R is called the corresponding tolerance relation of covering $C = \{C_i, i = 1, \ldots, n\}$.

Definition 1.3. Assume that function $R : X \times X \to \{0, 1\}([0, 1])$ and satisfies: 1) reflexivity: $\forall x \in X, R(x, x) = 1$; 2) symmetry: $\forall x, y \in X, R(x, y) = R(y, x)$. R is called a (fuzzy) tolerance relation on X.

Definition 1.4. R is a fuzzy tolerance relation on X. For $0 \leqslant \lambda \leqslant 1$, define

$$R_\lambda(x,y) = \begin{cases} 1, & R(x,y) \geqslant \lambda, \\ 0, & R(x,y) < \lambda, \end{cases}$$

R_λ is called an λ_- cut set of R. Obviously, R_λ is a common tolerance relation.

Proposition 1.1. R is a fuzzy tolerance relation on X. For $0 \leqslant \lambda_1 < \lambda_2 \leqslant 1$, we have two cut tolerance relations R_{λ_2} and R_{λ_1}. Then, $R_{\lambda_2} \subseteq R_{\lambda_1}$.

Therefore, given a fuzzy tolerance relation, we have a corresponding chain of hierarchical tolerance relations: $\{R_\lambda, 0 \leqslant \lambda \leqslant 1\}$.

We have the following theorem:

Theorem 1.1. The following statements are equivalent.

1) Given a fuzzy tolerance relation R on X.

2) Given a chain of hierarchical coverings (tolerance) on X: $\{R_\lambda, 0 \leqslant \lambda \leqslant 1\}$.

3) Given a symmetry [0,1] matrix on $X \times X$ with one as its diagonal elements.

4) Given a normalized weight network with X as its nodes.

These four statements depict the fuzzy tolerance relation roundly. The first one is its definition. The second one is its geometrical representation and provides a proper formalized model for hierarchical granular computing. The third representation is convenient for computation. The fourth representation is easy of constructing a fuzzy tolerance quotient space. The four representations supplement each other. In fact, the representation of chains of hierarchical coverings is the essential one whereas the matrix and weighted network representations are not unique. That is, different matrices may correspond to the same chain of hierarchical coverings. We will mainly discuss the structure of families of tolerance relations, i.e., the relationship among every fuzzy tolerance relations and its application to the structural definition of fuzzy subsets.

Definition 1.5. Assume R^1 and R^2 are two fuzzy tolerance relations on X. If their corresponding chains of hierarchical tolerance relations are the same, then R^1 and R^2 are called isomorphic.

2 The structure of chains of fuzzy tolerance relations

2.1 Foreword

In [3], we presented the quotient space theory based on equivalence relation. In [19], we extended the theory to the fuzzy quotient space. In [21], we discussed the four equivalent statements of the fuzzy tolerance relation and pointed out the numerical measure of a fuzzy tolerance relation is not essential. Contrarily, the "semi-order" relation among its elements denoted by the numerical values is essential, i.e., a chain of hierarchical coverings.

When using the matrix representation of the fuzzy tolerance relation, the semi-order relation among its elements can be defined via matrix inequations directly and the supremum and the infimum of subsets can be found by the min and max operations easily. On the other hand, alike to a fuzzy membership function, the fuzzy measurement of a fuzzy tolerance relation is also subjective. The fuzzy value of its every element is relative and only provides the semi-order relation among the elements of the fuzzy tolerance relation. So it is rational to represent the relation among fuzzy tolerance relations by their structure. It is known that a fuzzy tolerance relation is equivalent to a set of hierarchical coverings. The latter is essential. Therefore, we will expatiate how to represent the semi-order relation among fuzzy tolerance relations and how to compute their supremum and infimum under the form of chains of hierarchical coverings

First, let us see an example.

Example 2.1. Two fuzzy tolerance relations R^1 and R^2 are as follows:

$$R^1 = \begin{pmatrix} 1 & \frac{3}{4} & \frac{2}{4} & \frac{1}{4} \\ \frac{3}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{2}{4} & \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix}, \quad R^2 = \begin{pmatrix} 1 & \frac{2}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{2}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix},$$

$$\min(R^1, R^2) = R_* = \begin{pmatrix} 1 & \frac{2}{4} & \frac{2}{4} & \frac{1}{4} \\ \frac{2}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{2}{4} & \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix}, \quad \max(R^1, R^2) = R^* = \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix}.$$

R^1, R^2, R_*, and R^* correspond to the following structures (chains of coverings). Here, we use the same symbol R to indicate both matrix and covering forms of the fuzzy tolerance relations.

$$R^1 = (R^1_1, R^1_{3/4}, R^1_{2/4}, R^1_{1/4}) = (\{1,2,3,4\}, \{(1,2),3,4\}, \{(1,2),(1,3),4\}, \{(1,2,3,4)\}),$$

$$R^1_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R^1_{3/4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R^1_{2/4} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R^1_{1/4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$R^2 = (\{1,2,3,4\}, \{(1,3),2,4\}, \{(1,2),(1,3),4\}, \{(1,2,3,4)\}),$$
$$R_* = (\{1,2,3,4\}, \{(1,3),(1,2),4\}, \{(1,2,3,4)\}),$$
$$R^* = (\{1,2,3,4\}, \{(1,3),(1,2),4\}, \{(1,2,3,4)\}),$$

where R^2_λ, R^*_λ and $R_{*\lambda}$ are represented in covering forms only. For simplicity, we omit the matrix forms.

We have $R_* \subseteq R^1 \subseteq R^*$, $R_* \subseteq R^2 \subseteq R^*$. Then, R_* and R^* actually have the same structure. This means that if some relation of fuzzy tolerance relations is defined by matrix inequations, the relation may not necessarily satisfy the "reflex" condition. So the semi-order relation among fuzzy tolerance relations cannot be simply defined by matrix inequations.

We will present the necessary and sufficient condition of the isomorphism of two fuzzy tolerance relations.

2.2 The discriminant of isomorphic fuzzy tolerance relations

We will discuss in what condition that two fuzzy tolerance relations are isomorphic below, where the isomorphism is defined by Definition 1.5.

Theorem 2.1 (Isomorphism Discriminant). R^1 and R^2 are isomorphic on X. It is equivalent to

$$\forall x, y, u, v \in X, R^1(x,y) < R^1(u,v) \Leftrightarrow R^2(x,y) < R^2(u,v),$$
$$\text{and } R^1(x,y) = R^1(u,v) \Leftrightarrow R^2(x,y) = R^2(u,v).$$

Proof. $\Rightarrow$: Assume $R^1(x,y) < R^1(u,v)$. For λ_1: $R^1(x,y) < \lambda_1 < R^1(u,v)$. Let $X_1(\lambda_1) = \{(x,y)|R^1(x,y) \geqslant \lambda_1$. Then, u and v are tolerant in $X_1(\lambda_1)$ but x and y are not tolerant in $X_1(\lambda_1)$.

Since R^1 and R^2 are isomorphic, there exists $\lambda_2 : X_2(\lambda_2) = X_1(\lambda_1)$. x and y are not tolerant in $X_2(\lambda_2)$ but u and v are tolerant in $X_2(\lambda_2)$. We have $R^2(u,v) \geqslant \lambda_2$, $R^2(x,y) < \lambda_2$. Then, $R^2(x,y) < R^2(u,v)$.

Similarly, we have $R^2(x,y) < R^2(u,v) \Leftrightarrow R^1(x,y) < R^1(u,v)$.

Now, we prove that $R^2(x,y) = R^2(u,v)$. Then, we have $R^1(x,y) = R^1(u,v)$.

By reduction to absurdity, otherwise, we assume that $R^2(x,y) = R^2(u,v)$ and $R^1(x,y) \neq R^1(u,v)$. Assuming $R^1(x,y) < R^1(u,v)$, from the above result we have $R^2(x,y) < R^2(u,v)$. This contradicts with $R^2(x,y) = R^2(u,v)$. Thus, $R^1(x,y) = R^1(u,v)$.

$\Leftarrow$: Let $I_1 = \{\lambda|\exists(x,y), R^1(x,y) = \lambda, 0 \leqslant \lambda \leqslant 1\}$. $\forall\lambda \in I_1$, by letting $D_1(\lambda) = \{(x,y)|R^1(x,y) = \lambda\}$, from the assumption $R^2(x,y) = R^2(u,v) \Leftrightarrow R^1(x,y) = R^1(u,v)$, we have that all values of R^2 on $D_1(\lambda)$ are the same. Letting the values be μ, we may define a function $f(\lambda) = \mu$ on I_1.

$\forall\lambda \in I_1$, letting $X_1(\lambda) = \{(x,y)|R^1(x,y) \geqslant \lambda\}$, we have a hierarchical covering structure $\{X_1(\lambda), \lambda \in I_1\}$.

$\forall\lambda \in I_1$, letting $X_2(f(\lambda)) = \{(x,y)|R^2(x,y) \geqslant f(\lambda)\}$, we have a hierarchical covering structure $\{X_2(f(\lambda)), \lambda \in I_1\}$.

We will prove that the hierarchical covering structures $\{X_1(\lambda), \lambda \in I_1\}$ and $\{X_2(f(\lambda)), \lambda \in I_1\}$ are the same.

For $X_1(\lambda) = \{(x,y)|R(x,y) \geqslant \lambda\}$, assume $(u,v) \in X_1(\lambda)$. We have $R^1(u,v) \geqslant \lambda = R^1(x_1,y_1)$.

On the other hand, from $R^1(u,v) \geqslant R^1(x_1,y_1) \Leftrightarrow R^2(u,v) \geqslant R^2(x_1,y_1) = f(\lambda)$, we have $(u,v) \in X_2(f(\lambda))$.

Similarly, we have that when $(u,v) \in X_2(f(\lambda))$, $(u,v) \in X_1(\lambda)$, that is, the chains of hierarchical covering structures $\{X_1(\lambda), \lambda \in I_1\}$ and $\{X_2(f(\lambda)), \lambda \in I_1\}$ are the same.

Definition 2.1. R^1 and R^2 are two fuzzy tolerance relations. $\forall(x,y) \in X \times X$, define a function $F: F(R^1(x,y)) = R^2(x,y)$. For any $(x,y),(u,v) \in X \times X$, when $R^1(x,y) < R^1(u,v)$, $F(R^1(x,y)) < F(R^1(u,v))$ holds, then F is called strictly increasing.

Corollary 2.1. R^1 and R^2 are two fuzzy tolerance relations. Assume that F defined by Definition 2.1 is a strictly increasing function, fuzzy tolerance relations R^1 and R^2 are isomorphic.

Proof. From the definition of function F and Theorem 2.1, we have the corollary.

Proposition 2.1. R^1 and R^2 are isomorphic. $\Leftrightarrow$ the corresponding function F is a strictly increasing function and $F(R^1(x,y)) = R^2(x,y)$.

The proof is omitted.

3 Fuzzy tolerance relations and fuzzy subsets

We will discuss the relation among all subsets that generated from two isomorphic fuzzy tolerance relations below.

Definition 3.1. Given a fuzzy tolerance relation R on X. For any set A on X, define a corresponding fuzzy set $\underline{A}$ and its membership function $A(x)$:

$$A(x) = \sup\{R(x,y)|y \in A\}. \tag{3.1}$$

The definition shows that a common set A can be extended to a fuzzy subset with A as its core via fuzzy tolerance relation R. The definition also indicates the relation between a common set and its corresponding fuzzy set. This helps us intuitively understand the concept of fuzzy sets.

Definition 3.2. For any fuzzy subset $\underline{A}$, $\mu_{\underline{A}}(x)$ is its membership function. Define an equivalence relation on X, i.e., $R : x \sim y \Leftrightarrow \mu_{\underline{A}}(x) = \mu_{\underline{A}}(y)$. We have a quotient space $[X]_{\underline{A}}$ corresponding to R. Now we define an order relation "$\prec$" on $[X]_{\underline{A}}$ such that $[x] \prec [y] \Leftrightarrow \mu_{\underline{A}}(x) \leqslant \mu_{\underline{A}}(y), x \in [x], y \in [y]$. Then we have a quotient space $([X]_{\underline{A}}, \prec)$, and $([X]_{\underline{A}}, \prec)$ is called a total order quotient space corresponding to fuzzy subset $\underline{A}$.

Definition 3.3. For fuzzy subsets $\underline{A}$ and $\underline{B}$, if their corresponding total order quotient spaces are the same, then $\underline{A}$ and $\underline{B}$ are called isomorphic.

Proposition 3.1. If two fuzzy tolerance relations R^1 and R^2 are isomorphic, then for any set A, two fuzzy subsets $\underline{A}_1$ and $\underline{A}_2$ defined by Definition 3.1 based on R^1 and R^2 respectively are isomorphic.

Proposition 3.2. Assume that R^1, R^2 are two isomorphic fuzzy tolerance relations on X and A, B are two common sets on X. Fuzzy subsets $\underline{A}_1$, $\underline{B}_1$ and $\underline{A}_2$, $\underline{B}_2$ are defined by Definition 3.1 based on R^1 and R^2 respectively. Then $\underline{A}_1 \cup \underline{B}_1$ and $\underline{A}_2 \cup \underline{B}_2$ ($\underline{A}_1 \cap \underline{B}_1$ and $\underline{A}_2 \cap \underline{B}_2$) are isomorphic.

Proof. Assume there are four fuzzy subsets and their corresponding membership functions are $A_1(x)$, $A_2(x)$, $B_1(x)$, and $B_2(x)$, respectively. Let the corresponding membership functions of $\underline{A}_1 \cup \underline{B}_1$ and $\underline{A}_2 \cup \underline{B}_2$ be $C_1(x)$ and $C_2(x)$ respectively. From the definition of logical operations, we have $C_1(x) = \max[A_1(x), B_1(x)]$ and $C_2(x) = \max[A_2(x), B_2(x)]$.

Let $I_{A_1} = \{x|C_1(x) = A_1(x)\}$, $I_{A_2} = \{x|C_2(x) = A_2(x)\}$ We will prove that $I_{A_1} = I_{A_2}$ below.

For $x \in I_1$, $A_1(x) \geqslant B_1(x)$. From Proposition 1.3, it is known that $A_1(x)$ and $A_2(x)$, and $B_1(x)$ and $B_2(x)$ are isomorphic.

Assume $A_1(x) \geqslant B_1(x)$. From the definition of $A_1(x)$, for $\varepsilon > 0$, there exists $y \in A$ such that $R^1(x, y) > A_1(x) - \varepsilon$.

On the other hand, $\forall z \in B$, we have $R^1(x, z) \leqslant B_1(x)$. Let ε be a small number such that $R^1(x, y) > R^1(x, z) - \varepsilon, \forall z \in B$. Since R^1 and R^2 are isomorphic, $R^2(x, y) > R^2(x, z) - \varepsilon, \forall z \in B$.

The supremum of the right-hand side of the above formula is as follows:

$$R^2(x, y) \geqslant \sup\{R^2(x, z) - \varepsilon, \forall z \in B\} = B_2(x) - \varepsilon.$$

Letting $\varepsilon \to 0$ and implementing the supremum operation over y in the above formula, we have

$$A_2(x) = \sup_y\{R(x, y)\} \geqslant B_2(x) \text{ and } A_2(x) \geqslant B_2(x), \text{ i.e., } x \in I_{A_2}.$$

Similarly, when $x \in I_{A_2}$, we have $x \in I_{A_1}$. Finally, $I_{A_1} = I_{A_2}$.

We will prove that $C_1(x)$ and $C_2(x)$ are isomorphic below.

For x, y, $C_1(x) > C_2(y)$, if for $x, y \in I_{A_1}$, $C_1(x) = A_1(x)$ and $C_1(y) = A_1(y)$, then we have $A_1(x) > A_1(y)$.

From $I_{A_1} = I_{A_2}$, we have $C_1(x) = A_1(x)$ and $C_1(y) = A_1(y)$. Since A_1 and A_2 are isomorphic, we have

$$C_2(x) = A_2(x) > A_2(y) = C_2(y).$$

Similarly, when x and y do not belong to I_{A_1}, we can prove the same result.

Finally, we only consider the case: $x \in I_{A_1}$ and $y \notin I_{A_1}$.

Assume $C_1(x) > C_1(y)$. Then $A_1(x) > B_1(y)$. From their definitions, for $\varepsilon > 0$, there exist $z_1 \in A$ and $z_2 \in B$ such that $A_1(x) - \varepsilon < R^1(x, z_1) \leqslant A_1(x)$ and $B_1(x) - \varepsilon < R^1(x, z_2) \leqslant B_1(x)$, respectively. If ε is small enough, then $R^1(y, z_2) \leqslant B_1(y) < A_1(x) - \varepsilon < R^1(x, z_1) \leqslant A_1(x)$, i.e., $\forall z \in B, R^1(y, z) < R^1(x, z_1)$. Since R^1 and R^2 are isomorphic, for $\forall z \in B, R^2(y, z) < R^2(x, z_1)$, we have $C_2(y) = B_2(y) \leqslant A_2(x) = C_2(x)$. If $B_2(y) = A_2(x)$, we have $y \in I_{A_2} = I_{A_1}$. This contradicts $y \notin I_{A_1}$. Finally, we have $C_2(y) = B_2(y) < A_2(x) = C_2(x)$.

Similarly, when $C_1(x) < C_1(y) \Leftrightarrow C_2(x) < C_2(y)$ and $C_1(x) = C_1(y) \Leftrightarrow C_2(x) = C_2(y)$, we have $\underline{A}_1 \cup \underline{B}_1$ and $\underline{A}_2 \cup \underline{B}_2$ are isomorphic. Similarly, $\underline{A}_1 \cap \underline{B}_1$ and $\underline{A}_2 \cap \underline{B}_2$ are isomorphic.

Proposition 3.3. R^1 and R^2 are two isomorphic fuzzy tolerance relations on X. A is a common set on X. Two fuzzy subsets $\underline{A}_1$ and $\underline{A}_2$ are defined by Definition 3.1 based on R^1 and R^2, respectively. Thus, $\underline{A}_1^-$ and $\underline{A}_2^-$ are isomorphic, where $\underline{A}_1^-$ is the complement set of $\underline{A}_1$ and its membership function $\underline{A}_1^-(x) = 1 - \underline{A}_1(x)$.

Theorem 3.1 (Isomorphism principle). R^1 and R^2 are two isomorphic fuzzy tolerance relations on X. $A = \{A_1, \ldots, A_n\}$ is a collection of common subsets. Using R^1 and R^2 to define a collection of fuzzy subsets, we have $\underline{A} = \{\underline{A}_1, \ldots, \underline{A}_n\}$ and $\underline{B} = \{\underline{B}_1, \ldots, \underline{B}_n\}$, respectively. Implementing a finite number of operations (union, intersection, complement) on them, we have a collection of sets of fuzzy subsets $\underline{C} = \{\underline{C}_1, \ldots, \underline{C}_m\}$ and $\underline{D} = \{\underline{D}_1, \ldots, \underline{D}_m\}$, respectively. Thus $\underline{C}$ and $\underline{D}$ are isomorphic as well.

Proof. The conclusion can be obtained from Propositions 3.2 and 3.3.

From the theorem, it is known that the total order quotient space of a fuzzy subset represents its essential property. Thus, when implementing a set of logical operations over the isomorphic fuzzy subsets, although they adapt different membership functions, the fuzzy subsets obtained after the operations are still isomorphic. This means that for a fuzzy subset, although it may be represented by different membership functions, as long as it has the same structure (isomorphic), this does not bring the fuzzy analysis an obvious effect.

Contrarily, if two fuzzy subsets are isomorphic, their corresponding fuzzy tolerance relations do not necessarily isomorphic generally, i.e., the converse theorem of Theorem 3.1 does not hold generally. We give a counterexample below.

Example 3.1.

$$R^1 = \begin{pmatrix} 1 & 0.2 & 0.3 & 0.3 \\ 0.2 & 1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 1 & 0.1 \\ 0.3 & 0.3 & 0.1 & 1 \end{pmatrix} \text{ and } R^2 = \begin{pmatrix} 1 & 0.1 & 0.3 & 0.3 \\ 0.1 & 1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 1 & 0.2 \\ 0.3 & 0.3 & 0.2 & 1 \end{pmatrix}$$

are not isomorphic fuzzy tolerance relations, since $R^1(1,2) = 0.2 > R^1(3,4) = 0.1, R^2(1,2) = 0.1 < R^2(3,4) = 0.2$ does not satisfy the isomorphism discriminant (Theorem 2.1). But we will show below that all fuzzy subsets defined by R^1 and R^2 are isomorphic.

According to Definition 3.1, the structures of fuzzy subsets defined by R^1 and R^2 are as follows:

$$\begin{aligned}
&\mu^1_{\{1\}} = \{1 > (3,4) > 2\},\ \mu^1_{\{2\}} = \{2 > (3,4) > 1\},\ \mu^1_{\{3\}} = \{3 > (1,2) > 4\},\ \mu^1_{\{4\}} = \{4 > (1,2) > 3\}.\\
&\mu^1_{\{1,2\}} = \{(1,2) > (3,4)\},\ \mu^1_{\{1,3\}} = \{(1,3) > (2,4)\}, \ldots,\ \mu^2_{\{1\}} = \{1 > (3,4) > 2\},\\
&\mu^2_{\{2\}} = \{2 > (3,4) > 1\},\ \mu^2_{\{3\}} = \{3 > (1,2) > 4\},\ \mu^2_{\{4\}} = \{4 > (1,2) > 3\}.\\
&\mu^2_{\{1,2\}} = \{(1,2) > (3,4)\},\ \mu^2_{\{1,3\}} = \{(1,3) > (2,4)\}, \ldots,
\end{aligned}$$

where $\mu^1_{\{1\}} = \{1 > (3,4) > 2\}$ denotes the total order set of fuzzy subset $\{1\}_1$ as $(1) > (3,4) > (2)$ (i.e., element 1 in the forefront, followed by elements 3 and 4, element 2 at the end).

It is easy to verify that all fuzzy subsets above are isomorphic. This means that fuzzy tolerance relations with different structures can generate fuzzy subsets with the same structure.

4 The isomorphism analysis of fuzzy tolerance relations and fuzzy subsets

For two fuzzy tolerance relations, it is needed to answer in what condition their corresponding fuzzy subsets defined by Definition 3.1 are isomorphic.

Theorem 4.1. R^1 and R^2 are two fuzzy tolerance relations on X. For any common subset on X, there exists the following necessary and sufficient condition such that all fuzzy subsets defined by Definition 3.1 based on R^1 and R^2 are isomorphic.

1) $\forall x, y, u \in X, R^1(x,y) < R^1(x,u) \Leftrightarrow R^2(x,y) < R^2(x,u)$ and $R^1(x,y) = R^1(x,u) \Leftrightarrow R^2(x,y) = R^2(x,u)$.

2) If $\forall x, y, u, v \in X, R^1(x,y) \geqslant R^1(u,y),\ R^1(x,v) \leqslant R^1(u,v)$ then

$$R^1(x,y) > R^1(u,v) \Leftrightarrow R^2(x,y) > R^2(u,v), \quad R^1(x,y) = R^1(u,v) \Leftrightarrow R^2(x,y) = R^2(u,v).$$

Proof. $\Rightarrow$: From condition 1), when $A = \{x\}$ is a singleton, according to its definition

$$\mu_A^1(y) = R^1(x,y),\ \mu_A^1(v) = R^1(x,v),\ \text{if } \mu_A^1(y) < \mu_A^1(v),\ \text{i.e., } R^1(x,y) < R^1(x,v),$$

and from condition 1), we have $\mu_A^2(y) < \mu_A^2(v)$.

Similarly, we have if $\mu_A^1(y) = \mu_A^1(v)$, then $\mu_A^2(y) = \mu_A^2(v)$. Finally, $\mu_A^1 = \mu_A^2$.

Assume A is a common subset and $\mu_A^1(y) < \mu_A^1(v)$.

$$\begin{aligned}\mu_A^1(y) &= R^1(x,y) = \max\{R^1(z,y)|z \in A\}, x \in A,\\ \mu_A^1(v) &= R^1(u,v) = \max\{R^1(z,v)|z \in A\}, u \in A.\end{aligned} \tag{4.1}$$

Then, we have

$$R^1(x,y) \geqslant R^1(x,v),\ R^1(u,v) \geqslant R^1(x,v),\ R^1(x,y) < R^1(u,v).$$

From condition 1) and formula (4.1), we have

$$\begin{aligned}\mu_A^2(y) &= R^2(x,y) = \max\{R^2(z,y)|z \in A\},\ x \in A,\\ \mu_A^2(v) &= R^2(u,v) = \max\{R^2(z,v)|z \in A\}, u \in A.\end{aligned} \tag{4.2}$$

From condition 2), $R^2(x,y) < R^2(u,v)$. From formula (4.2),

$$\mu_A^2(y) = R^2(x,y) < \mu_A^2(v) = R^2(u,v).$$

Similarly, when $\mu_A^1(y) = \mu_A^1(v)$, $\mu_A^2(y) = \mu_A^2(v)$. Finally, we have that μ_A^1 and μ_A^2 are isomorphic.

$\Leftarrow$: For any A, μ_A^1 and μ_A^2 are isomorphic. Let A be a singleton. It is easy to show that condition 1) in the theorem holds.

If condition 2) of the theorem does not hold, there exist

$$R^1(x,y) \geqslant R^1(u,y), \quad R^1(x,v) \leqslant R^1(u,v),\ \text{and } R^1(x,y) > R^1(u,v). \tag{4.3}$$

But

$$R^2(x,y) \leqslant R^2(u,v). \tag{4.4}$$

For subset $A = \{x,u\}$, from formula (4.3), we have $\mu_A^1(y) = R^1(x,y)$, $\mu_A^1(v) = R^1(u,v)$. Then, we have

$$\mu_A^1(y) > \mu_A^1(v).$$

On the other hand, from the frontal part of formula (4.3) and condition 1), we have

$$R^2(x,y) \geqslant R^2(u,y), \quad R^2(x,v) \leqslant R^2(u,v),$$

then

$$\mu_A^2(y) = R^2(x,y), \quad \mu_A^2(v) = R^2(u,v).$$

From formula (4.4), $\mu_A^2(y) \leqslant \mu_A^2(v)$.

This contradicts the fact that μ_A^1 and $\mu_A^2(y)$ are isomorphic thus condition 2) holds.

Definition 4.1. R^1 and R^2 are two fuzzy tolerance relations. For any common subset A, if all fuzzy subsets $\underline{A}_1$ and $\underline{A}_2$ defined by Definition 3.1 based on R^1 and R^2 are isomorphic, then R^1 and R^2 are called subset-isomorphic.

Theorem 4.1 shows that using two fuzzy tolerance relations to define a set of fuzzy subsets such that the fuzzy subsets after a set of logical operations still remain isomorphic, the necessary and sufficient condition is that the two fuzzy tolerance relations are subset-isomorphic. But Theorem 2.1 only denotes the sufficient condition.

Theorem 4.2 (Isomorphism principle). R^1 and R^2 are two subset-isomorphic fuzzy tolerance relations on X. $S = \{A_1, \ldots, A_n\}$ is a set of common sets. Using R^1 and R^2 to define a set of fuzzy subsets, respectively, we have $\underline{A} = \{\underline{A}_1, \dots, \underline{A}_n\}$ and $\underline{B} = \{\underline{B}_1, \dots, \underline{B}_n\}$. Implementing a set of logical operations (union, intersection, complement) over the subsets, we have a family of sets of fuzzy subsets $\underline{C} = \{\underline{C}_1, \dots, \underline{C}_m\}$, $\underline{D} = \{\underline{D}_1, \dots, \underline{D}_m\}$. Thus, $\underline{C}_i$ and $\underline{D}_i, i = 1, \dots, m$ are isomorphic as well.

Proof. We only deal with the following three cases: $S = \{A, B\}$, $\underline{C} = \{\underline{A}_1 \cup \underline{A}_2\}$, and $\underline{D} = \{\underline{B}_1 \cup \underline{B}_2\}$.

From Theorem 1.2, $\underline{A}_1$, $\underline{A}_2$ and $\underline{B}_1$, $\underline{B}_2$ are isomorphic. We will prove below that $\underline{A}_1 \cup \underline{A}_2$ and $\underline{B}_1 \cup \underline{B}_2$ are isomorphic.

1) When $\mu^1_{A\cup B}(y) = R^1(x_1, y)$, we have $\mu^2_{A\cup B}(y) = R^2(x_1, y)$. Assume that $\mu^1_A(y) = R^1(x_1, y)$, $\mu^1_B(y) = R^1(x_2, y)$, $\mu^2_A(y) = R^2(x_3, y)$, $\mu^2_B(y) = R^2(x_4, y)$, and $\mu^1_A(y) = R^1(x_1, y) \geqslant \mu^1_B(y)$. Since $\mu^1_A(y) = R^1(x_1, y) = \max\{R^1(x, y)|x \in A\}$, and $\underline{A}_1$, $\underline{B}_1$ are isomorphic, we have $\mu^2_A(y) = R^2(x_1, y) = \max\{R^2(x, y)|x \in A\}$.

Similarly, $\mu^2_B(y) = R^2(x_2, y)$.

Since $\mu^1_{A\cup B}(y) = R^1(x_1, y)$, i.e., $\mu^1_A(y) = R^1(x_1, y) \geqslant \mu^1_B(y) = R^1(x_2, y)$, we have

$$\mu^2_A(y) = R^2(x_1, y) \geqslant \mu^2_B(y) = R^2(x_2, y),$$

then

$$\mu^2_{A\cup B}(y) = \mu^2_A(y) = R^2(x_1, y). \tag{4.5}$$

2) Assume $\mu^1_{A\cup B}(y) < \mu^1_{A\cup B}(v)$. If $\mu^1_{A\cup B}(y) = \mu^1_A(y), \mu^1_{A\cup B}(v) = \mu^1_A(v)$,

$$\mu^1_{A\cup B}(y) = \mu^1_A(y) < \mu^1_{A\cup B}(v) = \mu^1_A(v).$$

Therefore,

$$\mu^2_{A\cup B}(y) = \mu^2_A(y) < \mu^2_{A\cup B}(v) = \mu^2_A(v), \text{ i.e., } \mu^2_{A\cup B}(y) < \mu^2_{A\cup B}(v).$$

Now, it is only needed to prove the following case:

$$\mu^1_{A\cup B}(y) = \mu^1_A(y) < \mu^1_{A\cup B}(v) = \mu^1_B(v).$$

From the definitions of x_1, x_2, x_3 and x_4, we have

$$\mu^1_A(y) = R^1(x_1, y) \geqslant \mu^1_B(y) = R^1(x_2, y) \geqslant R^1(x_4, y).$$

Then,

$$R^1(x_1, y) \geqslant R^1(x_4, y), R^1(x_1, v) \leqslant R^1(x_3, v) \leqslant R^1(x_4, v), \text{ and } R^1(x_1, y) < R^1(x_4, v).$$

From condition 2) in Theorem 4.1 and formula (4.6), we have

$$\mu^2_{A\cup B}(y) = R^2(x_1, y) < R^2(x_4, v) = \mu^2_{A\cup B}(v), \text{ then } \mu^2_{A\cup B}(y) < \mu^2_{A\cup B}(v).$$

Similarly, if $\mu^1_{A\cup B}(y) = \mu^1_{A\cup B}(v)$, then $\mu^2_{A\cup B}(y) = \mu^2_{A\cup B}(v)$. We have that $\mu^1_{A\cup B}$ and $\mu^2_{A\cup B}$ are isomorphic.

Similarly, we have that $\mu^1_{A\cap B}$ and $\mu^2_{A\cap B}$, $\mu^1_{\bar{A}}$ and $\mu^2_{\bar{A}}$ are isomorphic, respectively, where $\bar{A}(x) = 1 - A(x)$.

Theorem 4.3. The necessary and sufficient condition that two fuzzy tolerance relations R^1, R^2 are subset-isomorphic is as follows:

The corresponding fuzzy sets of all subsets with cardinality $\leqslant 2$ are isomorphic.

Proof. $\Rightarrow$: Obtained from the definition of subset-isomorphic directly.

$\Leftarrow$: Theorem 4.1 shows that we only need to prove that conditions 1) and 2) of Theorem 4.1 hold.

First, consider condition 1). For $x, y, u \in X$, assume $R^1(x, y) < R^1(x, u)$. Letting $A = \{x\}$, we have $\mu^1_A(y) = R^1(x, y) < R^1(x, u) = \mu^1_A(u)$. Since μ^1_A and μ^2_A are isomorphic, $\mu^2_A(y) < \mu^2_A(u)$, and $\mu^2_A(y) = R^2(x, y), \mu^2_A(u) = R^2(x, u)$, we have $R^2(x, y) < R^2(x, u)$.

Thus, condition 1) holds.

Next, we prove that condition 2) holds as well. By reduction to absurdity, if condition 2) does not hold, then

$$\exists x, y, u, v \in X, \ R^1(x, y) \geqslant R^1(u, y), \ R^1(x, v) \leqslant R^1(u, v). \tag{4.6}$$

But when

$$R^1(x,y) > R^1(u,v),\ R^2(x,y) \leqslant R^2(u,v), \tag{4.7}$$

$$(\text{Or when } R^1(x,y) = R^1(u,v),\ R^2(x,y) \neq R^2(u,v)).$$

For $A = \{x,u\}$, from (4.6), (4.7), we have $\mu^1_{\underline{A}}(y) = R^1(x,y)$, $\mu^1_{\underline{A}}(v) = R^1(u,v)$, and $\mu^1_{\underline{A}}(y) > \mu^1_{\underline{A}}(v)$.
From (4.6) and condition 1) of Theorem 4.1 (we have proved that condition 1) holds), we have

$$R^2(x,y) \geqslant R^2(u,y), \quad R^2(x,v) \leqslant R^2(u,v), \text{ and } \mu^2_{\underline{A}}(y) = R^2(x,y),\ \mu^2_{\underline{A}}(v) = R^2(u,v).$$

From formula (4.7) again, we have $\mu^2_{\underline{A}}(y) \leqslant \mu^2_{\underline{A}}(v)$.

This contradicts the fact that $\mu^1_{\underline{A}}$ and $\mu^2_{\underline{A}}$ are isomorphic. Therefore, condition 2) of Theorem 4.1 holds.

Finally, as a result of Theorem 4.1, fuzzy tolerance relations R^1 and R^2 are subset-isomorphic.

The theorem indicates that in order to show that two fuzzy tolerance relations are subset-isomorphic; it is only needed to verify that all subsets with cardinality $\leqslant 2$ are isomorphic. This will reduce its computational complexity greatly. The conditions of Theorem 4.2 cannot be cut down, for example, the conditions cannot be replaced by "the corresponding fuzzy sunsets of all singletons are isomorphic". The following is a counterexample.

Example 4.1. Assume

$$R^1 = \begin{pmatrix} 1 & 0.4 & 0.3 & 0.3 \\ 0.4 & 1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 1 & 0.5 \\ 0.3 & 0.3 & 0.5 & 1 \end{pmatrix} \text{ and } R^2 = \begin{pmatrix} 1 & 0.5 & 0.3 & 0.3 \\ 0.5 & 1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 1 \end{pmatrix}.$$

Then, $[1]_1 = \{(1),(2),(3,4)\} = [1]_2$, $[2]_1 = \{(2),(1),(3,4)\} = [2]_2$, $[3]_1 = \{(3),(4),(1,2)\} = [3]_2, [4]_1 = \{(4),(3),(1,2)\} = [4]_2$. The two structures are the same.

But $[1,4]_1 = \{(1,4) > (3) > (2)\} \neq [1,4]_2 = \{(1,4) > (2) > (3)\}$.

Therefore, we cannot have the conclusion: "The corresponding fuzzy subsets of all subsets with cardinality $\leqslant 2$ are isomorphic".

5 Conclusions

We analyze the structure and characteristics of fuzzy sets by using the concepts of granularity and hierarchy in the quotient space theory in order to solve the puzzle in fuzzy set theory. First, we discuss several equivalent statements of fuzzy tolerance relations and show that the hierarchical representation is essential. We present the isomorphic definition of two fuzzy tolerance relations and their isomorphic discriminant. Second, based on the analysis of the structure and characteristics of fuzzy sets, we present the definition of fuzzy subsets by using fuzzy tolerance relations and the necessary and sufficient condition that two fuzzy tolerance relations are subset-isomorphic. Third, we show that after implementing a finite number of logical operations over the above defined fuzzy subsets, the obtained fuzzy subsets are still isomorphic. The result answers the following question: why the fuzzy processing is robust? i.e., for a fuzzy set, although users adopt different membership functions, after implementing a set of logical operations we still can have the same structure as long as we have the same understanding of its structure.

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