# Bifurcations of rough heteroclinic loop with two saddle points

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Received April 16, 2002; revised September 30, 2002

**Abstract** The bifurcation problems of rough 2-point-loop are studied for the case $\rho_1^1 > \lambda_1^1$, $\rho_2^1 < \lambda_2^1$, $\rho_1^1\rho_2^1 < \lambda_1^1\lambda_2^1$, where $-\rho_i^1 < 0$ and $\lambda_i^1 > 0$ are the pair of principal eigenvalues of unperturbed system at saddle point $p_i$, $i = 1, 2$. Under the transversal and nontwisted conditions, the authors obtain some results of the existence of one 1-periodic orbit, one 1-periodic and one 1-homoclinic loop, two 1-periodic orbits and one 2-fold 1-periodic orbit. Moreover, the bifurcation surfaces and the existence regions are given, and the corresponding bifurcation graph is drawn.

**Keywords: local coordinates, Poincaré map, 1-homoclinic loop, 1-periodic orbit, 2-fold 1-periodic orbit, bifurcation surface.**

## 1 Introduction and hypotheses

As well known, in the last several decades, lots of papers and books were devoted to the study of the bifurcation problems of homoclinic and heteroclinic loops of planar systems[1−10].

In recent years, the bifurcations of homoclinic and heteroclinic loops for higher dimensional cases were also studied extensively[11−18]. In this paper, we consider the bifurcations of rough heteroclinic loop with two hyperbolic saddle points (abbr. 2-point-loop) in higher dimensional space.

Consider the following $C^r$ system

$$\dot{z} = f(z) + g(z, \mu), \tag{1.1}$$

and its unperturbed system

$$\dot{z} = f(z), \tag{1.2}$$

where $r \geqslant 4$, $z \in \mathbb{R}^{m+n}$, $\mu \in \mathbb{R}^l$, $l \geqslant 2$, $0 \leqslant |\mu| \ll 1$, $g(z, 0) = 0$. For $i = 1, 2$, we assume that $f(p_i) = 0$, $g(p_i, \mu) = 0$ and

**(H1)** $z = p_i$ is a hyperbolic critical point of (1.2). The $C^r$ stable manifold $W_i^s$ and the unstable manifold $W_i^u$ of $z = p_i$ are $m$-dimensional and $n$-dimensional, respectively. Moreover, $-\rho_i^1$ and $\lambda_i^1$ are the simple real eigenvalues of $D_z f(p_i)$ such that any remaining eigenvalues $-\rho_i^j$ and $\lambda_i^k$ of $D_z f(p_i)$ satisfy

$$-\mathrm{Re}\rho_i^j < -\rho_i^0 < -\rho_i^1 < 0 < \lambda_i^1 < \lambda_i^0 < \mathrm{Re}\lambda_i^k, \tag{1.3}$$

where $2 \leqslant j \leqslant m$, $2 \leqslant k \leqslant n$, $\rho_i^0$ and $\lambda_i^0$ are some positive constants.

**(H2)** System (1.2) has a heteroclinic loop $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i = \{z = r_i(t) : t \in \mathbb{R}\}$, $r_i(+\infty) = r_{i+1}(-\infty) = p_{i+1}$, $r_3(t) = r_1(t)$, $p_3 = p_1$. For any point $P_i \in \Gamma_i$, $\dim(T_{P_i}W_i^u \cap T_{P_i}W_{i+1}^s) = 1$, $W_3^s = W_1^s$.

**(H3)** Define $e_i^{\pm} = \lim_{t\to\mp\infty} \dot{r}_i(t)/|\dot{r}_i(t)|$. Then, $e_i^+ \in T_{p_i}W_i^u$ and $e_i^- \in T_{p_{i+1}}W_{i+1}^s$ are the unit eigenvectors corresponding to $\lambda_i^1$ and $-\rho_{i+1}^1$, respectively.

**(H4)** $\operatorname{span}(T_{r_i(t)}W_i^u, T_{r_i(t)}W_{i+1}^s, e_{i+1}^+) = \mathbb{R}^{m+n}, \quad t \gg 1,$

$\operatorname{span}(T_{r_i(t)}W_i^u, T_{r_i(t)}W_{i+1}^s, e_{i-1}^-) = \mathbb{R}^{m+n}, \quad t \ll -1,$

where $e_3^+ = e_1^+$, $e_0^- = e_2^-$.

(H4) is called the strong inclination property, which is equivalent to

$$\lim_{t\to+\infty}(T_{r_i(t)}W_i^u + T_{r_i(t)}W_{i+1}^s) = T_{p_{i+1}}W_{i+1}^{uu} + T_{p_{i+1}}W_{i+1}^s,$$
$$\lim_{t\to-\infty}(T_{r_i(t)}W_i^u + T_{r_i(t)}W_{i+1}^s) = T_{p_i}W_i^u + T_{p_i}W_i^{ss},$$

where $W_i^{ss}$ and $W_i^{uu}$ are the strong stable and the strong unstable manifolds of $p_i$, respectively. $W_i^{ss} \subset W_i^s$ is the $(m-1)$-dimensional solution manifold tangent to the generalized eigenspace corresponding to those eigenvalues with smaller real part than $-\rho_i^0$, and $W_i^{uu} \subset W_i^u$ is the $(n-1)$-dimensional solution manifold tangent to the generalized eigenspace corresponding to those eigenvalues with larger real part than $\lambda_i^0$.

It is easy to see that under hypothesis (H1), hypotheses (H2)—(H4) are generic.

Under hypotheses (H1)——(H4), Zhu and Xia[17] studied the bifurcation problems of 2-point-loop for the case $\rho_i^1 > \lambda_i^1$, $i = 1, 2$. And in ref. [18], Tian and Zhu studied the fine 2-point-loop. In this paper, we study the bifurcations of rough 2-point-loop for the case $\rho_1^1 > \lambda_1^1$, $\rho_2^1 < \lambda_2^1$, and $(\rho_1^1\rho_2^1)/(\lambda_1^1\lambda_2^1) < 1$. Under some transversal conditions and the nontwisted condition, we discuss the existence of one 1-periodic orbit, one 1-periodic and one 1-homoclinic loop, two 1-periodic orbits and one 2-fold 1-periodic orbit. Moreover, the relative bifurcation surfaces and the existence regions are given, and the corresponding bifurcation graph is drawn.

Our results show that the bifurcation pattern of the rough loop in case $(\rho_1^1 - \lambda_1^1)(\rho_2^1 - \lambda_2^1) < 0$ studied here is much more complicated than that in case $(\rho_1^1 - \lambda_1^1)(\rho_2^1 - \lambda_2^1) > 0$ studied by ref. [17].

## 2 Local coordinates and bifurcation equations

In this section, we will select the linear independent solutions of the linear variational equation along $\Gamma_i$ as the demanded local coordinates to construct the Poincaré map $F_i$ which will be the composition of two maps. One of the maps, $F_i^0$, will be induced by the flow near $p_i$ (approximately, which will be taken as the flow generated by the linearization of (1.1) at $p_i$). The other map, $F_i^1$, will be constructed from the flow outside a neighborhood of $p_i$ and in a sufficiently small tube neighborhood of $\Gamma_i$. The Poincaré map, $F_i$, will then be given by $F_i = F_i^1 \circ F_i^0$. This method is similar to that of ref. [17], but is much easier than that of ref. [17].

As well known, $W_i^s$ and $W_i^u$ are $C^r$ (refs. [19, 20]). Suppose that the neighborhood $U_i$ of $p_i$ is small enough, then there always exists a $C^r$ transformation such that system (1.1) has the following form in $U_i$:

$$\begin{aligned}
\dot{x} &= [\lambda_i^1(\mu)+\cdots\cdots]\,x+O(u)[O(u)+O(y)+O(v)],\\
\dot{y} &= [-\rho_i^1(\mu)+\cdots\cdots]\,y+O(v)[O(v)+O(x)+O(u)],\\
\dot{u} &= [B_i^1(\mu)+\cdots\cdots]\,u+O(x)[O(x)+O(y)+O(v)],\\
\dot{v} &= [-B_i^2(\mu)+\cdots\cdots]\,v+O(y)[O(y)+O(x)+O(u)],
\end{aligned} \tag{2.1}$$

for $|\mu|$ sufficiently small, where $\lambda_i^1(0)=\lambda_i^1$, $\rho_i^1(0)=\rho_i^1$, $\mathrm{Re}\sigma(B_i^1(0))>\lambda_i^0$, $\mathrm{Re}\sigma(-B_i^2(0))<-\rho_i^0$, $z=(x,y,u^*,v^*)^*$, $x\in\mathbb{R}^1$, $y\in\mathbb{R}^1$, $u\in\mathbb{R}^{n-1}$, $v\in\mathbb{R}^{m-1}$, and (2.1) is $C^{r-1}$. Here, the sign $*$ means transposition. In $U_i$, we have

$$\begin{aligned}
&W_i^u=\{z:y=0,v=0\}, && W_i^s=\{z:x=0,u=0\},\\
&W_i^{uu}=\{z:x=x(u),y=0,v=0\}, && W_i^{ss}=\{z:x=0,u=0,y=y(v)\},\\
&\Gamma\cap W_i^u=\{z:u=u(x),y=0,v=0\}, && \Gamma\cap W_i^s=\{z:x=0,u=0,v=v(y)\},
\end{aligned}$$

where $u(0)=\dot{u}(0)=0$, $v(0)=\dot{v}(0)=0$, $x(0)=\dot{x}(0)=0$, $y(0)=\dot{y}(0)=0$.

Denote $r_i(t)=(r_i^x(t),r_i^y(t),(r_i^u(t))^*,(r_i^v(t))^*)^*$. Assume $r_i(-T_i^0)=(\delta,0,\delta_{u_i}^*,0^*)^*$, $r_i(T_i^1)=(0,\delta,0^*,\delta_{v_i}^*)^*$, where $\delta>0$ is small enough such that $\{(x,y,u^*,v^*)^*:|x|,\ |y|,\ |u|,\ |v|<2\delta\}\subset U_i$. Obviously, $|\delta_{u_i}|=o(\delta)$, $|\delta_{v_i}|=o(\delta)$. Consider the linear variational system

$$\dot{z}=Df(r_i(t))z \tag{2.2}$$

and its adjoint system

$$\dot{\phi}=-(Df(r_i(t)))^*\phi. \tag{2.3}$$

By ref. [16], we see (2.2) and (2.3) have exponential dichotomies in both $\mathbb{R}^-$ and $\mathbb{R}^+$. Similar to that of ref. [17], system (2.2) has a fundamental solution matrix $Z_i(t)=(z_i^1(t),z_i^2(t),\ z_i^3(t),z_i^4(t))$ satisfying

$$\begin{aligned}
&z_i^1(t)\in(T_{r_i(t)}W_i^u)^c\cap(T_{r_i(t)}W_{i+1}^s)^c, z_i^2(t)=-\dot{r}_i(t)/|\dot{r}_i^y(T_i^1)|\in T_{r_i(t)}W_i^u\cap T_{r_i(t)}W_{i+1}^s,\\
&z_i^3(t)=(z_i^{3,1}(t),\cdots,z_i^{3,n-1}(t))\in T_{r_i(t)}W_i^u\cap(T_{r_i(t)}W_{i+1}^s)^c,\\
&z_i^4(t)=(z_i^{4,1}(t),\cdots,z_i^{4,m-1}(t))\in(T_{r_i(t)}W_i^u)^c\cap T_{r_i(t)}W_{i+1}^s,
\end{aligned}$$

$$Z_i(-T_i^0)=\begin{pmatrix} w_i^{11} & w_i^{21} & 0 & w_i^{41}\\ w_i^{12} & 0 & 0 & w_i^{42}\\ w_i^{13} & w_i^{23} & I & w_i^{43}\\ 0 & 0 & 0 & w_i^{44}\end{pmatrix},\quad Z_i(T_i^1)=\begin{pmatrix} 1 & 0 & w_i^{31} & 0\\ 0 & 1 & w_i^{32} & 0\\ 0 & 0 & w_i^{33} & 0\\ w_i^{14} & w_i^{24} & w_i^{34} & I\end{pmatrix},$$

where $W_3^{ss}=W_1^{ss}$, $w_i^{21}<0$, $w_i^{12}\neq 0$, $\det w_i^{44}\neq 0$, $\det w_i^{33}\neq 0$. Moreover, for $\delta$ small enough, $||w_i^{1j}(w_i^{12})^{-1}||\ll 1$ for $j\neq 2$, $||w_i^{2j}(w_i^{21})^{-1}||\ll 1$ for $j=3,4$, $||w_i^{3j}(w_i^{33})^{-1}||\ll 1$ for $j\neq 3$, $||w_i^{4j}(w_i^{44})^{-1}||\ll 1$ for $j\neq 4$.

Thus, we may select $z_i^1(t)$, $z_i^2(t)$, $z_i^3(t)$, $z_i^4(t)$ as a local coordinate system along $\Gamma_i$. Let $\Phi_i(t)=(\phi_i^1(t),\phi_i^2(t),\phi_i^3(t),\phi_i^4(t))=(Z_i^{-1}(t))^*$ be the fundamental solution matrix of (2.3).

Denote $w_i^{12}=\Delta_i|w_i^{12}|$. We say that $\Gamma$ is nontwisted as $\Delta=\Delta_1\Delta_2=1$, and twisted as $\Delta=-1$. In this paper, we only consider the case $\Delta=1$. The other cases are similar.

Let $N_i=(n_i^1,0,(n_i^3)^*,(n_i^4)^*)^*$, $n_i^3=(n_i^{3,1},\cdots,n_i^{3,n-1})^*$, $n_i^4=(n_i^{4,1},\cdots,n_i^{4,m-1})^*$, $h_i(t)=r_i(t)+Z_i(t)N_i$. We define $S_i^0=\{z=h_i(-T_i^0):|x|,|y|,|u|,|v|<2\delta\}$, $S_i^1=\{z=h_i(T_i^1):|x|,|y|,|u|,|v|<2\delta\}$ as the cross sections of $\Gamma_i$ at $t=-T_i^0$ and $t=T_i^1$, respectively.

Now, we consider the map $F_i^0 : q_{i-1}^1 \in S_{i-1}^1 \mapsto q_i^0 \in S_i^0$, and the map $F_i^1 : q_i^0 \in S_i^0 \mapsto q_i^1 \in S_i^1$, where $S_0^1 = S_2^1$, $q_0^1 = q_2^1$. Denote

$$q_i^0 = (x_i^0, y_i^0, (u_i^0)^*, (v_i^0)^*)^* = r_i(-T_i^0) + Z_i(-T_i^0)N_i^0, N_i^0 = (n_i^{0,1}, 0, (n_i^{0,3})^*, (n_i^{0,4})^*)^*,$$
$$q_i^1 = (x_i^1, y_i^1, (u_i^1)^*, (v_i^1)^*)^* = r_i(T_i^1) + Z_i(T_i^1)N_i^1, N_i^1 = (n_i^{1,1}, 0, (n_i^{1,3})^*, (n_i^{1,4})^*)^*.$$

By the expressions of $Z_i(-T_i^0)$ and $Z_i(T_i^1)$, $i = 1, 2$, we get $y_i^1 \approx \delta$, $x_i^0 \approx \delta$, and

$$\begin{aligned} n_i^{1,1} &= x_i^1 - w_i^{31}(w_i^{33})^{-1}u_i^1, \\ n_i^{1,3} &= (w_i^{33})^{-1}u_i^1, \\ n_i^{1,4} &= -w_i^{14}x_i^1 + (w_i^{14}w_i^{31} + w_i^{24}w_i^{32} - w_i^{34})(w_i^{33})^{-1}u_i^1 + v_i^1 - \delta_{v_i}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} n_i^{0,1} &= (w_i^{12})^{-1}(y_i^0 - w_i^{42}(w_i^{44})^{-1}v_i^0), \\ n_i^{0,3} &= u_i^0 - \delta_{u_i} + b_i(w_i^{12})^{-1}y_i^0 + [a_i^3 - w_i^{23}(w_i^{21})^{-1}a_i^1](w_i^{44})^{-1}v_i^0, \\ n_i^{0,4} &= (w_i^{44})^{-1}v_i^0, \end{aligned} \tag{2.5}$$

where $b_i = w_i^{11}w_i^{23}(w_i^{21})^{-1} - w_i^{13}$, $a_i^j = w_i^{1j}(w_i^{12})^{-1}w_i^{42} - w_i^{4j}$, $j = 1, 3$.

Suppose that $z = h_i(t)$ is the solution of (1.1) in some small tube neighborhood of $\Gamma_i$. Substituting it into (1.1) and using $\dot{r}_i(t) = f(r_i(t))$, $\dot{Z}_i(t) = Df(r_i(t))Z_i(t)$ and some simple calculation, we obtain the maps $F_i^1$ defined by

$$n_i^{1,j} = n_i^{0,j} + M_i^j\mu + h.o.t., \quad j = 1, 3, 4, \tag{2.6}$$

where $M_i^j = \int_{-\infty}^{+\infty} \phi_i^{j*}(t)g_\mu(r_i(t), 0)dt$, $i = 1, 2$, $j = 1, 3, 4$ are called Melnikov vectors.

Suppose that

**(H5)** $\beta_1 = \rho_1^1/\lambda_1^1 > 1$, $\beta_2 = \rho_2^1/\lambda_2^1 < 1$, $\beta_1\beta_2 < 1$.

By (H5) and the continuity of $\beta_i(\mu) = \rho_i^1(\mu)/\lambda_i^1(\mu)$, $i = 1, 2$, we have $\beta_1(\mu) > 1$, $\beta_2(\mu) < 1$ and $\beta_1(\mu)\beta_2(\mu) < 1$ for $|\mu| \ll 1$. We may as well denote $\beta_i(\mu) := \beta_i$, $i = 1, 2$. Let $\tau_i$ be the flying time from $q_{i-1}^1$ to $q_i^0$, $s_1 = e^{-\lambda_1^1(\mu)\tau_1}$, $s_2 = e^{-\rho_2^1(\mu)\tau_2}$. Then, using the linearization of (1.1) at $p_i$, we can easily get $F_1^0$ defined by

$$x_0^1 \approx s_1\delta, \quad y_1^0 \approx s_1^{\beta_1}\delta, \quad u_0^1 \approx s_1^{B_1^1(\mu)/\lambda_1^1(\mu)}u_1^0, \quad v_1^0 \approx s_1^{B_1^2(\mu)/\lambda_1^1(\mu)}v_0^1, \tag{2.7}$$

and $F_2^0$ defined by

$$x_1^1 \approx s_2^{1/\beta_2}\delta, \quad y_2^0 \approx s_2\delta, \quad u_1^1 \approx s_2^{B_2^1(\mu)/\rho_2^1(\mu)}u_2^0, \quad v_2^0 \approx s_2^{B_2^2(\mu)/\rho_2^1(\mu)}v_1^1, \tag{2.8}$$

if we neglect the higher order terms, $(s_i, u_i^0, v_{i-1}^1)$, $i = 1, 2$ are called Silnikov coordinates.

Thus, we have defined the Poincaré map $F_1 = F_1^1 \circ F_1^0$: $S_2^1 \mapsto S_1^1$ as

$$\begin{aligned} n_1^{1,1} &= (w_1^{12})^{-1}\delta s_1^{\beta_1} + M_1^1\mu + \text{h.o.t.}, \\ n_1^{1,3} &= u_1^0 - \delta_{u_1} + b_1(w_1^{12})^{-1}\delta s_1^{\beta_1} + M_1^3\mu + \text{h.o.t.}, \\ n_1^{1,4} &= (w_1^{44})^{-1}s_1^{B_1^2(\mu)/\lambda_1^1(\mu)}v_0^1 + M_1^4\mu + \text{h.o.t.}, \end{aligned} \tag{2.9}$$

and $F_2 = F_2^1 \circ F_2^0$: $S_1^1 \mapsto S_2^1$ as

$$\begin{aligned} n_2^{1,1} &= (w_2^{12})^{-1}\delta s_2 + M_2^1\mu + \text{h.o.t.}, \\ n_2^{1,3} &= u_2^0 - \delta_{u_2} + b_2(w_2^{12})^{-1}\delta s_2 + M_2^3\mu + \text{h.o.t.}, \\ n_2^{1,4} &= (w_2^{44})^{-1}s_2^{B_2^2(\mu)/\rho_2^1(\mu)}v_1^1 + M_2^4\mu + \text{h.o.t.}. \end{aligned} \tag{2.10}$$

Let $G_i(q^1_{i-1}) = (G^1_i, G^3_i, G^4_i) = F_i(q^1_{i-1}) - q^1_i$, $q^1_0 = q^1_2$. Owing to (2.4), (2.7)—(2.10), we get the successor functions $G_i$ as follows:

$$
\begin{aligned}
G^1_1 &= \delta[(w^{12}_1)^{-1} s_1^{\beta_1} - s_2^{1/\beta_2}] + M^1_1\mu + \text{h.o.t.},\\
G^3_1 &= u^0_1 - \delta_{u_1} + b_1(w^{12}_1)^{-1}\delta s_1^{\beta_1} - (w^{33}_1)^{-1} s_2^{B^1_2(\mu)/\rho^1_2(\mu)} u^0_2 + M^3_1\mu + \text{h.o.t.},\\
G^4_1 &= -v^1_1 + \delta_{v_1} + w^{14}_1\delta s_2^{1/\beta_2} + (w^{44}_1)^{-1} s_1^{B^2_1(\mu)/\lambda^1_1(\mu)} v^1_0 + M^4_1\mu + \text{h.o.t.},\\
G^1_2 &= \delta[(w^{12}_2)^{-1} s_2 - s_1] + M^1_2\mu + \text{h.o.t.},\\
G^3_2 &= u^0_2 - \delta_{u_2} + b_2(w^{12}_2)^{-1}\delta s_2 - (w^{33}_2)^{-1} s_1^{B^1_1(\mu)/\lambda^1_1(\mu)} u^0_1 + M^3_2\mu + \text{h.o.t.},\\
G^4_2 &= -v^1_0 + \delta_{v_2} + w^{14}_2\delta s_1 + (w^{44}_2)^{-1} s_2^{B^2_2(\mu)/\rho^1_2(\mu)} v^1_1 + M^4_2\mu + \text{h.o.t.}.
\end{aligned}
\tag{2.11}
$$

We call the following equation the bifurcation equation:

$$(G^1_1, G^3_1, G^4_1, G^1_2, G^3_2, G^4_2) = 0. \tag{2.12}$$

Thus, there is a 1-1 correspondence between the 2-point-loop, 1-homoclinic and 1-periodic orbit of (1.1) and the solution $Q = (s_1, s_2, u^0_1, u^0_2, v^1_1, v^1_2)$ of (2.12) with $s_1 \geqslant 0$, $s_2 \geqslant 0$.

**Remark 2.1.** Under hypothesis (H5), $G$ is $C^{r-2}$ with respect to $Q$ and $\mu$ in the region $s_1 \geqslant 0$, $s_2 \geqslant 0$.

## 3 Nontwisted bifurcations

Now, we consider the bifurcations near $\varGamma$ under hypotheses (H1)—(H5). Consider the solution of (2.12). It is easy to see that the equation $(G^3_1, G^4_1, G^3_2, G^4_2) = 0$ always has a solution $u^0_i = u^0_i(s_1, s_2, \mu)$, $v^1_i = v^1_i(s_1, s_2, \mu)$ $i = 1, 2$ for $\delta$, $|\mu|$, $s_1$, $s_2$ sufficiently small. Substituting it into $(G^1_1, G^1_2) = 0$, we obtain

$$
\begin{aligned}
s_2^{1/\beta_2} &= (w^{12}_1)^{-1} s_1^{\beta_1} + \delta^{-1} M^1_1\mu + \text{h.o.t.},\\
s_1 &= (w^{12}_2)^{-1} s_2 + \delta^{-1} M^1_2\mu + \text{h.o.t.}.
\end{aligned}
\tag{3.1}
$$

Firstly, if (3.1) has solution $s_1 = s_2 = 0$, then we have

$$M^1_i\mu + h.o.t. = 0, \quad i = 1, 2. \tag{3.2}$$

If $M^1_i \neq 0$, then by (3.2) and the implicit function theorem, there exists a $(l-1)$-dimensional surface $L_i$ with a normal vector $M^1_i$ at $\mu = 0$, such that the $i$th equation of (3.1) has solution $s_1 = s_2 = 0$ as $\mu \in L_i$ and $|\mu| \ll 1$, that is, $\varGamma_i$ is persistent.

If $\operatorname{rank}(M^1_1, M^1_2) = 2$, then $L_{12} = L_1 \cap L_2$ is a $(l-2)$-dimensional surface (see ref. [21]) such that (3.1) has solution $s_1 = s_2 = 0$ as $\mu \in L_{12}$ and $|\mu| \ll 1$, equivalently, system (1.1) has a 1-heteroclinic loop near $\varGamma$.

Secondly, suppose that (3.1) has solution $s_1 = 0$, $s_2 > 0$. Then we have

$$
\begin{aligned}
&s_2 = -\delta^{-1} w^{12}_2 M^1_2\mu + \text{h.o.t.},\\
&(-\delta^{-1} w^{12}_2 M^1_2\mu + h.o.t.)^{1/\beta_2} = \delta^{-1} M^1_1\mu + \text{h.o.t.}.
\end{aligned}
\tag{3.3}
$$

If $M^1_1\mu > 0$, $\varDelta_2 M^1_2\mu < 0$, then following from the implicit function theorem, there exists a $(l-1)$-dimensional surface $L^2_1$ with a normal vector $M^1_1$ at $\mu = 0$, which means $L^2_1$ is tangent to $L_1$ at $\mu = 0$, such that (3.1) has a solution $s_1 = 0$, $s_2 = s_2(\mu) > 0$ as $\mu \in L^2_1$ and $|\mu| \ll 1$. That is to say, system (1.1) has a 1-homoclinic loop $\varGamma^2_1$ near $\varGamma$ homoclinic to $p_1$ as $\mu \in L^2_1$ and $|\mu| \ll 1$.

In the same way, we can discuss the case $s_1 > 0$, $s_2 = 0$ and obtain the surface $L_2^1$ in the region $M_2^1\mu > 0$, $\Delta_1 M_1^1\mu < 0$ and $|\mu| \ll 1$ which is tangent to $L_1$ at $\mu = 0$ such that system (1.1) has a 1-homoclinic loop $\Gamma_2^1$ near $\Gamma$ homoclinic to $p_2$ as $\mu \in L_2^1$.

Denote $R_1^2 = \{\mu : M_1^1\mu > 0,\ \Delta_2 M_2^1\mu < 0,\ |\mu| \ll 1\}$, $R_2^1 = \{\mu : \Delta_1 M_1^1\mu < 0,\ M_2^1\mu > 0,\ |\mu| \ll 1\}$. Then, we have shown the following theorem.

**Theorem 3.1.** Suppose that hypotheses (H1)—(H5) are valid, and $\mathrm{rank}(M_1^1, M_2^1) = 2$. Then

(1) There exists a $(l-1)$-dimensional surface $L_i$ with normal vector $M_i^1$ at $\mu = 0$, such that (1.1) has a heteroclinic orbit joining $p_1$ and $p_2$ near $\Gamma_i$ if and only if $\mu \in L_i$ and $|\mu| \ll 1$, $i = 1, 2$. Moreover, (1.1) has a 1-heteroclinic loop near $\Gamma$ if and only if $|\mu| \ll 1$ and $\mu \in L_{12} = L_1 \cap L_2$ which is a $(l-2)$-dimensional surface.

(2) There exists a $(l-1)$-dimensional surface $L_1^2 \subset R_1^2$ which is tangent to $L_1$ at $\mu = 0$ such that (1.1) has a unique 1-homoclinic loop $\Gamma_1^2$ connecting $p_1$ for $\mu \in L_1^2$. Meanwhile, there also exists a $(l-1)$-dimensional surface $L_2^1 \subset R_2^1$ which is tangent to $L_1$ at $\mu = 0$ such that (1.1) has a unique 1-homoclinic loop $\Gamma_2^1$ connecting $p_2$ for $\mu \in L_2^1$.

Now, we consider the 1-periodic orbits bifurcating from $\Gamma$. That is, consider the solutions of (3.1) which satisfy $s_1 > 0$, $s_2 > 0$. For simplicity, we assume

**(H6)** $\Delta_1 = \Delta_2 = 1$.

Obviously, for $\Delta_1 = \Delta_2 = 1$, $R_1^2$ and $R_2^1$ read as $\{\mu : M_1^1\mu > 0, M_2^1\mu < 0, |\mu| \ll 1\}$ and $\{\mu : M_1^1\mu < 0, M_2^1\mu > 0, |\mu| \ll 1\}$, respectively.

**Lemma 3.1.** Suppose that hypotheses (H1)—(H6) are valid, then, in addition to the 1-homoclinic loop $\Gamma_1^2$, system (1.1) has exactly one simple 1-periodic orbit near $\Gamma$ for $\mu \in L_1^2$. Moreover, the 1-periodic orbit is persistent for $\mu$ changes in the neighborhood of $L_1^2$.

**Proof.** If $\mu \in L_1^2$ and $|\mu| \ll 1$, then we know that (1.1) has one 1-homoclinic loop $\Gamma_1^2$ homoclinic to $p_1$. Now, following (3.1), we have

$$s_1^{\beta_1} + \delta^{-1} w_1^{12} M_1^1\mu + \text{h.o.t.} = w_1^{12}(w_2^{12})^{1/\beta_2}(s_1 - \delta^{-1}M_2^1\mu + \text{h.o.t.})^{1/\beta_2}. \tag{3.4}$$

Let $V_1(s_1)$ and $N_1(s_1)$ be the left and right hand of (3.4), respectively. Then, by (3.3), we get $V_1(0) = N_1(0)$ as $\mu \in L_1^2$. Moreover,

$$\dot V_1(s_1) = \beta_1 s_1^{\beta_1 - 1}, \quad \dot N_1(s_1) = (1/\beta_2) w_1^{12}(w_2^{12})^{1/\beta_2}(s_1 - \delta^{-1}M_2^1\mu + \text{h.o.t.})^{1/\beta_2 - 1},$$

so $0 = \dot V_1(s_1)|_{s_1=0} < \dot N_1(s_1)|_{s_1=0}$ for $\beta_1 > 1$, $\beta_2 < 1$. Therefore, there is a $\tilde s_1$, $0 < \tilde s_1 \ll 1$ such that for $0 < s_1 < \tilde s_1$, $V_1(s_1) < N_1(s_1)$.

Denote $\bar s_1 = -\delta^{-1}M_2^1\mu$. Then

$$V_1(\bar s_1) = \bar s_1^{\beta_1} + \delta^{-1} w_1^{12} M_1^1\mu + \text{h.o.t.}, \quad N_1(\bar s_1) = w_1^{12}(2w_2^{12})^{1/\beta_2}(\bar s_1)^{1/\beta_2} + \text{h.o.t.}.$$

It is not difficult to check that $V_1(\bar s_1) > N_1(\bar s_1)$ for the reason that $|\mu|$, $\bar s_2 \ll 1$ and $1 < \beta_1 < \frac{1}{\beta_2}$.

Thus, we get $V_1(s_1) = N_1(s_1)$ has at least one solution $s_1^*$ satisfying $0 < s_1^* < \bar s_1$. That is to say, (1.1) has at least one 1-periodic orbit near $\Gamma$ for $\mu \in L_1^2$ and $0 < |\mu| \ll 1$.

It is easy to see that $V_1(s_1) > N_1(s_1)$ as $\bar s_1 \leqslant s_1 \ll 1$. In fact,

$$\begin{aligned}\dot V_1(s_1) = \beta_1 s_1^{\beta_1 - 1} &> [1/(2\beta_2)] w_1^{12}(2w_2^{12})^{1/\beta_2} s_1^{1/\beta_2 - 1} + \text{h.o.t.} \\ &\geqslant (1/\beta_2) w_1^{12}(w_2^{12})^{1/\beta_2}(s_1 - \delta^{-1}M_2^1\mu + h.o.t.)^{1/\beta_2 - 1} = \dot N_1(s_1).\end{aligned}$$

Combining it with $V_1(\bar{s}_1) > N_1(\bar{s}_1)$, we can obtain $V_1(s_1) > N_1(s_1)$ as $\bar{s}_1 \leqslant s_1 \ll 1$ immediately.

Next, we prove the uniqueness of the sufficiently small positive solution of equation $V_1(s_1) = N_1(s_1)$.

Noticing that $V_1(s_1) - N_1(s_1) = 0$ has solutions $s_1 = 0$ and $s_1 = s_1^*$, we know that $\dot{V}_1(s_1) - \dot{N}_1(s_1) = 0$ has surely a solution $s_1 = \hat{s}_1$ in $(0, \bar{s}_1)$ according to the Role middle value theorem. Thus we have

$$\beta_1 \hat{s}_1^{\beta_1 - 1} - (1/\beta_2) w_1^{12} (w_2^{12})^{1/\beta_2} (\hat{s}_1 + \bar{s}_1)^{1/\beta_2 - 1} + \text{h.o.t.} = 0,$$

$$\frac{\hat{s}_1}{\hat{s}_1 + \bar{s}_1} = (\beta_1\beta_2)^{\frac{\beta_2}{\beta_2 - 1}} [(w_1^{12})^{\beta_2} w_2^{12}]^{\frac{1}{1-\beta_2}} (\hat{s}_1)^{\frac{1-\beta_1\beta_2}{1-\beta_2}} \ll 1 \ \text{ for } \ 0 < |\mu| \ll 1.$$

Therefore,

$$\begin{aligned} \frac{d^2[V_1(\hat{s}_1) - N_1(\hat{s}_1)]}{d{s_1}^2} &= (\beta_1 - 1)\beta_1 \hat{s}_1^{\beta_1 - 2} - \frac{1-\beta_2}{\beta_2}\frac{1}{\beta_2} w_1^{12} (w_2^{12})^{1/\beta_2} (\hat{s}_1 + \bar{s}_1)^{1/\beta_2 - 2} + \text{h.o.t.} \\ &= \frac{\beta_1 - 1}{\hat{s}_1}\left[1 - \frac{1-\beta_2}{\beta_2(\beta_1 - 1)}\frac{\hat{s}_1}{\hat{s}_1 + \bar{s}_1}\right] \dot{N}_1(\hat{s}_1) > 0. \end{aligned}$$

This turns out that $s_1 = s_1^*$ is the unique sufficiently small positive solution of equation $V_1(s_1) = N_1(s_1)$. Due to (3.1) and (3.3), (3.1) has sufficiently small positive solution $s_2 = s_2^*$ corresponding to $s_1 = s_1^*$. Moreover, $s_1^*$ is a simple zero of $V_1(s_1) = N_1(s_1)$ which is persistent under small perturbation of $\mu$.

The proof is complete.

**Lemma 3.2.** Suppose that hypotheses (H1)—(H6) hold, then, system (1.1) has no any 1-periodic orbit near $\Gamma$ for $\mu \in L_2^1$ except the 1-homoclinic loop $\Gamma_2^1$.

**Proof.** The condition $\mu \in L_2^1$ and $|\mu| \ll 1$ means that (1.1) has one 1-homoclinic loop $\Gamma_2^1$ homoclinic to $p_2$. Based on (3.1), we have

$$(s_2 + \delta^{-1} w_2^{12} M_2^1 \mu + \text{h.o.t.})^{\beta_1} = w_1^{12} (w_2^{12})^{\beta_1} (s_2^{1/\beta_2} - \delta^{-1} M_1^1 \mu) + \text{h.o.t.} \tag{3.4$'$}$$

Let $V_2(s_2)$ and $N_2(s_2)$ be the left and right hand of (3.4)$'$, respectively. Then $V_2(0) = N_2(0)$ as $\mu \in L_2^1$. Moreover,

$$\dot{V}_2(s_2) = \beta_1 (s_2 + \delta^{-1} w_2^{12} M_2^1 \mu + \text{h.o.t.})^{\beta_1 - 1}, \quad \dot{N}_2(s_2) = (1/\beta_2) w_1^{12} (w_2^{12})^{\beta_1} s_2^{1/\beta_2 - 1}.$$

Taking note of $1 < \beta_1 < 1/\beta_2$, we get

$$\dot{V}_2(s_2) > \beta_1 {s_2}^{\beta_1 - 1} > \dot{N}_2(s_2) \text{ for } 0 \leqslant s_2 \ll 1.$$

Therefore, $V_2(s_2) > N_2(s_2)$ for $0 < s_2 \ll 1$. The proof is complete.

**Lemma 3.3.** Suppose that hypotheses (H1)—(H6) hold. Then, the curve $h = V_1(s_1)$ is tangent to $h = N_1(s_1)$ at some point $s_1$ satisfying $0 \leqslant s_1 \ll 1$ if and only if $\mu \in R_1^2$, $|M_1^1\mu| \ll |M_2^1\mu|$ and

$$-\delta^{-1} M_2^1 \mu = (w_2^{12})^{\frac{1}{\beta_2 - 1}} \left(-\frac{M_1^1\mu}{M_2^1\mu}\right)^{\frac{\beta_2}{1-\beta_2}} - \left(\frac{w_1^{12}}{\beta_1\beta_2}\right)^{\frac{1}{\beta_1 - 1}} \left(-\frac{M_1^1\mu}{M_2^1\mu}\right)^{\frac{1}{\beta_1 - 1}}. \tag{3.5}$$

**Proof.** It is easy to see that $h = V_1(s_1)$ is tangent to $h = N_1(s_1)$ at point $s_1$ if and only if $V_1(s_1) = N_1(s_1)$ and $\dot{V}_1(s_1) = \dot{N}_1(s_1)$, that is

$$s_1^{\beta_1} + \delta^{-1} w_1^{12} M_1^1 \mu + \text{h.o.t.} = w_1^{12} (w_2^{12})^{1/\beta_2} (s_1 - \delta^{-1} M_2^1 \mu + \text{h.o.t.})^{1/\beta_2}, \tag{3.4}$$

$$\beta_1\beta_2 s_1^{\beta_1-1} = w_1^{12}(w_2^{12})^{1/\beta_2}(s_1 - \delta^{-1}M_2^1\mu + \text{h.o.t.})^{1/\beta_2-1}. \tag{3.6}$$

It is not difficult to get that (3.4) and (3.6) have a unique small positive solution

$$s_1 = \left(-\frac{w_1^{12}M_1^1\mu}{\beta_1\beta_2 M_2^1\mu}\right)^{\frac{1}{\beta_1-1}} + \text{h.o.t.} \tag{3.7}$$

only when $\mu \in R_1^2$ and $|M_1^1\mu| \ll |M_2^1\mu|$. Substituting (3.7) into (3.6), we get the lemma.

**Lemma 3.4.** Suppose that hypotheses (H1)—(H6) hold, then, the curve $h = V_2(s_2)$ cannot be tangent to $h = N_2(s_2)$ at any point $s_2$ satisfying $0 \leqslant s_2 \ll 1$.

**Proof.** By some elementary computation, one can see that $V_2(s_2) = N_2(s_2)$ and $\dot V_2(s_2) = \dot N_2(s_2)$ are equivalent to

$$(s_2 + \delta^{-1}w_2^{12}M_2^1\mu + \text{h.o.t.})^{\beta_1} = w_1^{12}(w_2^{12})^{\beta_1}(s_2^{1/\beta_2} - \delta^{-1}M_1^1\mu) + \text{h.o.t.}, \tag{3.4$'$}$$

$$\beta_1(s_2 + \delta^{-1}w_2^{12}M_2^1\mu + \text{h.o.t.})^{\beta_1-1} = (1/\beta_2)w_1^{12}(w_2^{12})^{\beta_1}s_2^{1/\beta_2-1} + \text{h.o.t.}. \tag{3.6$'$}$$

They have a small positive solution

$$s_2 = \left(-\frac{\beta_1\beta_2 M_1^1\mu}{w_2^{12}M_2^1\mu}\right)^{\frac{\beta_2}{1-\beta_2}} + \text{h.o.t.} \tag{3.7$'$}$$

only when $\mu \in R_2^1$ and $|M_1^1\mu| \ll |M_2^1\mu|$. Substituting (3.7)$'$ into (3.6)$'$, we get

$$\delta^{-1}M_2^1\mu = (w_1^{12})^{\frac{1}{\beta_1-1}}\left(-\frac{M_1^1\mu}{M_2^1\mu}\right)^{\frac{1}{\beta_1-1}} - (w_2^{12})^{\frac{1}{\beta_2-1}}\left(-\frac{\beta_1\beta_2M_1^1\mu}{M_2^1\mu}\right)^{\frac{\beta_2}{1-\beta_2}}. \tag{3.5$'$}$$

Owing to (H5), we have $\frac{1}{\beta_1-1} > \frac{\beta_2}{1-\beta_2}$. Thus, for $|M_1^1\mu| \ll |M_2^1\mu|$, the right hand of (3.5)$'$ will be negative, but $\delta^{-1}M_2^1\mu > 0$. This is a contradiction.

The proof of Lemma 3.4 is over.

If $M_1^1$ and $M_2^1$ are linearly independent, then (3.5) has a solution which defines a $(l-1)$-dimensional surface $\tilde L_1^2$ in the neighborhood of $\mu = 0$. There is no difficulty to see that $\tilde L_1^2$ is tangent to $L_1$ at $\mu = 0$.

Following from (H5), we get $\frac{\beta_2}{1-\beta_2} < \frac{1}{\beta_1-1}$, and so we have

$$-\delta^{-1}M_2^1\mu|_{L_1^2} = (w_2^{12})^{\frac{1}{\beta_2-1}}\left(-\frac{M_1^1\mu}{M_2^1\mu}\right)^{\frac{\beta_2}{1-\beta_2}}$$

$$> (w_2^{12})^{\frac{1}{\beta_2-1}}\left(-\frac{M_1^1\mu}{M_2^1\mu}\right)^{\frac{\beta_2}{1-\beta_2}} - \left(\frac{w_1^{12}}{\beta_1\beta_2}\right)^{\frac{1}{\beta_1-1}}\left(-\frac{M_1^1\mu}{M_2^1\mu}\right)^{\frac{1}{\beta_1-1}} = -\delta^{-1}M_2^1\mu|_{\tilde L_1^2}$$

$$> 0 \approx -\delta^{-1}M_2^1\mu|_{L_2}$$

for $\mu \in R_1^2$ and $|M_1^1\mu| \ll |M_2^1\mu|$. By the definitions of $L_2$, $L_1^2$, $\tilde L_1^2$, $L_1$, and the above inequality, we get $\tilde L_1^2$ will be in the open region that is bounded by $L_2$ and $L_1^2$. Thus, we can define the following three open regions:

$(R_1^2)_1$ is that whose boundaries are $L_1$ and $L_1^2$, and has nonempty intersection with $R_1^2$. $(R_1^2)_2$ is that whose boundaries are $L_1^2$ and $\tilde L_1^2$, and has nonempty intersection with $R_1^2$. $(R_1^2)_0$ is that whose boundaries are $\tilde L_1^2$ and $L_2$, and has nonempty intersection with $R_1^2$.

Now, we consider the non-negative solutions of $V_1(s_1) = N_1(s_1)$ which is defined by (3.4). Using Lemmas 3.1 and 3.3, it is not difficult to prove that:

If $\mu \in (R_1^2)_1$, then $V_1(s_1) = N_1(s_1)$ has exactly one small positive solution.

If $\mu \in L_1^2$, then $V_1(s_1) = N_1(s_1)$ has exactly one small positive and one zero solution.

If $\mu \in (R_1^2)_2$, then $V_1(s_1) = N_1(s_1)$ has exactly two small positive solutions.

If $\mu \in \tilde{L}_1^2$, then $V_1(s_1) = N_1(s_1)$ has exactly one small two-fold positive solution.

If $\mu \in (R_1^2)_0$, then $V_1(s_1) = N_1(s_1)$ has no any small non-negative solution.

Thus, we have shown the following theorem.

**Theorem 3.2.** Suppose that hypotheses (H1)—(H6) hold, $\mu \in R_1^2$. Then the following conclusions are true.

(1) System (1.1) has exactly one simple 1-periodic orbit near $\Gamma$ as $\mu \in (R_1^2)_1$.

(2) System (1.1) has exactly one simple 1-periodic orbit and one 1-homoclinic loop homoclinic to $p_1$ near $\Gamma$ as $\mu \in L_1^2$.

(3) System (1.1) has exactly two simple 1-periodic orbits near $\Gamma$ as $\mu \in (R_1^2)_2$.

(4) System (1.1) has a unique two-fold 1-periodic orbit near $\Gamma$ as $\mu \in \tilde{L}_1^2$.

(5) System (1.1) has not any 1-periodic and 1-homoclinic loop near $\Gamma$ as $\mu \in (R_1^2)_0$.

Similarly, we can define two open regions $(R_2^1)_0$ and $(R_2^1)_1$ in $R_2^1$ and obtain the following Theorem 3.3, where $(R_2^1)_0$ is that whose boundaries are $L_1$ and $L_2^1$, $(R_2^1)_1$ is that whose boundaries are $L_2^1$ and $L_2$.

**Theorem 3.3.** Suppose that hypotheses (H1)—(H6) hold. Then system (1.1) has not any 1-periodic and 1-homoclinic loop near $\Gamma$ as $\mu \in (R_2^1)_0$, has exactly one 1-homoclinic loop homoclinic to $p_2$ near $\Gamma$ as $\mu \in L_2^1$, and has exactly one simple 1-periodic orbits near $\Gamma$ as $\mu \in (R_2^1)_1$, respectively.

Denote $D_1^2$ as the open region whose boundaries are $L_1$ and $L_2$, such that $D_1^2 \cap \{\mu : M_1^1\mu > 0, M_2^1\mu > 0, |\mu| \ll 1\} \neq \emptyset$. $D_2^1$ is the open region whose boundaries are $L_2$ and $L_1$, such that $D_2^1 \cap \{\mu : M_1^1\mu < 0, M_2^1\mu < 0, |\mu| \ll 1\} \neq \emptyset$. By Lemmas 3.3 and 3.4, we obtain the following valid theorem.

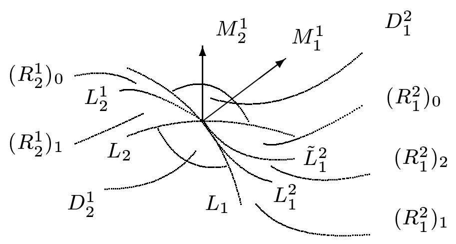


Fig. 1. $\Delta_1 = \Delta_2 = 1$

**Theorem 3.4.** Suppose that hypotheses (H1)—(H6) hold. Then

(1) system (1.1) has exactly one simple 1-periodic orbits near $\Gamma$ as $\mu \in D_2^1$;

(2) system (1.1) has no any 1-periodic orbits near $\Gamma$ as $\mu \in D_1^2$.

Now, we assume that

**(H7)** $\Delta_1 = \Delta_2 = -1$.

By Theorems 3.1—3.4, we obtain the bifurcation graph (fig. 1).

In this case, we have $R_1^2 = R_2^1 = \{\mu : M_1^1\mu > 0, M_2^1\mu > 0, |\mu| \ll 1\} := R$. Thus, we can discuss in a similar way and obtain the following theorem.

**Theorem 3.5.** Suppose that hypotheses (H1)—(H5) and (H7) hold. Then, in the open region $R$, there are three $(l-1)$-dimensional surfaces $L_1^2$, $\tilde{L}_1^2$ and $L_2^1$, all tangent to $L_1$ at $\mu = 0$, and the following four open regions: $R_1$ is that whose boundaries are $L_1$ and $L_1^2$, $R_2$ is that whose boundaries are $L_1^2$ and $\tilde{L}_1^2$, $R_3$ is that whose boundaries are $\tilde{L}_1^2$ and $L_2^1$, and $R_4$ is that whose boundaries are $L_2^1$ and $L_2$, all have nonempty intersection with $R$, such that the following conclusions are true.

(1) System (1.1) has exactly one simple 1-periodic orbit near $\Gamma$ as $\mu \in R_1 \cup R_4$.

(2) System (1.1) has exactly one simple 1-periodic orbit and one 1-homoclinic loop homoclinic to $p_1$ near $\Gamma$ as $\mu \in L_1^2$.

(3) System (1.1) has exactly two simple 1-periodic orbits near $\Gamma$ as $\mu \in R_2$.

(4) System (1.1) has a unique two-fold 1-periodic orbit near $\Gamma$ as $\mu \in \tilde{L}_1^2$.

(5) System (1.1) has exactly one 1-homoclinic loop homoclinic to $p_2$ near $\Gamma$ as $\mu \in L_2^1$.

(6) System (1.1) has no any 1-periodic orbits and 1-homoclinic loop near $\Gamma$ as $\mu \in R_3$ or $\mu \bar{\in} R$.

**Remark 3.1.** For the case $\beta_1\beta_2 > 1$, we only need to make a transformation of time, $t \to -t$.

**Acknowledgements** This work was supported by the National Natural Science Foundation of China (Grant No. 10071022) and the Shanghai Priority Academic Discipline.

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