A note on noncommutative moment problems

MA Xiujuan

School of Sciences, Hebei University of Technology, Tianjin 300130, China; Mathematics Institute, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China

(email: xiujuanm@math.unh.edu)

Received December 4, 2004

Abstract Noncommutative moment problems for C^*-algebras are studied. We generalize a result of Hadwin on tracial states to nontracial case. Our results are applied to obtain simple solutions to moment problems on the square and the circle as well as extend the positive unital functionals from a (discrete) complex group algebra to states on the group C^*-algebra.

Keywords: moment, free product, state, group C^*-algebra.

DOI: 10.1360/022004-104

1 Introduction

The classical Hamburger moment theorem gives conditions on a sequence $\{\kappa_n\}$ of numbers so that there is a positive Borel measure on $\mathbb{R}$ whose sequence of moments is $\{\kappa_n\}$. The following is an immediate consequence of Hamburger's result.

Theorem 1. Given a sequence $\{\kappa_n\}_{n\geqslant 0}$ of real numbers, there is a probability measure μ on $[-1,1]$ such that

$$\int_{[-1,1]} t^n d\mu(t) = \kappa_n \text{ for } n = 0,1,2,\ldots$$

if and only if the linear functional $\varphi : \mathbb{C}[t] \to \mathbb{C}$ defined by $\varphi(t^n) = \kappa_n$ for $n \geqslant 0$ satisfies

(1) $\varphi(1) = 1,$

(2) $\varphi(|p|^2) \geqslant 0$ for every $p \in \mathbb{C}[t]$, and

(3) $\liminf\limits_{n\to\infty} \kappa_{2n} < \infty.$

Note that the probability measures on $[-1,1]$ correspond to the states on the C^*-algebra $C[-1,1]$ of continuous functions on $[-1,1]$. In Theorem 1.3 of ref. [1], Don Hadwin generalized the preceding theorem to the noncommutative setting of tracial states on free products of copies of $C[-1,1]$. As a tool in his proof he gave a simple proof of a generalized Hölder's inequality. For background on C^*-algebras and free products the reader can consult, respectively, refs. [2] and [3].

In this paper we give a simpler proof of Hadwin's theorem, and we show that the theorem and Hadwin's version of Hölder's inequality are not true without the tracial assumption ($\tau(ab) = \tau(ba)$). We prove a version of Hadwin's result when $C[-1,1]$ is replaced by $C(\mathbb{T})$ (where $\mathbb{T}$ is the unit circle). We then prove an analogue of Hadwin's result in the nontracial setting, and we prove an extension theorem for (nontracial) functionals on a group algebra $\mathbb{C}(G)$ to states on the group C^*-algebra $C^*(G)$, when G is a discrete group.

2 Noncommutative moment problem for states

Suppose $\mathcal{A}_n$ is the C^*-algebraic free product of n copies of $C[-1,1]$. If we consider that $C[-1,1]$ is the universal unital C^*-algebra generated by an element t subject to the relations $t = t^*$ and $-1 \leqslant t \leqslant 1$, then $\mathcal{A}_n$ is naturally generated by elements $t_1, \ldots, t_n$, where t_k is the generator of the kth copy of $C[-1,1]$. Let $\mathbb{P}_n$ denote the vector space of all noncommutative polynomials in $t_1, t_2, \ldots, t_n$, and let $\mathbb{M}_n$ denote all of the monomials in $t_1, t_2, \ldots, t_n$. We then have $\mathbb{P}_n \subset \mathcal{A}_n$. When $n = 1$, let $t = t_1$. We have $\mathbb{P}_n$=$\mathbb{C}[t]$, $\mathbb{M}_n = \{1, t, t^2, \ldots\}$ and $\mathcal{A}_n = C[-1,1]$. It is clear that $\mathbb{M}_n$ is a linear basis for the vector space $\mathbb{P}_n$, so there is a bijection between the functions on $\mathbb{M}_n$ and the linear functionals on $\mathbb{P}_n$. A functional φ is tracial if $\varphi(ab) = \varphi(ba)$ always holds. Here is Don Hadwin's noncommutative moment theorem with our simplified proof.

Theorem 2. A tracial linear functional φ on $\mathbb{P}_n$ can be extended to a (tracial) state on $\mathcal{A}_n$ if and only if

(1) $\varphi(1) = 1$,

(2) $\liminf\limits_{m\to\infty} |\varphi(t_j^{2m})| < \infty$ for $1 \leqslant j \leqslant n$, and

(3) $\varphi(p^*p) \geqslant 0$ for every $p \in \mathbb{P}_n[t]$.

Proof. "$\Rightarrow$" Suppose (1), (2) and (3) are true. Define a semi-inner product $(\cdot,\cdot)$ on $\mathbb{P}_n$ by $(p,q) = \varphi(q^*p) = \varphi(pq^*)$ and, for $1 \leqslant j \leqslant n$, define a linear transformation L_{t_j} on $\mathbb{P}_n$ by $L_{t_j}p = t_jp$. We want to show that each L_{t_j} induces a contractive linear mapping on the semi-inner product space, i.e.

$$(L_{t_j}p, L_{t_j}p) = \varphi(p^*t_j^2p) \leqslant \varphi(p^*p) = (p,p)$$

for every $p \in \mathbb{P}_n$.

Clearly, we can assume that $\varphi(p^*p) > 0$. Define a functional ψ on $\mathbb{C}[t_j]$ by

$$\psi(q) = \varphi(qp^*p)/\varphi(p^*p).$$

Then $\psi(1) = 1$ and

$$\psi(q^*q) = \varphi(q^*qp^*p)/\varphi(p^*p) = \varphi((qp)^*(qp))/\varphi(p^*p) \geqslant 0.$$

By the Cauchy-Schwarz inequality, we have

$$\psi(t_j^{2m}) = \varphi(t_j^{2m}p^*p)/\varphi(p^*p) \leqslant \varphi(t_j^{4m})^{\frac{1}{2}}\varphi((p^*p)^2)^{\frac{1}{2}}/\varphi(p^*p)$$

for $1 \leqslant j \leqslant n$, $0 \leqslant m < \infty$. Since for each j and m, $|\varphi(t_j^m)| \leqslant 1$, it follows that

$$\psi(t_j^{2m}) = \varphi(t_j^{2m}p^*p)/\varphi(p^*p) < \infty,$$

for $1 \leqslant j \leqslant n$, $0 \leqslant m < \infty$. Hence, ψ satisfies the conditions of classical moment theorem, then we have $|\psi(t_j^2)| \leqslant 1$, which is the desired inequality

$$\varphi(t_j^2 p^* p) \leqslant \varphi(p^* p).$$

Thus, each L_{t_j} defines a selfadjoint contraction operator on the Hilbert-space completion of our semi-inner product space $\mathbb{P}_n$. It follows from the definition of $\mathcal{A}_n$ that there is a unital*-homomorphism π on $\mathcal{A}_n$ such that $\pi(t_j) = L_{t_j}$ for $1 \leqslant j \leqslant n$. Clearly, for each $p \in \mathbb{P}_n$, $\varphi(p) = (\pi(p)[1], [1])$, which completes the proof. □

Next, we will show that without the tracial assumption on φ, none of the above results remains true.

Example 1. Define $\varphi : \mathbb{P}(t_1, t_2) \to \mathbb{C}$ by $\varphi(p(t_1, t_2)) = (p(T_1, T_2)e, e)$, where $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Clearly, φ is linear and $\varphi(1) = 1$. Moreover, for every $p \in \mathbb{P}(t_1, t_2)$,

$$\begin{aligned} \varphi(p^* p) &= (p(T_1, T_2)^* p(T_1, T_2)e, e) \\ &= (p(T_1, T_2)e, p(T_1, T_2)e) = ||p(T_1, T_2)e||^2 \geqslant 0. \end{aligned}$$

Also, $\varphi(t_1^{2n}) = \varphi(t_2^{2n}) = 1$, for $n = 1, 2, 3 \ldots$. Therefore, except for the tracial property, φ satisfies all of the conditions in Theorem 1.3 of ref. [1]. However, $\varphi(t_2 t_1^n t_2) = 2^n$ for $n = 1, 2, 3 \ldots$. This implies that the version of Hölder's inequality in the proof of Theorem 1.3 of ref. [1] is not true. Also, in $C[-1, 1] * C[-1, 1]$

$$||t_2 t_1^n t_2|| \leqslant 1 \text{ and } 2^n = \varphi(t_2 t_1^n t_2),$$

which implies that φ cannot be extended to a continuous linear functional on $C[-1, 1] * C[-1, 1]$.

The previous counter-example shows that without tracial assumption the theorem may fail. But with some modifications of the conditions, we will get the following theorem:

Theorem 3. Suppose φ is a linear functional on $\mathbb{P}_n$ satisfying $\varphi(1) = 1$, and $\varphi(p^* p) \geqslant 0$. Then the following are equivalent:

(1) $\sup_{x \in \mathbb{M}_n} |\varphi(x)| < \infty$;

(2) $\liminf\limits_{k \to \infty} \varphi\left((x^* x)^{2k}\right) < \infty$ for every $x \in \mathbb{M}_n$;

(3) $\sup_{x \in \mathbb{M}_n} |\varphi(x)| \leqslant 1$;

(4) φ can be extended to a state on $\mathcal{A}_n$.

Proof. The proofs of $(4) \Rightarrow (1) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (3)$. It follows from Theorem 1 that if $t = x^* x$, then there is a probability measure μ on $[-1, 1]$ such that

$$\varphi\left(t^{2k}\right) = \int_{-1}^{1} t^{2k} d\mu \leqslant 1$$

for $k \geqslant 1$.

$(3) \Rightarrow (4)$. As in the GNS construction, define a semi-inner product (,) on $\mathbb{P}_n$ by

$$(p, q) = \varphi(q^* p)$$

and, for $1 \leqslant j \leqslant n$, define a linear transformation L_{t_j} on $\mathbb{P}_n$ by $L_{t_j} p = t_j p$ $(where$ $t_j^* = t_j$ for each j). We want to show that each L_{t_j} induces a contractive linear mapping on the semi-inner product space, i.e.

$$(L_{t_j} p, L_{t_j} p) = (t_j p, t_j p) = \varphi(p^* t_j^2 p) \leqslant \varphi(p^* p) = (p, p)$$

for every $p \in \mathbb{P}_n$. Clearly, we can assume that $\varphi(p^* p) > 0$. Define a linear functional ψ on $\mathbb{C}[t_j]$ by

$$\psi(q) = \varphi(p^* q p)/\varphi(p^* p)$$

for each $q \in \mathbb{C}[t_j]$ and $p \in \mathbb{P}_n$.

Then $\psi(1) = 1$ and

$$\psi(q^* q) = \varphi(p^* q^* q p)/\varphi(p^* p) = \varphi((qp)^*(qp))/\varphi(p^* p) \geqslant 0.$$

Since any non-commutative polynomial $p \in \mathbb{P}_n$ can be written as the finite sum of monomials, suppose

$$p = \sum_{i=1}^{N} \lambda_i m_i,$$

where each $m_i \in \mathbb{M}_n$. Then

$$p^* = \sum\nolimits_{i=1}^{N} \overline{\lambda}_i m_i.$$

Thus, for $\alpha = 1, 2, 3 \ldots$

$$\psi(t_j^\alpha) = \varphi(p^* t_j^\alpha p)/\varphi(p^* p) = \varphi\left(\sum_{i=1}^{N}\sum_{k=1}^{N} \overline{\lambda}_i \lambda_k m_i t_j^\alpha m_k\right) \Big/ \varphi(p^* p)$$
$$= \sum_{i=1}^{N}\sum_{k=1}^{N} \overline{\lambda}_i \lambda_k \varphi(m_i t_j^\alpha m_k)/\varphi(p^* p).$$

Since $m_i t_j^\alpha m_k$ is a monomial, by (1) , we have $\psi(t_j^\alpha) < \infty$. That means $\psi(t_j^{2m}) < \infty$ for $1 \leqslant j \leqslant n$, $0 \leqslant m < \infty$. Hence, ψ satisfies the conditions of classical moment theorem, then we have $|\psi(t_j^2)| \leqslant 1$, which is the desired inequality

$$\varphi(p^* t_j^2 p) \leqslant \varphi(p^* p).$$

Thus, each L_{t_j} defines a selfadjoint contraction operator on the Hilbert-space completion of our semi-inner product space $\mathbb{P}_n$. It follows from the definition of $\mathcal{A}_n$ that there is a unital*-homomorphism π on $\mathcal{A}_n$ such that $\pi(t_j) = L_{t_j}$ for $1 \leqslant j \leqslant n$. Clearly, for each $p \in \mathbb{P}_n$, $\varphi(p) = (\pi(p)[1], [1])$, which completes the proof. □

3 Unitary generator case

Theorem 2 can be used to show that any positive unital tracial linear functional on the algebraic free product of C^*-algebras extends to a tracial state on the C^*-algebraic free product completion.

Considering GNS construction from the states on the free product of copies of $C[-1,1]$, we have C^*-algebras generated by selfadjoint contractions. It is often more convenient to look at C^*-algebras generated by unitaries. We next use Theorem 2 to prove a version of Theorem 1 with $C[-1,1]$ replaced by $C(\mathbb{T})$, where

$$\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

To do this we need a few simple lemmas.

Lemma 1. Suppose $\mathcal{A} = C[-1,1]$, with a variable s and $\mathcal{B} = C[-1,1]$ with a variable t. Let $\mathcal{J}$ be the closed ideal in $\mathcal{A} * \mathcal{B}$ generated by $st - ts$. Let x, y be the coordinate functions in $C([-1,1] \times [-1,1])$. Then the map that sends $1 \mapsto 1$, $s \mapsto x$, $t \mapsto y$ generates a $*$-isomorphism from $(\mathcal{A} * \mathcal{B})/\mathcal{J}$ onto $C([-1,1] \times [-1,1])$.

Proof. This follows intuitively from the fact that $\mathcal{A} * \mathcal{B}$ is the universal unital C^*-algebra generated by s,t subject to the conditions $-1 \leqslant s,t \leqslant 1$, while $C([-1,1] \times [-1,1])$ is the universal unital C^*-algebra generated by s,t subject to the conditions $-1 \leqslant s,t \leqslant 1$ and $st = ts$. It is easy to check that the map defined in the lemma is an isomorphism. □

We can now use the Hadwin's result to obtain the well-known version of the Hamburger moment problem for the square.

Corollary 1. If $\varphi : \mathbb{C}[x,y] \mapsto \mathbb{C}$ is a positive linear functional such that $\varphi(1) = 1$,

$$\liminf_{k\to\infty} \varphi\left(x^{2k}\right) < \infty$$

and

$$\liminf_{k\to\infty} \varphi(y^{2k}) < \infty,$$

then there is a probability measure μ on $[-1,1] \times [-1,1]$ such that for any $p \in \mathbb{C}[x,y]$,

$$\varphi(p) = \int_{[-1,1]\times[-1,1]} p d\mu.$$

Proof. Define $\pi : \mathbb{P}_2(s,t) \to \mathbb{C}[x,y]$ by $\pi(p(s,t)) = p(x,y)$. Then π is a unital $*$-homomorphism. If we take $\Psi = \varphi \circ \pi$ then Ψ satisfies the hypothesis of Theorem 2. Therefore, there exists a state $\widehat{\Psi}$ on $C[-1,1] * C[-1,1]$, such that $\widehat{\Psi}|_{\mathbb{P}_2} = \varphi \circ \pi$. Then by the GNS construction, there exist a Hilbert space $\mathcal{H}$ and $e \in \mathcal{H}$, $||e|| = 1$, $\rho : C[-1,1] * C[-1,1] \to \mathcal{B}(\mathcal{H})$ so that $\widehat{\Psi}(A) = (\rho(A)e, e)$, and $\overline{\rho(C[-1,1] * C[-1,1])e} = \mathcal{H}$. Suppose $p, q \in \mathbb{P}_2(s,t)$, then

$$\begin{aligned}\widehat{\Psi}\left(p^*(st-ts)q\right) &= \Psi\left(p^*\left(st-ts\right)q\right) = \varphi\left(\pi\left(p^*\left(st-ts\right)q\right)\right)\\ &= \varphi\left(\pi\left(p^*\right)\pi\left(st-ts\right)\pi\left(q\right)\right) = \varphi\left(\pi\left(p^*\right)0\pi\left(q\right)\right) = 0.\end{aligned}$$

So, for any $p, q \in \mathbb{P}_2$,

$$\left(\rho\left(p^*\left(st-ts\right)q\right)e, e\right) = \left(\rho\left(st-ts\right)\rho\left(q\right)e, \rho\left(p\right)e\right) = 0,$$

that implies $(\rho(st-ts)u, v) = 0$ for any $u, v \in \mathcal{H}$. Thus, $\rho(st-ts) = 0$. Therefore, $st - ts \in \ker\rho$. Then the closed ideal $\mathcal{J}$ generated by $st - ts$ is contained in $\ker\rho$. Suppose η is the natural quotient map from $\mathcal{A} * \mathcal{B}$ to $\mathcal{A} * \mathcal{B}/\mathcal{J}$, then there is a unique homomorphism $\sigma : \mathcal{A} * \mathcal{B}/\mathcal{J} \to \mathcal{B}(\mathcal{H})$ such that $\sigma \circ \eta = \rho$. By Lemma 1, we know that $\mathcal{A} * \mathcal{B}/\mathcal{J}$ is isomorphic to $C([-1,1] \times [-1,1])$. For every $f \in C([-1,1] \times [-1,1])$, define

$$\delta(f) = (\sigma(f)e, e).$$

Then δ is a state on $C([-1,1] \times [-1,1])$. So there exists a probability measure μ on $C[-1,1] \times [-1,1]$ such that

$$\delta(f) = \int f d\mu.$$

Thus, for any commutative polynomial $p(x,y)$,

$$\delta(p(x,y)) = \int\limits_{[-1,1]\times[-1,1]} p(x,y)\, d\mu.$$

Since

$$\begin{aligned}\delta(p(x,y)) &= (\sigma(\eta(p(s,t)))e, e) = ((\rho(p(s,t)))e, e) \\ &= \Psi(p(s,t)) = \varphi(p(x,y)),\end{aligned}$$

we see that

$$\varphi(p(x,y)) = \int\limits_{[-1,1]\times[-1,1]} p(x,y)\, d\mu. \qquad \square$$

The preceding corollary gives us an easy proof of the moment problem for the square.

Corollary 2. If $\varphi : \mathbb{C}[x,y] \to \mathbb{C}$ is a positive linear functional, $\varphi(1) = 1$, and $\varphi((x^2+y^2-1)^2) = 0$, then there exists a probability measure μ on the unit circle $\mathbb{T}$ such that, for any $p \in \mathbb{C}[x,y]$,

$$\varphi(p) = \int\limits_{\mathbb{T}} p d\mu,$$

i.e. φ extends to a state on $C(\mathbb{T})$.

Proof. For any $n \in \mathbb{N}$,

$$\left|\varphi\left(\left(x^2+y^2\right)^n - 1\right)\right| = \left|\varphi\left(\left(x^2+y^2-1\right)q(x,y)\right)\right|,$$

where

$$q(x,y) = 1 + (x^2+y^2) + (x^2+y^2)^2 + \cdots + (x^2+y^2)^{n-1}.$$

It follows from the Cauchy-Schwarz inequality that,

$$\left|\varphi((x^2+y^2)^n - 1)\right| \leqslant \varphi((x^2+y^2-1)^2)^{\frac{1}{2}}\varphi(q^*(x,y)q(x,y))^{\frac{1}{2}} = 0.$$

Thus,

$$\varphi\left(\left(x^2+y^2\right)^n\right) = 1$$

for any $n \in \mathbb{N}$. Also,

$$\left(x^2+y^2\right)^n - x^{2n} = \sum_{k=0}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} \left(x^2\right)^k \left(y^2\right)^{n-k},$$

so the positivity of φ on $\mathbb{C}[x,y]$ implies

$$\varphi\left(\left(x^2+y^2\right)^n - x^{2n}\right) = \varphi\left(\sum_{k=0}^{n-1}\begin{pmatrix} n \\ k \end{pmatrix}\left(x^k y^{n-k}\right)^*\left(x^k y^{n-k}\right)\right) \geqslant 0.$$

Thus,

$$\varphi\left(x^{2n}\right) \leqslant \varphi\left(\left(x^2+y^2\right)^n\right) = 1.$$

Similarly, we have

$$\varphi\left(y^{2n}\right) \leqslant 1.$$

That tells us that φ satisfies the hypothesis of Theorem 2, and by arguing as in the proof of Corollary 1, there is a probability measure μ on $[-1,1]\times[-1,1]$ such that for any $p \in \mathbb{C}[x,y]$,

$$\varphi(p) = \int\limits_{[-1,1]\times[-1,1]} p d\mu.$$

Since

$$0 = \varphi\left(\left(x^2+y^2-1\right)^2\right) = \int\limits_{[-1,1]\times[-1,1]} \left(x^2+y^2-1\right)^2 d\mu,$$

μ is supported on the unit circle $\mathbb{T}$. Therefore, for any $p \in \mathbb{C}[x,y]$,

$$\varphi(p) = \int\limits_{[-1,1]\times[-1,1]} p d\mu = \int\limits_{\mathbb{T}} p d\mu. \qquad \square$$

With the identification $z = x+iy$, $z^* = \overline{z} = x - iy$, we see that $\mathbb{C}[x,y] = \mathbb{C}[z,\overline{z}]$.

Corollary 3. If $\varphi : \mathbb{C}[z,\overline{z}] \to \mathbb{C}$ is a positive linear functional with $\varphi(1) = 1$, and $\varphi\left((1-\overline{z}z)^2\right) = 0$, then there exists a probability measure μ on $\mathbb{T}$ such that for any $p \in \mathbb{C}[z,\overline{z}]$,

$$\varphi(p) = \int\limits_{\mathbb{T}} p(z,\overline{z})\, d\mu.$$

If we define $z^* = \frac{1}{z}$ on $\mathbb{C}\left[z,\frac{1}{z}\right]$, we get a unital $*$-algebra.

Corollary 4. Suppose $\varphi : \mathbb{C}\left[z,\frac{1}{z}\right] \to \mathbb{C}$ is a positive linear functional, $\varphi(1) = 1$. Then there exists a probability measure μ on $\mathbb{T}$, such that for any $p \in \mathbb{C}\left[z,\frac{1}{z}\right]$,

$$\varphi(p) = \int\limits_{\mathbb{T}} p\left(z,\frac{1}{z}\right) d\mu.$$

Proof. It is clear that the map $z \mapsto z, \overline{z} \mapsto \frac{1}{z}, 1 \mapsto 1$ extends to a $*$-homomorphism ρ from $\mathbb{C}[z,\overline{z}]$ onto $\mathbb{C}\left[z,\frac{1}{z}\right]$. Then $\varphi\circ\rho$ is a positive linear functional on $\mathbb{C}[z,\overline{z}]$ with $(\varphi\circ\rho)(1) = 1$ and

$$\begin{aligned}\varphi\circ\rho((1-\overline{z}z)^2) &= \varphi(\rho((1-\overline{z}z)^2)) \\ &= \varphi((\rho(1-\overline{z}z))^2) = \varphi\left(\left(1-\frac{1}{z}z\right)^2\right) = 0.\end{aligned}$$

Thus, by Corollary 3, there is a probability measure μ on $\mathbb{T}$ such that for any $p \in \mathbb{C}[z, \overline{z}]$,

$$(\varphi \circ \rho)(p) = \int_{\mathbb{T}} p(z, \overline{z})\, d\mu = \int_{\mathbb{T}} p\left(z, \frac{1}{z}\right) d\mu,$$

since on the unit circle $\overline{z} = \frac{1}{z}$. □

We can actually get Corollary 4 much more easily. This is because with unitaries, the boundedness of the norm is automatic, so the version of Theorem 2 for unitaries is easy. Suppose G is a discrete group with an identity element 1. We let $\mathbb{C}[G]$ denote the group algebra over $\mathbb{C}$ generated by G. With $g^* = g^{-1}$ for each $g \in G$, we see that $\mathbb{C}[G]$ becomes a unital $*$-algebra that is contained in the group C^*-algebra $C^*(G)$ generated by G. It is tempting to hope that somehow Theorem 2 might be recaptured from this simple proposition, but $\mathbb{C}[t]$ contains no unitary elements that are not scalars.

Proposition 1. Suppose G is a discrete group and $\varphi : \mathbb{C}[G] \to \mathbb{C}$ is a positive linear functional with $\varphi(1) = 1$, then φ extends to a state on $C^*(G)$.

Proof. When we use the GNS construction on $\mathbb{C}[G]$, we see that L_g is unitary for every $g \in G$, so the mapping $g \to L_g$ is a unitary representation of G, which, by definition, extends to a unital $*$-homomorphism on $C^*(G)$. □

Acknowledgements The author thanks professor Don Hadwin for numerous insightful discussions.

References

1. Hadwin, D. W., A noncommutative moment problem, Proc. Amer. Math. Soc., 2001, 129(6): 1785–1791.
2. Kadison, R. V., Ringrose, J. R., Fundamentals of the Theory of Operator Algebras, Vol. 2, New York: Academic Press, 1983.
3. Voiculescu, D., Dykema, K., Nica, A., Free random variables, CRM Monograph Series, Vol. 1, Providence: AMS, 1992.