Estimation of the Bezout number for piecewise algebraic curve

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Abstract A piecewise algebraic curve is a curve determined by the zero set of a bivariate spline function. In this paper, a conjecture on triangulation is confirmed. The relation between the piecewise linear algebraic curve and four-color conjecture is also presented. By Morgan-Scott triangulation, we will show the instability of Bezout number of piecewise algebraic curves. By using the combinatorial optimization method, an upper bound of the Bezout number defined as the maximum finite number of intersection points of two piecewise algebraic curves is presented.

Keywords: **piecewise algebraic curve, bezout theorem, triangulation, bivariate splines.**

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Let Δ be a regular triangulation of a region $\Omega \subset \Re^2$. Denote by $T_1, \cdots, T_N$ the triangles of Δ where N is the number of triangles in Δ. The space $S_d^r(\Delta)$ of bivariate splines with smoothness r and total degree d on Δ is defined as

$$S_d^r(\Delta) = \{s \in C^r(\Omega) : s|_{T_i} \in P_d, i = 1, \cdots, N\},$$

where P_d denotes the collection of bivariate polynomials with total degree d. The curve

$$\Gamma : \{(x, y)|s(x, y) = 0, s(x, y) \in S_d^r(\Delta)\}$$

is called a piecewise algebraic curve. It is obvious that the piecewise algebraic curve is a generalization of the classical algebraic curve. The piecewise algebraic curve is not only very important for the interpolation by the bivariate splines[1], but also a useful tool for studying traditional algebraic curves[2]. In the paper, we will discuss some properties on the piecewise algebraic curves.

Bezout's theorem is an important and classical theorem in the algebraic geometry[3]. Its weak form means that two algebraic curves will have infinitely many intersection points provided that the number of their intersection points is larger than the product of their degrees. Denote by $BN = BN(m, r; n, t; \Delta)$ the so-called Bezout's number. Any two piecewise algebraic curves

$$f(x, y) = 0, g(x, y) = 0, f \in S_m^r(\Delta), g \in S_n^t(\Delta)$$

must have infinitely many intersection points provided that they have more than BN intersection points. A fundamental problem of the piecewise algebraic curve is how to find the Bezout's number. In this paper, we show that the Bezout number $BN(m, r; n, t; \Delta)$ depends heavily on the geometry of Δ. An upper bound of $BN(m, 0; n, 0; \Delta)$ was shown in ref. [4]. In this paper, we convert this problem into a combinatorial optimization problem and give an upper bound of $BN(m,r;n,t;\Delta)$.

1 Several results on triangulation and piecewise algebraic curve

For a given triangulation Δ, let E_0 be the number of edges, $E_{\rm I}$ the number of interior edges, $E_{\rm B}$ the number of boundary edges, V_0 the number of vertices, $V_{\rm B}$ the number of boundary vertices and $V_{\rm I}$ the number of interior vertices. A triangulation Δ is said to be of 2-vertex signs if each vertex in Δ can be marked by -1 or 1 so that no vertices of the triangle in Δ are marked by the same number. An interesting conjecture was shown in ref. [4]: any triangulation is of 2-vertex signs. In the paper, we will prove that the conjecture is true.

To prove the conjecture, we need some notations and theorems in the graph theory. A graph G is an ordered pair of disjoint sets (V,E) such that E is a subset of the set $V^{(2)}$ of unordered pairs of V. V is the set of edges and E is the set of edges. A cut of the graph $G=(V,E)$ is the set of edges with one end in X_1 and one in X_2, where (X_1,X_2) is a nontrivial partition of V. A graph is k-edge-connected if it has no cut of cardinality less than k. Let v be a vertex of G. The number of edges of G incident with v is called the degree of v in G. The degree of v is denoted by $d(v)$. If every vertex of G has degree k, then G is said to be k-regular. A 3-regular graph is said to be cubic. Any connected graph that has no cycles is called a tree. Given a graph $G=(V,E)$, a matching is defined as a set E' of edges such that no two edges of E' are adjacent. A matching that saturates all vertices of G is called a perfect matching. If G is cubic and 2-edge-connected, then there exits a perfect matching of G. The triangulation Δ is considered as a planar graph $\Delta=(V,E)$. Given a triangulation, we construct another graph $\Delta'=(V',E')$. Inside each cell T_j of Δ we choose a point v'_j. We draw a line e'_i corresponding to each interior edge e_i which crosses e_i (but no other edge of Δ), and joins the vertices v'_j in the cells T_j adjoining e_i. At last, draw a loop O to surround Δ. For each boundary edge e_i, we draw a line e'_i which crosses e_i and intersects loop O at $o_i, o_i\neq o_j, i\neq j$. The set of v' and o_i is the vertices of Δ'. The set of e' is the vertices of Δ'. The edges e_i and e'_i are said to be dual to each other. The Δ and Δ' are said to be dual to each other also. For a given Δ, its geometric dual graph is shown in fig. 1(a).

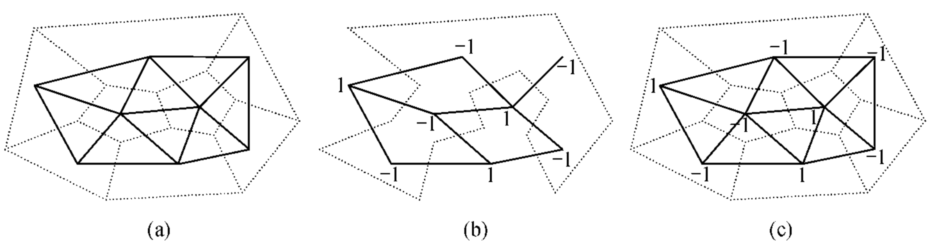

Fig. 1. The sketch map of proof of Proposition 1.

Lemma 1.1. Each vertex in any quadrangulation can be marked by -1 or 1, so that adjacent vertices have distinct numbers.

By induction, the Lemma can be proved easily. Hence we omit it.

Proposition 1.1. Any triangulation Δ is of 2-vertex signs.

Proof. Obviously, for any Δ, Δ' is cubic and 2-edge-connected. Hence there is a perfect matching in graph Δ', say, E'_0. The dual edge set of E'_0 is denoted by E_0. We consider the graph $D(\Delta) = (V, E\backslash E_0)$, and $D(\Delta') = (V', E'\backslash E'_0)$(fig. 1(b)). $D(\Delta)$ is the union of quadrangulation and some trees. By Lemma 1.1, we can mark each vertex in $D(\Delta)$ by -1 and 1, so that the adjoining vertices in $D(\Delta)$ are marked by different numbers (fig. 1(b)). For any cell T_i in Δ, the edge set adjoining with T_i is denoted by E_i. According to the definition, the perfect matching, $\#E_i \cap E_0 = 1, 1 \leqslant i \leqslant N$, where $\#E_i \cap E_0$ denotes the cardinality of the set $E_i \cap E_0$. So no vertices of triangle in Δ are marked by the same number. Hence any triangulation Δ is of 2-vertex signs.

By the proposition, some results, such as Theorem 3.1, Lemmas 3.2, and 3.3, in ref. [4] can be improved.

Moreover, we have the following corollary.

Corollary 1.1. For any triangulation Δ and $m \geqslant 1$ there exists a piecewise algebraic curve $\Gamma : f(x, y) = 0, f(x, y) \in S_m^0(\Delta)\backslash 0$ such that $\Gamma \cap T_i \neq \varnothing, 1 \leqslant i \leqslant N$.

Remark 1.1. In general, we are interested in the following problem: when $\dim S_d^r(\Delta) \geqslant N$, is there $s(x, y) \in S_d^r(\Delta)\backslash 0$ such that $\Gamma \cap T_i \neq \emptyset, 1 \leqslant i \leqslant N$, where $\Gamma : f(x, y) = 0$?

It is well known that four-color conjecture is a very famous unsolved problem. The following theorem shows the relation between four-color conjecture and the piecewise algebraic curves. To state the theorem, we first introduce several definitions. For a triangle ΔABC, suppose that the middle points of BC, AC and AB are a, b and c respectively. Denote by $M(\Delta ABC)$ the triangle Δabc, called the middle triangle of ΔABC. For a given triangulation Δ, $M(\Delta)$ is defined as $\bigcup_{i=1}^N M(T_i)$. A triangulation Δ is said to be of 3-edge signs, if each edge in Δ can be marked by $1, 2$ or 3 so that the edges incident with any triangle in Δ has distinct numbers.

Theorem 1.1. Four-color conjecture holds if and only if for any triangulation Δ there exist three piecewise algebraic curves $\Gamma_1 : f_1(x, y) = 0, \Gamma_2 : f_2(x, y) = 0, \Gamma_3 : f_3(x, y) = 0, f_i(x, y) \in S_1^0(\Delta), i = 1, 2, 3$ such that $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = M(\Delta)$.

Proof. By the graph theory, four-color conjecture holds, if and only if any triangulation is of 3-edge signs. Suppose that four-color conjecture holds. Then each edge in Δ can be marked by $1, 2, 3$ so that 3 edges of any triangle in Δ has distinct numbers. Let E_i be the edges marked by i. Consider $D_i(\Delta) = (V, E\backslash E_i)$. Obviously, $D_i(\Delta)$ is the union of quadrangulation and some trees. By Lemma 1.1, we can mark each vertex in $D_i(\Delta)$ by -1 and 1, so that the adjoining vertices in $D_i(\Delta)$ are marked by different numbers. We define $f_i \in S_1^0(\Delta)$ by $f_i(v) = w$, if the vertex v is marked by w in $D_i(\Delta)$. Obviously, $\bigcup_{i=1}^3 \Gamma_i = M(\Delta)$, where the piecewise linear algebraic curve Γ_i defined by $f_i(x, y) = 0$.

Suppose that for any triangulation Δ, there exist three piecewise linear algebraic curves $\Gamma_i : f_i(x, y) = 0, i = 1, 2, 3$ such that $\bigcup_{i=1}^3 \Gamma_i = M(\Delta)$. Consider a cell in Δ, say, T. The edge of T parallell to $\Gamma_i|_T$ is denoted by $e^{(i)}(T)$. $\Gamma_i|_T$ denotes the restriction of Γ_i on T. Let $E^{(i)} = \bigcup_T e^{(i)}(T)$. The edges in $E^{(i)}$ are marked by i. It is easy to prove that each edge in Δ can be marked by $1, 2, 3$ and the edges of a triangle in Δ have a distinct number. Hence any triangulation is of 3-edge sign. So four-color conjecture holds.

2 Bezout number $BN(m,1;n,1;\Delta)$

In ref. [4], the Bezout number of piecewise algebraic curve was considered and the result $BN(m,0;n,0;\Delta)\leqslant mnN-[\frac{V_{\rm odd}+2}{3}]$ was presented, where $V_{\rm odd}$ is the number of odd vertices in Δ, and $[x]$ means the greatest integer $\leqslant x$. Obviously, the result in ref. [4] only depends on the topology structure of Δ. Here we show that the Bezout number may depend on complicated geometric conditions of Δ. Let $\Delta_{\rm ms}$ be the triangulation with boundary vertices of V_1, V_2 and V_3 and interior vertices of $\widehat{V}_1, \widehat{V}_2$ and $\widehat{V}_3$ (fig. 2). The triangulation $\Delta_{\rm ms}$ is called Morgan-Scott triangulation.

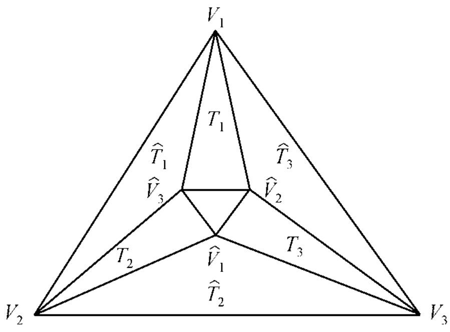

Fig. 2. The Morgan-Scott triangulation.

It is well known that $\dim S_2^1(\Delta_{\rm ms})$ is instable1). In ref. [6], the following results were presented.

$$\dim S_2^1(\Delta_{\rm ms})=\begin{cases}7, & V_1\widehat{V}_1, V_2\widehat{V}_2 \text{ and } V_3\widehat{V}_3 \text{ have a common point of intersection,}\\ 6, & \text{otherwise.}\end{cases}$$

When $\dim S_2^1(\Delta_{\rm ms})=6$, $S_2^1(\Delta_{\rm ms})$ coincides with P_2. Hence, when $\dim S_2^1(\Delta_{\rm ms})=6$, $BN(2,1;2,1;\Delta_{\rm ms})=4$; when $\dim S_2^1(\Delta_{\rm ms})=7$, we may suppose that the smoothing cofactor across $\widehat{V}_1\widehat{V}_2$ is free[1]. It is easy to construct two spline functions $f_1(x,y), f_2(x,y)\in S_2^1(\Delta_{ms})$ such that the intersection number of two piecewise algebraic curves $f_1(x,y)=0$ and $f_2(x,y)=0$ is finite and greater than 6. So we have

$$BN(2,1;2,1;\Delta_{\rm ms})=\begin{cases}\geqslant 6, & V_1\widehat{V}_1, V_2\widehat{V}_2 \text{ and } V_3\widehat{V}_3 \text{ have a common point of intersection,}\\ 4, & \text{otherwise.}\end{cases}$$

The example shows that the Bezout number may depend on the geometric property of the given triangulation. So we think that the Bezout number $BN(m,r;n,t;\Delta)$ is very complicated and depends on m,r,n,t, the topological structure of Δ and the geometrical property of Δ and $\dim S_m^r(\Delta), \dim S_n^t(\Delta)$.

To obtain an upper bound of Bezout number, several lemmas are needed. Denote by Δ_k the partition containing only k parallel lines.

Lemma 2.12)**.**

$$BN(m,r;n,\mu;\Delta_k)\leqslant (k+1)mn-\min(r,\mu)k.$$

If no triangle in the triangulation Δ is an obtuse triangle, the triangulation is called non-obtuse triangulation. Using the result on the polar coordinates, we have

Lemma 2.2[7]**.** For any given interior vertex in the non-obtuse triangulation Δ, say, v, the Bezout number on $R(v)$ satisfies

$$BN(m,1;n,1;R(v))\leqslant d(v)mn-(d(v)-1). \tag{1}$$

Using a method similar to that in ref. [7], we have

1) Morgan, J., Scott, R., The dimension of the space of C^1 piecewise polynomials, unpublished manuscript, 1975.

2) Luo, Z. X., Nonlinear spline function, Ph. D Dissertation, Dalian University of Technology, 1991.

Lemma 2.3. For any given boundary vertex v in the non-obtuse triangulation Δ, the Bezout number on $R(v)$ satisfies

$$BN(m,1;n,1;R(v)) \leqslant (d(v)-1)mn - (d(v)-2). \tag{2}$$

To obtain the upper bound of $BN(m,1;n,1;\Delta)$, we introduce a combinatorial optimization problem as follows. Let

$$\delta_1(v) = \begin{cases} d(v)-1, & \text{if } v \text{ is an interior vertex of } \Delta, \\ d(v)-2, & \text{if } v \text{ is a boundary vertex of } \Delta. \end{cases}$$

Let $\mathcal{P}$ be the set of points lying in Δ. Then we have

Problem 2.1.

$$\begin{array}{ll} \min_{\mathcal{P}} & \#\mathcal{P} \\ s.t.\mathcal{P} & \text{satisfies Condition 2.1.} \end{array}$$

Condition 2.1.

(i) No point lies on any edges.

(ii) For each vertex v, the number of points lying in $R(v)$ is not less than $\delta_1(v)$.

(iii) The number of points lying in two adjacent cells is not less than 1.

In general, for any given triangulation Δ, denoting by $\mathcal{P}_1(\Delta)$ the solution of Problem 2.1, we have

Theorem 2.1. For any non-obtuse triangulation Δ,

$$BN(m,1;n,1;\Delta) \leqslant mnN - \#\mathcal{P}_1(\Delta).$$

Proof. Suppose that the number of intersection points of two piecewise algebraic curves $\Gamma_1 : f(x,y)=0, \Gamma_2 : g(x,y)=0, f(x,y) \in S_m^1(\Delta), g(x,y) \in S_n^1(\Delta)$ is finite and equals $BN(m,1;n,1;\Delta)$. Label the cells and vertexes in Δ. By Bezout Theorem in the algebraic geometry, there exits $k_i \geqslant 0$ such that the number of the intersection points of Γ_1 and Γ_2 in the ith cell is $mn-k_i$ (if an intersection point lies on a grid line, we count it only on a cell.). We scatter some points in Δ such that the ith cell contains k_i points. Denote by $\mathcal{P}_0$ the set of the points. To prove that $\mathcal{P}_0$ satisfies Condition 2.1, by Theorem 2.2, for the interior vertex v, we have $BN(m,1;n,1;R(v)) \leqslant mnd(v)-(d(v)-1)$. So $\sum_{i\in I(v)}(mn-k_i) \leqslant mnd(v)-(d(v)-1)$, where $I(v)$ is the index set such that the cells $T_i, i \in I(v)$ share v as a common point. Hence $\sum_{i\in I(v)} k_i \geqslant d(v)-1$ holds for any interior vertex v. Similarly, by Lemma 2.3, when v is a boundary vertex, $\sum_{i\in I(v)} k_i \geqslant d(v)-2$. So, for any vertex, $\sum_{i\in I(v)} k_i \geqslant \delta_1(v)$. By using the same method and Lemma 2.1, we can prove that the number of points lying in two adjacent cells is not less than 1. Hence the point set $\mathcal{P}_0$ satisfies Condition 2.1. Therefore, $BN(m,1;n,1;\Delta) = mnN - \sum_i k_i = mnN - \#P_0 \leqslant mnN - \#\mathcal{P}(\Delta)$.

Now, we consider how to solve Problem 2.1. Unfortunately, Problem 2.1 is an integer linear program problem and we cannot construct a "good" algorithm for solving it. But for triangulation Δ, we can obtain a lower bound of $\#\mathcal{P}_1(\Delta)$.

Lemma 2.4. For any given triangulation Δ,

$$\#\mathcal{P}_1(\Delta) \geqslant \frac{\sum_v \delta_1(v)}{3}.$$

Proof. Suppose that $\mathcal{P}_1(\Delta)$ is a solution of Problem 2.1 and the number of points lying in $R(v)$ is K_v. Since Δ is a triangulation, $\sum_v K_v = 3\sum_i k_i$, where k_i is the number of points lying in the ith cell. According to Condition 1, $K_v \geqslant \delta_1(v)$. So we have $\#\mathcal{P}_1(\Delta) = \sum_i k_i \geqslant \frac{\sum_v \delta_1(v)}{3}$.

Theorem 2.2. For any non-obtuse triangulation Δ,

$$BN(m,1;n,1;\Delta) \leqslant mnN - \frac{2E_0}{3} + \frac{V_0+V_b}{3}.$$

By Theorems 2.1 and 2.4, the theorem can be proved easily.

3 Bezout number $BN(m,r;n,t;\Delta)$

Lemma 3.1[1)]**.** For any given interior vertex v in Δ, the Bezout number on $R(v)$ satisfies the following inequality:

$$BN(m,r;n,r;R(v)) \leqslant d(v)mn - r\rho_1(v), \tag{3}$$

where $\rho_1(v) = \max(d(v)-4, [\frac{d(v)+1}{2}])$.

Proof. Firstly we prove

$$BN(m,r;n,r;R(v)) \leqslant d(v)mn - r[\frac{d(v)+1}{2}].$$

Let the union of cells containing the interior edge vv_j be Δ_{vv_j} (fig. 3). By Lemma 2.1, we have $BN(m,r;n,r;\Delta_1) \leqslant 2mn - r$. Hence $BN(m,r;n,r;\Delta_{vv_j}) \leqslant 2mn - r$. Using a similar approach in Theorem 2.1, it is easy to prove that $BN(m,r;n,r;R(v)) \leqslant d(v)mn - r[\frac{d(v)+1}{2}]$.

Secondly, we prove $BN(m,r;n,r;R(v)) \leqslant d(v)mn - r(d(v)-4)_+$. According to the definition of Bezout number, there exist two piecewise algebraic curves $\Gamma_1 : f_1(x,y)=0, \Gamma_2 : f_2(x,y)=0$ such that Γ_1 and Γ_2 have BN intersection points in $R(v)$. Without loss of generality, we assume that v is the origin, and no intersection point of Γ_1 and Γ_2 lies on y-axis and x-axis (fig. 3). Make use of the transformation, $\chi : X = \frac{y}{x}, Y = \frac{1}{x}$. Let $g_1(X,Y) = Y^m f_1(\frac{1}{Y}, \frac{X}{Y}), g_2(X,Y) = Y^n f_2(\frac{1}{Y}, \frac{X}{Y})$. The transformation χ transforms $R(v)$ into a subset of regular partition, say, $\chi R(v)$, which contains only $d(v)$ rays parallel to Y-axis (fig. 3). Since no intersection point of Γ_1 and Γ_2 lies on y-axis and x-axis, the number of intersection points of $g_1(X,Y)=0$ and $g_2(X,Y) = 0, (X,Y) \in R^2$ is equal to that of intersection points of $f_1(x,y)=0$ and $f_2(x,y)=0, (x,y) \in R(v)$. By Lemma 2.1, it is not difficult to prove $BN(m,r;n,r;R(v)) \leqslant mnd(v) - r(d(v)-4)_+$.

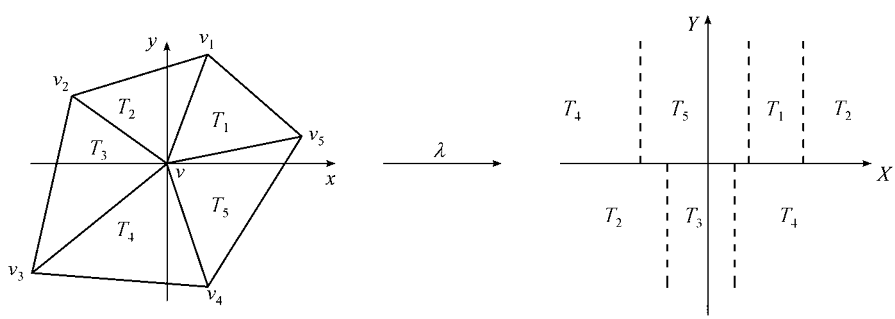

Fig. 3

1) Lai, Y. S., Piecewise algebraic curve and piecewise algebraic variety, Ph. D Dissertation, Dalian University of Technology, 2002.

Therefore, we have $BN(m,r;n,r;R(v)) \leqslant d(v)mn - r\rho_1(v)$.

Lemma 3.2. For any given boundary vertex v in Δ, the Bezout number on $R(v)$ satisfies the following inequality:

$$BN(m,r;n,r;R(v)) \leqslant d(v)mn - r\rho_2(v), \tag{4}$$

where $\rho_2(v) = \max(d(v)-2, [\frac{d(v)-1}{2}])$.

Let

$$\delta_2(v) = \begin{cases} \rho_1(v), & \text{if } v \text{ is an interior vertex of } \Delta, \\ \rho_2(v), & \text{if } v \text{ is a boundary vertex of } \Delta. \end{cases}$$

Let $\mathcal{P}$ be the set of points lying in Δ. Consider the following problem.

Problem 3.1.

$$\begin{array}{ll} \min_{\mathcal{P}} & \#\mathcal{P} \\ s.t. & \mathcal{P} \text{ satisfies Condition 3.1.} \end{array}$$

Condition 3.1.

(i) No point lies on any edges.

(ii) For each vertex v, the number of points lying in $R(v)$ is not less than $\delta_2(v)$.

(iii) The number of points lying in two adjacent cells is not less than 1.

For any given triangulation Δ, denoting by $\mathcal{P}_2(\Delta)$ the solution of Problem 3.1, we have

Theorem 3.1. For any given triangulation Δ,

$$BN(m,r;n,r;\Delta) \leqslant mnN - \#\mathcal{P}_2(\Delta)r.$$

The proof is similar to the proof of Theorem 2.1. Hence we omit it. Problem 3.1 is also NP-complete, but we can give a lower boundary of $\#\mathcal{P}_2(\Delta)$.

Lemma 3.3. For any given triangulation Δ,

$$\#\mathcal{P}_2(\Delta) \geqslant \frac{\sum_v \delta_2(v)}{3}.$$

Therefore, we have

Theorem 3.2. For any given triangulation Δ,

$$BN(m,r;n,t;\Delta) \leqslant mnN - \frac{\sum_v \delta_2(v)}{3}\mu,$$

where $\mu = \min(r,t)$.

Since $BN(m,r;n,t;\Delta) \leqslant BN(m,\mu;n,\mu;\Delta)$, the theorem holds.

Combining Theorems 2.2 and 3.2 gives

Theorem 3.3. For any given non-obtuse triangulation Δ,

$$BN(m,1;n,1;\Delta) \leqslant mnN - \frac{2E_0}{3} + \frac{V_0+V_b}{3}.$$

For any given triangulation Δ,

$$BN(m,r;n,t;\Delta) \leqslant mnN - \frac{\sum_v \delta_2(v)}{3} min\{r,t\}.$$

Remark 3.1. Theorem 3.1 also holds for any partition consisting of finite straight lines.

Remark 3.2. Two approaches can be used to improve the upper bound of Bezout number. One is to obtain the better lower bound of $\#\mathcal{P}_1$ and $\#\mathcal{P}_2$ by combinatorial optimization algorithm. The other is to improve the result in Lemma 3.1.

Remark 3.3. A further problem is to construct two piecewise algebraic curves such that the number of intersection points of them agrees with the upper bound.

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