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Borel probabilistic and quantitative logic

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Abstract The present paper introduces the notion of the probabilistic truth degree of a formula by means of Borel probability measures on the set of all valuations, endowed with the usual product topology, in classical two-valued propositional logic. This approach not only overcomes the limitations of quantitative logic, which require the probability measures on the set of all valuations to be the countably infinite product of uniform probability measures, but also remedies the drawback that probability logic behaves only locally. It is proved that the notions of truth degree, random truth degree in quantitative logic and the probability of formulas in probability logic can all be brought as special cases into the unified framework of the probabilistic truth degree. Thus quantitative logic and probability logic are unified. It also proves a one-to-one correspondence between deductively closed theories and topologically closed subsets of the space of all valuations, and a one-to-one correspondence between probabilistic truth degree functions and Borel probability measures on the space of all valuations. The second part of the present paper proposes an axiomatic definition of the probabilistic truth degree, and it is finally proved that each probabilistic truth degree function is represented by a unique Borel probability measure on the space of all valuations in the way given in the first part. Thus a theory which we call probabilistic and quantitative logic in the framework of classical propositional logic is established.

Keywords probabilistic truth degree, finite separation property, probability logic, quantitative logic, probabilistic and quantitative logic

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1 Introduction

To make logical deduction capable of expressing uncertainty, so as to accord with human reasoning, Hailperin [1] first introduced the idea of probability into classical propositional logic to measure the uncertainty of formulas. Soon afterwards, Nilsson [2] independently published this idea. For a systematic study on probability logic, see the monographs [3, 4]. In probability logic, the probability of a formula is uniquely determined by a probability distribution on the set of all state-descriptions of the formula. Given distinct probability distributions, one then gets different probabilities of the formula. Of course, this reflects the uncertainty of a formula well, but each formula has only finitely many state-descriptions and different formulas do not necessarily have the same state-descriptions. Hence, the probabilities of formulas in probability logic are only designed for concrete formulas, and the probabilities of different formulas are in general incomparable. The basic theory of probability logic concerns the probabilities of

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consequences of a specific valid inference, which consists of only finitely many formulas, while the classical propositional logic has infinitely many formulas. From this point of view, the theory of probability logic behaves only locally.

On the other hand, Wang et al. [5] introduced, by means of the countably infinite product of uniform probability spaces of cardinal 2, the notion of truth degree for all formulas in classical propositional logic in a unified way. This work has aroused great interest of many scholars and a series of important results has been created [6–12]. We summarized these results, which form a new branch of mathematics, called quantitative logic, in [13, 14]. In quantitative logic, all basic logical notions such as theorem, tautology, provable equivalence, logical equivalence and so on are graded. Quantitative logic has overall merits, but it lacks randomness. In fact, in quantitative logic, every formula is assigned a truth degree, but all atomic formulas have the same truth degree $\frac{1}{2}$ and the truth degree of the conjunction of any two atomic formulas is just equal to the product of the truth degrees of these formulas. From the point of view of probability, the probability of each atomic formula is $\frac{1}{2}$, and atomic formulas are mutually independent. This standpoint of viewing probabilities of atomic formulas as equal and independent seems to contradict actual facts in the real world. Keeping this situation in mind, Wang et al. [15] introduced the notion of random truth degree of formulas by means of a random sequence of numbers in the open unit interval. This remedies, to some extent, the limitation that quantitative logic lacks randomness. But it is unfortunate that this approach still requires atomic formulas to be independent. The reason is that each sequence of real numbers in (0, 1) uniquely determines a product probability measure (see (6)). As stated above, there exist in the real world a large number of propositions which are not independent. The reference [3] also discussed this problem in a special section. Then here arises a question: Can we find an effective way to unify probability logic and quantitative logic by learning from their strong points and offsetting their weaknesses?

The problem above is the starting point of this paper. The notion of the probabilistic truth degree of formulas in classical propositional logic is introduce by means of Borel probability measures on the space of all valuations endowed with the usual product topology. This approach not only overcomes the limitations of quantitative logic, which require the probability measures on the set of all valuations to be the countably infinite product of uniform probability measures, but also remedies the drawback that probability logic behaves only locally. It is also proved that the notion of the probabilistic truth degree of formulas can bring the notion of probability of formulas in probability logic, and the notions of truth degree and of random truth degree in quantitative logic into a unified framework as special cases. Hence all basic theories in probability logic and quantitative logic can be discussed in our setting. Moreover, the second part of the paper is devoted to an axiomatization of the probabilistic truth degree function by using Komogorov's axioms [16], and it is finally proved that each probabilistic truth degree function of this kind can be represented by a unique Borel probability measure on the space of all valuations in the same way as given in the first part.

2 Preliminaries

In this section we recall briefly some basic notions and results in probability logic and in quantitative logic. For preliminaries about classical propositional logic, we refer to [14, 17], and we follow the notation in [14] in this paper: for example, $\Omega = \{0,1\}^\omega$ denotes the set of all valuations and $F(S)$ denotes the set of all formulas, where $S = \{p_1, p_2, \ldots\}$ is the set of all atomic formulas.

Before introducing probabilities of formulas in probability logic, we first recall the notion of state-description.

Definition 2.1 [1] Let $A = A(p_1, \ldots, p_n)$ be a formula built up from atomic formulas $p_1, \ldots, p_n$, and assume that $p_i^1 = p_i$, $p_i^0 = \neg p_i$. Let $(x_1, \ldots, x_n) \in \{0,1\}^n$, then the basic conjunction $p_1^{x_1} \wedge \cdots \wedge p_n^{x_n}$ is called a **state-description** of A.

Obviously, $A = A(p_1, \ldots, p_n)$ has a total of 2^n pairwise different state-descriptions, denoted by $S_1, \ldots, S_{2^n}$.

Definition 2.2 [1] Let P be a probability distribution on the set $\{S_1, \ldots, S_{2^n}\}$ of state-descriptions of $A = A(p_1, \ldots, p_n)$. Then the **probability** $P(A)$ of A is defined as

$$P(A) = \sum \{P(\{S_i\}) \mid S_i \text{ logically entails } A\}. \tag{1}$$

By the definition of state-descriptions, state-descriptions $p_1^{x_1} \wedge \cdots \wedge p_n^{x_n}$ are in one-to-one correspondence with n-dimensional $0-1$ vectors $(x_1, \ldots, x_n)$. For convenience's sake, we call $(x_1, \ldots, x_n)$ a state-description. Then a probability distribution on $\{S_1, \ldots, S_{2^n}\}$ is just a probability distribution on $\{0,1\}^n$, and that $p_1^{x_1} \wedge \cdots \wedge p_n^{x_n}$ logically entails $A = A(p_1, \ldots, p_n)$ means $\bar{A}(x_1, \ldots, x_n) = 1$, where $\bar{A}$ is the Boolean function associated with A. From the analysis above we have

$$P(A) = \sum \{P(\{(x_1, \ldots, x_n)\}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\} = P(\bar{A}^{-1}(1)). \tag{2}$$

Example 2.3 (i) Let $A = p_1 \vee p_2$. Then A has four state-descriptions: $(1,1), (1,0), (0,1), (0,0)$. Let $P(\{(1,1)\}) = 0.1$, $P(\{(1,0)\}) = 0.6$, $P(\{(0,1)\}) = 0.2$, $P(\{(0,0)\}) = 0.1$ be a probability distribution on $\{0,1\}^2$. Then $P(A) = P(\bar{A}^{-1}(1)) = P(\{(1,1), (1,0), (0,1)\}) = 0.1 + 0.6 + 0.2 = 0.9$.

(ii) Let $B = p_1$. Then B has two state-descriptions: (1), (0). Let P be a probability distribution on $\{0,1\}$ with $P(\{1\}) = 0.8$, then $P(B) = P(\bar{B}^{-1}(1)) = P(\{(1)\}) = 0.8$.

Note that, in Example 2.3, $P(A) = 0.9$ and $P(B) = 0.8$ are incomparable even though 0.8 and 0.9, as real numbers, are comparable. This is because that the probability distributions are different.

In 2002, Wang et al. [5] introduced the notion of the truth degree of a formula in classical propositional logic with the intent of measuring the extent to which the formula is true; see the survey paper [13]. In contrast to probability logic, the truth degree function is defined on all formulas and consequently truth degrees of all formulas are comparable.

Definition 2.4 [5] Let $X_m = \{0,1\}$, and μ_m the uniform probability measure on X_m for $m = 1, 2, \ldots$. Let $X = \prod_{m=1}^{\infty} X_m$(i.e., $X = \Omega$), and $\mathscr{A}$ the (smallest) σ-algebra generated by the set of all subsets of X of the form $E_1 \times \cdots \times E_n \times X_{n+1} \times X_{n+2} \times \cdots$, where $E_m \subseteq X_m = \{0,1\}, m = 1, \ldots, n$, $n = 1, 2, \ldots$. Then there exists a unique probability measure μ on X (see e.g. [18]) such that

(i) $\mathscr{A}$ is just the set of all μ-measurable subsets,

(ii) For every subset E of $\prod_{m=1}^{n} X_m (n \in N)$, $E \times \prod_{m=n+1}^{\infty} X_m$ is μ-measurable, and

$$\mu\left(E \times \prod_{m=n+1}^{\infty} X_m\right) = (\mu_1 \times \cdots \times \mu_n)(E), n = 1, 2, \ldots .$$

In particular, if $E = E_1 \times \cdots \times E_n$, then

$$\mu\left(E \times \prod_{m=n+1}^{\infty} X_m\right) = (\mu_1 \times \cdots \times \mu_n)(E) = \mu_1(E_1) \times \cdots \times \mu_n(E_n),$$

where $E_m \in \mathscr{P}(X_m)$, $m = 1, \ldots, n$. Then μ is called the **product probability measure** of $\mu_1, \mu_2, \ldots$, denoted by $\mu = \mu_1 \times \mu_2 \times \cdots$. Let $A \in F(S)$, define

$$\tau_2(A) = \mu(A^{-1}(1)). \tag{3}$$

Then $\tau_2(A)$ is called the **truth degree** of A, where A is viewed as a function (in the same symbol) $A : \Omega \to \{0,1\} : A(v) = v(A), v \in \Omega$.

Let $A = A(p_1, \ldots, p_n) \in F(S)$, then the truth value $v(A)$ of A under a valuation v depends only on the truth values $v(p_1), \ldots, v(p_n)$ of the atomic formulas $p_1, \ldots, p_n$ occurring in A. Moreover, for two valuations $u, v \in \Omega$, we have $u(A) = v(A) = \bar{A}(v(p_1), \ldots, v(p_n))$ whenever $u(p_i) = v(p_i), i = 1, \ldots, n$. Hence $A^{-1}(1)$ must be of the form $\bar{A}^{-1}(1) \times \prod_{m=n+1}^{\infty} X_m$, i.e., $A^{-1}(1) = \bar{A}^{-1}(1) \times \prod_{m=n+1}^{\infty} X_m$. Thus (3) can be further simplified as

$$\tau_2(A) = \mu(A^{-1}(1)) = \mu\left(\bar{A}^{-1}(1) \times \prod_{m=n+1}^{\infty} X_m\right)$$

$$
\begin{aligned}
&= (\mu_1 \times \cdots \times \mu_n)(\bar{A}^{-1}(1)) \\
&= (\mu_1 \times \cdots \times \mu_n)(\{(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\}) \\
&= \sum \{(\mu_1 \times \cdots \times \mu_n)(\{(x_1, \ldots, x_n)\}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\} \\
&= \sum \{\mu_1(\{x_1\}) \times \cdots \times \mu_n(\{x_n\}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\} \\
&= \sum \left\{\frac{1}{2} \times \cdots \times \frac{1}{2} \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\right\} \\
&= \frac{1}{2^n}|\bar{A}^{-1}(1)|.
\end{aligned} \tag{4}
$$

(4) is just the expression given in [13].

Comparing (2) with (4), for a concrete formula $A = A(p_1, \ldots, p_n)$, if the P in (2) is uniform, then $P(A) = P(\bar{A}^{-1}(1)) = \sum\{P(\{(x_1, \ldots, x_n)\}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\} = \frac{1}{2^n}|\bar{A}^{-1}(1)| = \tau_2(A)$. For example, in Example 2.3 (i), if P is uniform, then $P(A) = 0.75 = \tau_2(A)$. Thus for a concrete formula A, $\tau_2(A)$ is only a special case of $P(A)$ when P is uniform. But it must be stressed that, τ_2 is defined on the whole set $F(S)$ of formulas, while P is only defined for a specific formula A, hence one cannot simply regard quantitative logic as a special case of probability logic, each has its own appropriate sphere of application.

Let us now recall the notion of a random truth degree of a formula proposed in [15].

Definition 2.5 [15] Let $D = (P_1, P_2, \ldots)$ be a sequence in $(0, 1)$, and $x = (x_1, x_2, \ldots) \in \Omega$ a valuation. $\forall i = 1, 2, \ldots$, let $Q_i^{x_i} = P_i$ if $x_i = 1$ and otherwise $Q_i^{x_i} = 1 - P_i$. $\forall A = A(p_1, \ldots, p_n) \in F(S)$, define

$$\tau_D(A) = \sum\{Q_1^{x_1} \times \cdots \times Q_n^{x_n} \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\}. \tag{5}$$

Then $\tau_D(A)$ is called the D-random truth degree of A.

Just from the definition of (5) one cannot see the distinction and relationship between $\tau_D(A)$ and $\tau_2(A)$, because the background of introducing $\tau_D(A)$ is not so clear, but it actually randomizes the $\tau_2(A)$. In fact, for each coordinate P_i in D, P_i determines a probability measure μ_i on $X_i = \{0, 1\}$ satisfying $\mu_i(\{1\}) = P_i$. Let $\mu = \mu_1 \times \mu_2 \times \cdots$ be the unique product probability measure on $\Omega = \{0, 1\}^\omega$ satisfying the conditions (i) and (ii) of Definition 2.4. Then we have

$$
\begin{aligned}
\tau_D(A) &= \sum\{Q_1^{x_1} \times \cdots \times Q_n^{x_n} \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\} \\
&= \sum\{\mu_1(\{x_1\}) \times \cdots \times \mu_n(\{x_n\}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\} \\
&= \sum\{(\mu_1 \times \cdots \times \mu_n)(\{(x_1, \ldots, x_n)\}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\} \\
&= (\mu_1 \times \cdots \times \mu_n)(\bar{A}^{-1}(1)) \\
&= \mu\left(\bar{A}^{-1}(1) \times \prod_{m=n+1}^{\infty} X_m\right) \\
&= \mu(A^{-1}(1)).
\end{aligned} \tag{6}
$$

It follows from (6) and (4) that $\tau_D(A)$ and $\tau_2(A)$ are defined in the same way, which both are based on the product probability measure μ on Ω except that the μ in (6) is not necessarily uniform. In particular, if $\mu_1, \mu_2, \ldots$ in (6) are all uniform, then $\tau_D(A) = \tau_2(A)$. But (6) still requires μ a product probability measure, and hence atomic formulas are still mutually independent. This is also unacceptable for particular applications.

In the next section we will introduce the notion of the probabilistic truth degree of formulas by means a Borel probability measure on Ω (which is not necessarily uniform or independent), which can bring $P(A)$, $\tau_2(A)$ and $\tau_D(A)$ as special cases into a unified framework.

3 Probabilistic truth degrees of formulas and their properties

3.1 Definitions and properties

Let $X_m = \{0,1\}$ be the discrete topological space for all $m = 1, 2, \ldots$, and $\Omega = \{0,1\}^\omega = \prod_{m=1}^{\infty} X_m$ the usual product topological space, called the valuation space. Let $\mathscr{B}(\Omega)$ and $\mathscr{B}(X_m)$ be the respective sets of Borel subsets of Ω and $X_m (m = 1, 2, \ldots)$. It is clear that $\mathscr{B}(X_m) = \mathscr{P}(X_m)$ for $m = 1, 2, \ldots$. By Lemma 8.2.4 in [19] we have that $\mathscr{B}(\Omega)$ is just the σ-algebra generated by the topological basis for Ω, i.e., $\mathscr{B}(\Omega) = \mathscr{A}$, where $\mathscr{A}$ is as defined in Definition 2.4. Note that for every $n \in \mathbb{N}$, $\{0,1\}^n$ is also endowed with the usual product topology and $\mathscr{B}(\{0,1\}^n) = \mathscr{P}(\{0,1\}^n)$. In this sense, the probability measures μ in (3) and (6) are all Borel.

Definition 3.1 Let $A \in F(S)$, μ be a Borel probability measure on Ω, and define

$$\tau_\mu(A) = \mu(A^{-1}(1)). \tag{7}$$

Then $\tau_\mu(A)$ is called the μ-probabilistic truth degree of A, or simply the μ-truth degree of A.

Remark 3.2 (i) As in Definition 2.4, every formula A is viewed as a function $A : \Omega \to \{0,1\}$, $A(v) = v(A)$, $v \in \Omega$. Note that, for every formula A, the function $A : \Omega \to \{0,1\}$ is continuous from the valuation space Ω into the discrete space $\{0,1\}$, therefore a Borel measurable function on Ω. Hence every formula can be viewed as a discrete function of random variables taking only values 0 and 1 on $(\Omega, \mathscr{B}(\Omega), \mu)$, and $\tau_\mu(A) = \mu(A^{-1}(1)) = 1 \cdot \mu(A^{-1}(1)) + 0 \cdot \mu(A^{-1}(0)) = \int_\Omega A(v)\mathrm{d}\mu$ is the mathematical expectation of A.

(ii) By (7), logically equivalent formulas have the same probabilistic truth degree. Without loss of generality, we assume every formula we are dealing with is built up from the first n atomic formulas $p_1, \ldots, p_n$ for some n.

(iii) Let $A = A(p_1, \ldots, p_n) \in F(S)$, then $A^{-1}(1) = \bar{A}^{-1}(1) \times \prod_{m=n+1}^{\infty} X_m$. Thus $\tau_\mu(A) = \mu(\bar{A}^{-1}(1) \times \prod_{m=n+1}^{\infty} X_m)$. If define $\mu(n) : \mathscr{P}(\{0,1\}^n) \to [0,1]$ by $\mu(n)(E) = \mu(E \times \prod_{m=n+1}^{\infty} X_m), E \in \mathscr{P}(\{0,1\}^n)$, then $\mu(n)$ is a Borel probability measure on the finite product space $\{0,1\}^n$, called the restriction of μ on $\{0,1\}^n$. Then (7) can be further simplified as

$$\tau_\mu(A) = \mu(A^{-1}(1)) = \mu\left(\bar{A}^{-1}(1) \times \prod_{m=n+1}^{\infty} X_m\right) = \mu(n)(\bar{A}^{-1}(1)). \tag{8}$$

Conversely, let P be a Borel probability measure on $\{0,1\}^n$ and μ_m the uniform probability measure on $X_m = \{0,1\}$ $(m = n+1, n+2, \ldots)$. Because Ω can also be viewed as the infinite product space of $\{0,1\}^n, X_{n+1}, X_{n+2}, \ldots$, then $P, \mu_{n+1}, \mu_{n+2}, \ldots$ determine a unique product probability measure μ on Ω. It is easy to check that $\mu(n) = P$.

(iv) Let $A = A(p_1, \ldots, p_n) \in F(S)$ and $\mu(n)$ be a probability measure on $\{0,1\}^n$ (of state-descriptions of A), then $\tau_\mu(A) = \mu(n)(\bar{A}^{-1}(1)) = \sum\{\mu(n)(\{(x_1, \ldots, x_n)\}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\}$. By (2), $\tau_\mu(A)$ is a probability of A in the sense of probability logic. But the most important difference is that τ_μ is defined on the whole set $F(S)$ of formulas.

(v) Let $A = A(p_1, \ldots, p_n) \in F(S)$, and μ be the product of the probability measures μ_m on the subspaces X_m of Ω, i.e., $\mu = \mu_1 \times \mu_2 \times \cdots$, then

$$\begin{aligned}\tau_\mu(A) &= \mu(n)(\bar{A}^{-1}(1)) \\ &= (\mu_1 \times \cdots \times \mu_n)(\bar{A}^{-1}(1)) \\ &= \sum\{(\mu_1 \times \cdots \times \mu_n)(\{(x_1, \ldots, x_n)\}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\} \\ &= \sum\{\mu_1(\{x_1\}) \times \cdots \times \mu_n(\{x_n\}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\}.\end{aligned} \tag{9}$$

(9) is just (6). Thus the random truth degree of formulas in [15] is only a special case of the probabilistic truth degree of formulas in the case where μ is a product probability measure. In particular, if each μ_m is uniform, then by (9), $\tau_\mu(A) = \frac{1}{2^n}|(\bar{A}^{-1}(1))| = \tau_2(A)$, which is just the expression (4).

(vi) Every valuation v is a probabilistic truth degree function τ_μ in the sense of Definition 2.1 when μ satisfies $\mu(E)=1$ whenever $v\in E$.

Example 3.3 Let $A=p_1, B=p_2\to p_3, C=p_1\vee p_2\vee p_3, D=p_1\wedge p_2\wedge p_3$. Find $\tau_\mu(A),\tau_\mu(B),\tau_\mu(C)$,$\tau_\mu(D)$.

Solution (i) Let $\mu(3)(\{(1,1,1)\})=\mu(3)(\{(1,1,0)\})=0.2,\ \mu(3)(\{(x_1,x_2,x_3)\})=0.1,(x_1,x_2,x_3)\in\{0,1\}^3$ and $(x_1,x_2,x_3)\neq(1,1,1),(x_1,x_2,x_3)\neq(1,1,0)$, then

$\tau_\mu(A)=\mu(3)(\bar{A}^{-1}(1))=\mu(3)(\{(1,1,1),(1,1,0),(1,0,1),(1,0,0)\})=0.6$;

$\tau_\mu(B)=\mu(3)(\bar{B}^{-1}(1))=\mu(3)(\{(1,1,1),(1,0,1),(1,0,0),(0,1,1),(0,0,1),(0,0,0)\}=0.7$;

$\tau_\mu(C)=\mu(3)(\bar{C}^{-1}(1))=\mu(3)(\{0,1\}^3-\{(0,0,0)\})=0.9$;

$\tau_\mu(D)=\mu(3)(\{(1,1,1)\})=0.2$.

(ii) Let $\mu(3)$ be a uniform probability measure on $\{0,1\}^3$, then $\tau_\mu(A)=0.5,\tau_\mu(B)=0.75,\tau_\mu(C)=0.875,\tau_\mu(D)=0.125$.

Proposition 3.4 Let μ be a Borel probability measure on Ω, and define

$$H_\mu=\{\tau_\mu(A)\mid A\in F(S)\}.$$

Then

(i) $H_\mu=\{\mu(n)(E)\mid E\subseteq\{0,1\}^n,n\in\mathbb{N}\}$.

(ii) If μ satisfies $\mu(\{v\})=0$ for every $v\in\Omega$, then H_μ is a countable dense subset of $[0,1]$.

(iii) If μ is generated by uniform probability measures on $\{0,1\}$, then $H_\mu=\{\frac{k}{2^n}\mid k=0,1,\ldots,2^n,n\in\mathbb{N}\}$.

Proof. (i) Take any $A\in F(S)$. Without loss of generality, we can assume that $A=A(p_1,\ldots,p_n)$. Then $\bar{A}^{-1}(1)\subseteq\{0,1\}^n$, and by (8), $\tau_\mu(A)=\mu(n)(\bar{A}^{-1}(1))\in\{\mu(n)(E)\mid E\subseteq\{0,1\}^n,n\in\mathbb{N}\}$. For the converse, take $E\subseteq\{0,1\}^n$ and construct a formula $A=\vee\{p_1^{x_1}\wedge\cdots\wedge p_n^{x_n}\mid(x_1,\ldots,x_n)\in E\}$ where we assume that $p^1=p$ and $p^0=\neg p$. Then it is easy to check that $\bar{A}^{-1}(1)=E$. Therefore $\mu(n)(E)\in H_\mu$ and $\{\mu(n)(E)\mid E\subseteq\{0,1\}^n,n\in\mathbb{N}\}\subseteq H_\mu$.

(ii) Suppose that $\mu(\{v\})=0$ for every $v\in\Omega=\{0,1\}^\omega$. Then $\forall\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $\mu((x_1,\ldots,x_n)\times\prod_{m=n+1}^{\infty}X_m)<\varepsilon$ for every $(x_1,\ldots,x_n)\in\{0,1\}^n$. $\forall E\subseteq\{0,1\}^n$, construct a formula $A_E=\vee\{p_1^{x_1}\wedge\cdots\wedge p_n^{x_n}\mid(x_1,\ldots,x_n)\in E\}$. Then $\{\mu(A_E)\mid E\subseteq\{0,1\}^n\}\subseteq H_\mu$, and so H_μ is dense in $[0,1]$.

(iii) An immediate consequence of (i) and (ii).

τ_μ has the following properties.

Proposition 3.5 Let μ be a Borel probability measure on Ω, $A,B\in F(S)$. Then

(i) $\tau_\mu(A)=\Sigma\{\tau_\mu(A\wedge S_i)\mid S_i$ is a state-description of $A\}=\Sigma\{\tau_\mu(S_i)\mid S_i$ logically entails $A\}$.

(ii) If A is a tautology (contradiction), then $\tau_\mu(A)=1(\tau_\mu(A)=0)$.

(iii) If A and B are logically equivalent, then $\tau_\mu(A)=\tau_\mu(B)$.

(iv) $\tau_\mu(\neg A)=1-\tau_\mu(A)$.

(v) $\tau_\mu(A)+\tau_\mu(B)=\tau_\mu(A\vee B)+\tau_\mu(A\wedge B)$.

(vi) $1+\tau_\mu(A\wedge B)=\tau_\mu(A)+\tau_\mu(A\to B)$.

(vii) $\tau_\mu(A)+\tau_\mu(A\to B)=\tau_\mu(B)+\tau_\mu(B\to A)$.

(viii) If $\models A\to B$, then $\tau_\mu(A)\leqslant\tau_\mu(B)$.

(ix) $\tau_\mu(B)\leqslant\tau_\mu(A\to B)$.

(x) $\tau_\mu(A\wedge B)\leqslant\min\{\tau_\mu(A),\tau_\mu(B)\}\leqslant\max\{\tau_\mu(A),\tau_\mu(B)\}\leqslant\tau_\mu(A\vee B)$.

Proof. A routine verification.

Corollary 3.6 Let $A,B,C\in F(S)$, and μ be a Borel probability measure on Ω, $\alpha,\beta\in[0,1]$. Then

(i) If $\tau_\mu(A)\geqslant\alpha,\tau_\mu(A\to B)\geqslant\beta$, then $\tau_\mu(B)\geqslant\alpha+\beta-1$.

(ii) If $\tau_\mu(A\to B)\geqslant\alpha,\tau_\mu(B\to C)\geqslant\beta$, then $\tau_\mu(A\to C)\geqslant\alpha+\beta-1$.

Corollary 3.7 Let $A,B,C\in F(S)$, and μ be a Borel probability measure on Ω. Then

(i) If $\tau_\mu(A\to B)=1$, then $\tau_\mu(A)\leqslant\tau_\mu(B)$.

(ii) If $\tau_\mu(A)=1,\tau_\mu(A\to B)=1$, then $\tau_\mu(B)=1$.

(iii) If $\tau_\mu(A \to B) = 1, \tau_\mu(B \to C) = 1$, then $\tau_\mu(A \to C) = 1$.

3.2 Deductively closed theories versus topologically closed sets

Definition 3.8 Let μ be a Borel probability measure on Ω, and define

$$\ker(\tau_\mu) = \{A \in F(S) \mid \tau_\mu(A) = 1\}.$$

Then $\ker(\tau_\mu)$ is called the **kernel** of τ_μ.

Theorem 3.9 $\ker(\tau_\mu)$ is a deductively closed consistent theory.

Proof. (i) Let $A \in F(S)$. If A is an axiom, then clearly we have $A \in \ker(\tau_\mu)$.

(ii) Let A, $A \to B \in \ker(\tau_\mu)$, then $\tau_\mu(A) = 1, \tau_\mu(A \to B) = 1$. By Corollary 3.7(ii) we have $\tau_\mu(B) = 1$, and consequently $B \in \ker(\tau_\mu)$.

It follows from (i) and (ii) that $\ker(\tau_\mu)$ is deductively closed. Obviously $\perp \notin \ker(\tau_\mu)$; this shows that $\ker(\tau_\mu)$ is consistent.

In the following we concentrate on the converse problem of Theorem 3.9, i.e, whether each deductively closed consistent theory has the form $\ker(\tau_\mu)$ for some τ_μ. To do this, we first study properties of models of a theory.

Definition 3.10 Let $\Sigma \subseteq \Omega = \{0,1\}^\omega$. If Σ satisfies: $\forall u = (u_1, u_2, \ldots) \in \Omega - \Sigma$, there exists $n \in \mathbb{N}$ such that $\forall v = (v_1, v_2, \ldots) \in \Sigma$, $(u_1, \ldots, u_n) \neq (v_1, \ldots, v_n)$, then Σ is said to have the finite separation property.

For example, $\Sigma = \emptyset$, $\Sigma = \{(1, 1, \ldots)\}$ and $\Sigma = \Omega$ all have this property; however, $\Sigma = \Omega - \{(1, 1, \ldots)\}$ does not. We shall prove a one-to-one correspondence between deductively closed theories and closed subsets of Ω by virtue of subsets having the finite separation property, and we finally give a positive answer to the converse of Theorem 3.9.

Theorem 3.11 Let Γ be a theory, and Σ the set of all models of Γ, then Σ has the finite separation property.

Proof. We show the claim above by contradiction. Suppose not, then there exists $u = (u_1, u_2, \ldots) \in \Omega - \Sigma$ such that, $\forall n \in \mathbb{N}$ there is $v = (v_1, v_2, \ldots) \in \Sigma$ with $(u_1, \ldots, u_n) = (v_1, \ldots, v_n)$. Take any $A \in \Gamma$, and assume $A = A(p_1, \ldots, p_n)$. Then

$$1 = v(A) = \bar{A}(v(p_1), \ldots, v(p_n)) = \bar{A}(u(p_1), \ldots, u(p_n)) = u(A).$$

By the arbitrariness of A, $u \in \Sigma$, a contradiction.

Theorem 3.12 Suppose that $\Sigma \subseteq \Omega$ has the finite separation property, then there exists a theory Γ such that Σ is just the set of all models of Γ.

Proof. If $\Sigma = \emptyset$, $\Gamma = \{p_1, \neg p_1\}$ is required. Suppose now $\Sigma \neq \emptyset$. $\forall n \in \mathbb{N}$, let $\Sigma(n) = \{(v_1, \ldots, v_n) \mid v = (v_1, v_2, \ldots) \in \Sigma\}$ and construct a theory

$$\Gamma = \{\vee\{p_1^{v_1} \wedge \cdots \wedge p_n^{v_n} \mid (v_1, \ldots, v_n) \in \Sigma(n)\} \mid n \in \mathbb{N}\}. \tag{10}$$

Note that because, $\forall n \in \mathbb{N}$, $\Sigma(n)$ is finite, every formula in Γ is well defined, and consequently Γ is a theory. It is easy to check that Σ is the set of all models of Γ.

For every subset Σ of Ω having the finite separation property, the proof of Theorem 3.12 not only points out that there exists a theory Γ such that Σ is exactly the set of all models of Γ but also gives the structure of Γ. The problem is how many such theories there are for a given Σ. Clearly, every theory Γ' with $\Gamma \subseteq \Gamma' \subseteq D(\Gamma)$ satisfies the requirement, where Γ is the theory defined in the proof of Theorem 3.12. But there is a unique deductively closed theory whose set of models is Σ. This is given precisely by the strong completeness theorem of classical propositional logic.

Proposition 3.13 Let Γ_1, Γ_2 be theories, and Σ_1, Σ_2 their sets of models, respectively. Then $D(\Gamma_1) = D(\Gamma_2)$ iff $\Sigma_1 = \Sigma_2$.

Corollary 3.14 There is a one-to-one correspondence between deductively closed theories and subsets of Ω having the finite separation property.

Next we shall show that subsets of Ω having the finite separation property are in one-to-one correspondence with the closed subsets of Ω. Note that Ω as the usual product topological space is metrizable; the metric ρ (see e.g. [20]) is defined by

$$\rho(u,v) = \max\left\{\frac{|u_i - v_i|}{i} \mid i = 1,2,\dots\right\}, \tag{11}$$

for all $u = (u_1, u_2, \dots), v = (v_1, v_2, \dots) \in \Omega$.

Theorem 3.15 Let $\Sigma \subseteq \Omega$, then Σ has the finite separation property iff Σ is closed in (Ω, ρ).

Proof. Suppose that Σ has the finite separation property. To show the topological closedness of Σ, it is enough to show that $\forall u \in \Omega - \Sigma$ there is $\varepsilon > 0$ such that $\rho(u, \Sigma) > \varepsilon$. Take any $u \in \Omega - \Sigma$. Then there exists $n \in \mathbb{N}$ such that $\forall v = (v_1, v_2, \dots) \in \Sigma$, $(u_1, \dots, u_n) \neq (v_1, \dots, v_n)$. Let $\varepsilon = \frac{1}{n+1}$, then $\forall v = (v_1, v_2, \dots)$, $\rho(u,v) \geqslant \frac{1}{n} > \frac{1}{n+1} = \varepsilon$, and hence $\rho(u, \Sigma) \geqslant \frac{1}{n} > \varepsilon$. This shows that Σ is closed.

Conversely, let Σ be a closed set. Then, $\forall u \in \Omega - \Sigma$ there is $\varepsilon(0 < \varepsilon < 1)$ such that $\rho(u, \Sigma) > \varepsilon$. Let $n = [\frac{1}{\varepsilon}] + 1$, then $\varepsilon > \frac{1}{n}$. Therefore $\forall v \in \Sigma$, $\rho(u,v) > \frac{1}{n}$, and so $(u_1, \dots, u_n) \neq (v_1, \dots, v_n)$. This shows that Σ has the finite separation property.

It follows from Theorems 3.14 and 3.15 that:

Theorem 3.16 There is a one-to-one correspondence between deductively closed theories and closed subsets of the space Ω.

By Theorem 3.16, minimal nonempty closed subsets of Ω (i.e., singleton sets) correspond to maximally consistent theories (because maximally consistent theories must be deductively closed [21]). Thus we get the structural and topological characterizations of maximally consistent theories in classical propositional logic.

Theorem 3.17 $\forall v = (v_1, v_2, \dots) \in \Omega$, define $\Gamma_v = D(\{p_1^{v_1}, p_1^{v_1} \wedge p_2^{v_2}, \dots\})$, then

(i) $M = \{\Gamma_v \mid v \in \Omega\}$ is the set of all maximally consistent theories in classical propositional logic.

(ii) Define on M, $d(\Gamma_u, \Gamma_v) = \rho(u,v), u, v \in \Omega$, where ρ is given by (11), then (M, d) is a Cantor space [22].

It is left for the reader to check that, $\forall v = (v_1, v_2, \dots) \in \Omega$, the maximally consistent theory Γ_v can be further characterized by $\Gamma_v = D(\{p_1^{v_1}, p_2^{v_2}, \dots\})$, which is the same as in [23].

We are now ready to prove the converse of Theorem 3.9.

Theorem 3.18 For each maximally consistent theory Γ in classical propositional logic, there exists a Borel probability measure μ on Ω such that $\Gamma = \ker(\tau_\mu)$.

Proof. Let Γ be a deductively closed consistent theory, and Σ the set of models of Γ. It follows from Theorems 3.11 and 3.15 that Σ is a nonempty closed subset of Ω. Consider Σ as a subspace of Ω, then $\mathscr{B}(\Sigma) = \mathscr{B}(\Omega) \cap \Sigma = \{\Delta \cap \Sigma \mid \Delta \in \mathscr{B}(\Omega)\}$. Take a Borel probability measure μ_0 on Σ satisfying $\mu_0(\Lambda) = 1$ iff $\Lambda = \Sigma$, where Λ is a closed subset of Σ. Define $\mu : \mathscr{B}(\Omega) \to [0,1]$ by

$$\mu(\Delta) = \mu_0(\Delta \cap \Sigma), \quad \Delta \in \mathscr{B}(\Omega).$$

Then μ is a Borel probability measure on Ω. Take any $A \in \Gamma$. Then it is clear that $\Sigma \subseteq A^{-1}(1)$, and $\tau_\mu(A) = \mu(A^{-1}(1)) = \mu(\Sigma) = 1$, this shows $\Gamma \subseteq \ker(\tau_\mu)$. Conversely, because both Γ and $\ker(\tau_\mu)$ are deductively closed, it suffices to show that $\forall B \in \ker(\tau_\mu)$, $\Sigma \subseteq B^{-1}(1)$. Suppose that $\Sigma \nsubseteq B^{-1}(1)$ for some $B \in \ker(\tau_\mu)$, then $B^{-1}(1) \cap \Sigma$ is a proper closed subset of Σ. Hence, $\tau_\mu(B) = \mu(B^{-1}(1)) = \mu_0(B^{-1}(1) \cap \Sigma) \neq 1$, a contradiction.

Theorems 3.9 and 3.18 give a one-to-one correspondence between deductively closed theories and kernels $\ker(\tau_\mu)$ of truth degree functions τ_μ. From the proof of Theorem 3.18 we obtain a characterization of truth degree functions with the same kernel:

Theorem 3.19 Let μ, ν be Borel probability measures on Ω, then $\ker(\tau_\mu) = \ker(\tau_\nu)$ iff $\mu(\Sigma) = \nu(\Sigma) = 1$ and for a topological closed subset Δ, $\Delta \subset \Sigma$ implies $\mu(\Delta) < 1, \nu(\Delta) < 1$, where Σ is the set of models of $\ker(\tau_\mu)$. In particular, if $\ker(\tau_\mu)$ is maximally consistent, then

$$\ker(\tau_\mu)=\ker(\tau_\nu) \text{ iff } \tau_\mu=\tau_\nu.$$

Proof. The first part follows from the proof of Theorem 3.18. Suppose now $\ker(\tau_\mu)$ is maximally consistent. By Theorem 3.17 there exists $u \in \Omega$ such that $\Sigma=\{u\}$, and consequently $\mu(\{u\})=\nu(\{u\})=1$. Thus, $\tau_\mu=u=\tau_\nu$.

When $\tau_\mu=\tau_\nu$ we have $\mu=\nu$, i.e..

Theorem 3.20 Let μ,ν be Borel probability measures on Ω, then

$$\tau_\mu=\tau_\nu \quad \text{iff } \mu=\nu.$$

To prove Theorem 3.20, we need a lemma.

Lemma 3.21 [18] Let μ,ν be Borel probability measures on Ω. If $\mu(\Sigma)=\nu(\Sigma)$ for all topological closed subsets $\Sigma\subseteq\Omega$, then $\mu=\nu$.

The proof of Theorem 3.20. Suppose now $\tau_\mu=\tau_\nu$. By Lemma 3.21 it is enough to show $\mu(\Sigma)=\nu(\Sigma)$ for any closed subset $\Sigma\subseteq\Omega$. Suppose to the contrary that there is a closed subset $\Sigma_0\subseteq\Omega$ with $\mu(\Sigma_0)\neq\nu(\Sigma_0)$. Then there exists $n_0\in\mathbb{N}$ such that $\mu(\Sigma_0(n_0)\times\prod_{m=n_0+1}^{\infty}X_m)\neq\nu(\Sigma_0(n_0)\times\prod_{m=n_0+1}^{\infty}X_m)$. By (10), construct a formula $A_0=\vee\{p_1^{v_1}\wedge\cdots\wedge p_{n_0}^{v_{n_0}} \mid (v_1,\ldots,v_{n_0})\in\Sigma_0(n_0)\}$. It is easy to see that $\overline{A_0}^{-1}(1)=\Sigma_0(n_0)$. Then $\tau_\mu(A_0)=\mu(\Sigma_0(n_0)\times\prod_{m=n_0+1}^{\infty}X_m)\neq\nu(\Sigma_0(n_0)\times\prod_{m=n_0+1}^{\infty}X_m)=\tau_\nu(A_0)$, a contradiction. The converse is obvious.

Theorem 3.20 gives a one-to-one correspondence between truth degree functions τ_μ defined by (7) and Borel probability measures μ on Ω. In section 4 we shall give an axiomatic definition of the probabilistic truth degree functions by using Kolmogorov axioms in probability theory, and finally prove the representations of such truth degree functions for Borel probability measures by (7).

4 Axiomatic definition of truth degree functions and their representations

Definition 4.1 (Kolmogorov axioms) Let $A,B\in F(S)$. A mapping $\tau: F(S)\to[0,1]$ is called a truth degree function if it satisfies

(K1) $0\leqslant\tau(A)\leqslant 1$.

(K2) If A is an axiom, then $\tau(A)=1$.

(K3) If $A\to B$ is a theorem, then $\tau(A)\leqslant\tau(B)$.

(K4) If $\{A,B\}$ is inconsistent (simply called A and B inconsistent), then $\tau(A\vee B)=\tau(A)+\tau(B)$.

Example 4.2 (i) Let μ be a Borel probability measure on Ω, then the τ_μ in (7) is a truth degree function in the sense of Definition 4.1.

(ii) Each $v\in\Omega$ is also a truth degree function in the sense of Definition 4.1.

In the following proposition we select basic properties of τ.

Proposition 4.3. (i) If $\vdash A$, then $\tau(A)=1$.

(ii) If A and B are provably equivalent, then $\tau(A)=\tau(B)$.

(iii) $\tau(\neg A)=1-\tau(A)$.

(iv) If A is the refutable formula $\perp$, then $\tau(A)=0$.

(v) $\tau(A)+\tau(B)=\tau(A\wedge B)+\tau(A\vee B)$.

(vi) $\tau(A\leftrightarrow B)=\tau(A\to B)+\tau(B\to A)-1$.

(vii) $1+\tau(A\wedge B)=\tau(A)+\tau(A\to B)$.

(viii) $\tau(A)+\tau(A\to B)=\tau(B)+\tau(B\to A)$.

(ix) $\tau(B)\leqslant\tau(A\to B)$.

(x) $\tau(A\wedge B)\leqslant\min\{\tau(A),\tau(B)\}\leqslant\max\{\tau(A),\tau(B)\}\leqslant\tau(A\vee B)$.

(xi) $\tau(A\wedge B)\geqslant\tau(A)+\tau(B)-1$.

(xii) $\tau(B)\geqslant\tau(A)+\tau(A\to B)-1$.

(xiii) If $\tau(A \to B) = 1$, then $\tau(A) \leqslant \tau(B)$.

(xiv) $\tau(A \to C) \geqslant \tau(A \to B) + \tau(B \to C) - 1$.

Proof. We limit ourselves to (v). Because $A \sim A \wedge (B \vee \neg B) \sim (A \wedge B) \vee (A \wedge \neg B)$, and $A \wedge B$ and $A \wedge \neg B$ are inconsistent, we have $\tau(A) = \tau((A \wedge B) \vee (A \wedge \neg B)) = \tau(A \wedge B) + \tau(A \wedge \neg B)$. Because $A \vee B$ and $(A \wedge \neg B) \vee B$ are provably equivalent, and $A \wedge \neg B$ and B are inconsistent, we have $\tau(A \vee B) = \tau((A \wedge \neg B) \vee B) = \tau(A \wedge \neg B) + \tau(B)$. Hence $\tau(A) = \tau(A \wedge B) + \tau(A \wedge \neg B) = \tau(A \wedge B) + \tau(A \vee B) - \tau(B)$.

The following are characterizations of truth degree functions.

Theorem 4.4 $\tau : F(S) \to [0, 1]$ is a truth degree function on $F(S)$ iff $\forall A, B \in F(S)$, τ satisfies

(i) If $\vdash A$, then $\tau(A) = 1$.

(ii) If $\vdash \neg A$, then $\tau(A) = 0$.

(iii) $\tau(A) + \tau(B) = \tau(A \vee B) + \tau(A \wedge B)$.

Proof. The necessity is given by Proposition 4.3 (i), (iv) and (v). We show now the sufficiency. (K1) and (K2) clearly hold. Because $A \wedge B$ is refutable whenever A and B are inconsistent, it follows from (ii) and (iii) that (K4) is true. It is left to show (K3). We first show $\tau(\neg A) = 1 - \tau(A)$. In fact, it follows from (i), (iii) and the facts that $\neg A \vee A$ is a theorem and $\neg A \wedge A$ is refutable that $1 = \tau(\neg A \vee A) = \tau(\neg A) + \tau(A) - \tau(\neg A \wedge A) = \tau(\neg A) + \tau(A)$. Hence $\tau(\neg A) = 1 - \tau(A)$. Let $\vdash A \to B$, then $\vdash \neg A \vee B$, it then follows from (i) and (iii) that $1 = \tau(\neg A \vee B) = \tau(\neg A) + \tau(B) - \tau(\neg A \wedge B) = 1 - \tau(A) + \tau(B) - \tau(\neg A \wedge B)$. This shows $\tau(A) \leqslant \tau(B)$.

Theorem 4.5 $\tau : F(S) \to [0, 1]$ is a truth degree function on $F(S)$ iff $\forall A, B \in F(S)$, τ satisfies

(i) If A is an axiom, then $\tau(A) = 1$.

(ii) $\tau(\neg A) = 1 - \tau(A)$.

(iii) $\tau(A) + \tau(A \to B) = \tau(B) + \tau(B \to A)$.

Proof. By Proposition 4.3, it suffices to show the sufficiency. By (iii), $\tau(B) = 1$ whenever $\tau(A) = \tau(A \to B) = 1$. It follows from (i) and this fact that $\tau(A) = 1$ for all theorems A. Let $\vdash A \to B$, then $\tau(A \to B) = 1$. By (iii) we have $\tau(A) = \tau(B) + \tau(B \to A) - \tau(A \to B) = \tau(B) + \tau(B \to A) - 1 \leqslant \tau(B)$, which shows (K3), and consequently, $\tau(A) = \tau(B)$ for provably equivalent formulas A, B. Next we show (K4). By (ii) and (iii), we have

$$\begin{aligned}
\tau(A \vee B) &= \tau(\neg A \to B) \\
&= \tau(B \to \neg A) + \tau(B) - \tau(\neg A) \\
&= \tau(B) + \tau(A) - 1 + \tau(B \to \neg A) \\
&= \tau(B) + \tau(A) - 1 + \tau(\neg B \vee \neg A) \\
&= \tau(B) + \tau(A) - \tau(A \wedge B).
\end{aligned}$$

By (i) and (ii), $\tau(A \wedge B) = 0$ whenever A and B are inconsistent, hence $\tau(A \vee B) = \tau(A) + \tau(B)$.

By Theorem 4.5, the truth degree function τ is equivalent to the so-called syntactic degree function τ^* introduced in [24]. In this paper we shall prove that such a truth degree function can be represented by (7) for a unique Borel probability measure. An alternate characterization of τ is given below.

Theorem 4.6 $\tau : F(S) \to [0, 1]$ is a truth degree function on $F(S)$ iff $\forall A, B \in F(S)$, τ satisfies

(i) If $\vdash A$, then $\tau(A) = 1$.

(ii) If $\vdash \neg A$, then $\tau(A) = 0$.

(iii) $\tau(A) + \tau(A \to B) = \tau(B) + \tau(B \to A)$.

Proof. By Proposition 4.3 it suffices to show the sufficiency. By Theorem 4.5 it is enough to show that $\forall A \in F(S)$, $\tau(\neg A) = 1 - \tau(A)$. By (i) and (iii) $\tau(E) = \tau(F)$ if E and F are provably equivalent. Then $\tau(A) + \tau(\neg A) = \tau(A) + \tau(A \to \bot) = \tau(\bot) + \tau(\top) = 0 + 1 = 1$ because $\neg A$ and $A \to \bot$ are provably equivalent and so are $\bot \to A$ and $\top$.

Before giving the representation of each truth degree function τ on $F(S)$, some preliminary work is needed.

Take any $n \in \mathbb{N}$ and any $(x_1, \ldots, x_n) \in \{0,1\}^n$, let $\delta_{(x_1,\ldots,x_n)}$ denote the basic conjunction $p_1^{x_1} \wedge \cdots \wedge p_n^{x_n}$ in (10), i.e., $\delta_{(x_1,\ldots,x_n)} = p_1^{x_1} \wedge \cdots \wedge p_n^{x_n}$.

Theorem 4.7 Let τ be a truth degree function on $F(S)$, and $(x_1, \ldots, x_n) \in \{0,1\}^n$, then

(i) $0 \leqslant \tau(\delta_{(x_1,\ldots,x_n)}) \leqslant 1$.

(ii) $\Sigma\{\tau(\delta_{(x_1,\ldots,x_n)}) \mid (x_1, \ldots, x_n) \in \{0,1\}^n\} = 1$.

(iii) $\tau(\delta_{(x_1,\ldots,x_n,0)}) + \tau(\delta_{(x_1,\ldots,x_n,1)}) = \tau(\delta_{(x_1,\ldots,x_n)})$.

Proof. (i) By (K1), (i) is trivial.

(ii) Take any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \{0,1\}^n$ with $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n)$, then $\delta_{(x_1,\ldots,x_n)}$ and $\delta_{(y_1,\ldots,y_n)}$ are inconsistent, and it then follows from (K4) that $\tau(\delta_{(x_1,\ldots,x_n)} \vee \delta_{(y_1,\ldots,y_n)}) = \tau(\delta_{(x_1,\ldots,x_n)}) + \tau(\delta_{(y_1,\ldots,y_n)})$. Because $\vee\{\delta_{(x_1,\ldots,x_n)} \mid (x_1, \ldots, x_n) \in \{0,1\}^n\}$ is a theorem, we have $\tau(\vee\{\delta_{(x_1,\ldots,x_n)} \mid (x_1, \ldots, x_n) \in \{0,1\}^n\}) = 1$. By several applications of (K4) we get (ii).

(iii) It follows from the provable equivalence of $\delta_{(x_1,\ldots,x_m,0)} \vee \delta_{(x_1,\ldots,x_m,1)}$ and $\delta_{(x_1,\ldots,x_m)}$ and (ii) that (iii) is true.

Theorem 4.8. Let τ be a truth degree function on $F(S)$, $A = A(p_1, \ldots, p_n) \in F(S)$, then

$$\tau(A) = \Sigma\{\tau(\delta_{(x_1,\ldots,x_n)}) \mid (x_1, \ldots, x_n) \in \bar{A}^{-1}(1)\}. \tag{12}$$

Proof. By Proposition 4.3 (ii), $\tau(A) = \tau(B)$ if A and B are provably equivalent. Without changing the value $\tau(A)$ of A, we assume that A is in disjunctive normal form, i.e., $A = \vee\{\delta_{(x_1,\ldots,x_n)} \mid (x_1, \ldots, x_n) \in \overline{A}^{-1}(1)\}$. By the proof of Theorem 4.7 we have $\tau(A) = \tau(\vee\{\delta_{(x_1,\ldots,x_n)} \mid (x_1, \ldots, x_n) \in \overline{A}^{-1}(1)\}) = \Sigma\{\tau(\delta_{(x_1,\ldots,x_n)}) \mid (x_1, \ldots, x_n) \in \overline{A}^{-1}(1)\}$, which is (12).

Theorem 4.9 (Representation theorem) Let τ be a truth degree function on $F(S)$, then there is a unique Borel probability measure μ on Ω such that for all $A \in F(S)$,

$$\tau(A) = \mu(A^{-1}(1)). \tag{13}$$

Proof. Construct a Borel probability measure μ satisfying (13). Take any $n \in \mathbb{N}$, let

$$\mu\left((x_1, \ldots, x_n) \times \prod_{m=n+1}^{\infty} X_m\right) = \tau(\delta_{(x_1,\ldots,x_n)}). \tag{14}$$

For any closed subset Σ of Ω, let

$$\mu\left(\Sigma(n) \times \prod_{m=n+1}^{\infty} X_m\right) = \Sigma\{\tau(\delta_{(x_1,\ldots,x_n)}) \mid (x_1, \ldots, x_n) \in \Sigma(n)\}, \tag{15}$$

and

$$\mu(\Sigma) = \lim_{n\to\infty} \mu\left(\Sigma(n) \times \prod_{m=n+1}^{\infty} X_m\right). \tag{16}$$

Take any Borel set $\Delta \in \mathscr{B}(\Omega)$, let

$$\mu(\Delta) = \sup\{\mu(\Sigma) \mid \Sigma \subseteq \Delta \text{ is topologically closed}\}. \tag{17}$$

Then it follows from (14)–(17) that μ is a Borel probability measure on Ω. For every $A \in F(S)$, it follows from Theorem 4.8 that

$$\begin{aligned}\tau(A) &= \Sigma\{\tau(\delta_{(x_1,\ldots,x_n)}) \mid (x_1, \ldots, x_n) \in \overline{A}^{-1}(1)\} \\ &= \mu\left(\overline{A}^{-1}(1) \times \prod_{m=n+1}^{\infty} X_m\right) = \mu(A^{-1}(1));\end{aligned}$$

this shows (13). Let ν be an arbitrary Borel probability measure on Ω satisfying (13), then ν satisfies (14)–(17). By Lemma 3.21 we must have $\mu = \nu$.

Theorem 4.9 tells us that there are only truth degree functions of the form given by (7). That is one reason why we use Borel probability measures instead of general probability measures on the valuation space Ω to introduce the notion of probabilistic truth degree in the very beginning of the paper. Of course, the most important reason is that using Borel probability measures can bring existing notions into a unified framework.

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