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MATHEMATICS

ON A PROBLEM IN THE THEORY OF ABSOLUTE SUMMABILITY FACTORS OF THE FOURIER SERIES

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Let $f(t)$ be a periodic function, with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Suppose that

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) = \mathfrak{S}[f] \qquad (1)$$

is the Fourier series of $f(t)$.

Cheng[1], Pati[2], and Prasad[3] raised the question: "under the condition

$$\int_0^t |f(x+u) + f(x-u) - 2f(x)|\,du = o(t) \quad (t \to 0), \qquad (2)$$

will $\sum_{n=1}^{\infty} \lambda_n A_n(x)$ be summable $|C, 1|$ for every convex sequence $\{\lambda_n\}$ with $\sum n^{-1}\lambda_n < \infty$?"

The author answers the question negatively in [4]. Indeed, the following theorem was shown there.

Theorem A. *Suppose* $0 < \eta < \frac{1}{2}$, *then there exists an integrable even function* $f(t)$ *with period* 2π, *such that* (2) *holds with* $x = 0$ *and that*

$$\sum_{n=2}^{\infty} (\log n)^{-1-\eta} A_n(0)$$

is not summable $|C, 1|$.

On the other hand, Cheng proves the following[1]

Theorem B. *If* $\lambda_n = 1/(\log n)^{\frac{3}{2}+\varepsilon}$, $(\varepsilon > 0)$, *then, under* (2) *the series* $\sum \lambda_n A_n(x)$ *is summable* $|C, 1|$ *at the point* x.

The aim of this paper is to show that the number ε in Theorem B is not allowed to be replaced by 0. We prove

Theorem 1. *There is an integrable even periodic function* $f(t)$ *with period* 2π, *such that* (2) *holds with* $x = 0$ *and that*

$$\sum_{n=2}^{\infty} (\log n)^{-\frac{3}{2}} A_n(0)$$

is not summable $|C, 1|$.

According to a theorem of Pati[2], to demonstrate Theorem 1 is equivalent to establishing

Theorem 2. *There exists an integrable even periodic function* $f(t)$ *with period* 2π *satisfying the conditions*

$$\int_0^t |f(u)|\,du = o(t), \quad (t \to 0),$$

$$f(0) = 0 \qquad (3)$$

and

$$\sum_{n=2}^{\infty} n^{-1} (\log n)^{-\frac{3}{2}} |S_n(0)| = \infty, \qquad (4)$$

$S_n(0)$ *being the partial sums of* $\mathfrak{S}[f]$.

If we write

$$S_n^* = a_1 + a_2 + \cdots + a_{n-1} + \frac{a_n}{2} = \frac{1}{2}(S_n(0) + S_{n-1}(0)),$$

$$b_n = n^{-1} S_n^*,$$

(4) is equivalent to

$$\sum_{n=2}^{\infty} n^{-1} (\log n)^{-\frac{3}{2}} |S_n^*| = \sum_{n=2}^{\infty} (\log n)^{-\frac{3}{2}} |b_n| = \infty. \qquad (5)$$

It is known[5] that the series

$$b_0 + \sum_{n=1}^{\infty} b_n \cos nt \qquad (6)$$

is a Fourier series of even function

$$g(t) = \frac{1}{2}\int_t^{\pi} f(u) \operatorname{ctg} \frac{u}{2}\, du, \ (t > 0). \quad (7)$$

From (3) and

$$g'(t) = -\frac{1}{2} f(t) \operatorname{ctg} t, \qquad (8)$$

we have

$$\int_0^t u|g'(u)|du = o(t), \quad (t \to 0). \quad (9)$$

Conversely, if there is a periodic integrable even function $g(u)$ such that $g'(u)$ exists p.p. that $ug'(u) \in L(0, 2\pi)$, and that both (9) and (5) hold true, then the function given by (8) satisfies the conditions in Theorem 2. Hence, Theorem 1 is also equivalent to

Theorem 3. *There is a periodic integrable even function*

$$g(t) \sim \frac{1}{2} b_0 + \sum_{n=1}^{\infty} b_n \cos nt, \quad (10)$$

such that the derivative $g'(t)$ satisfies (9) *and*

$$\int_0^{\pi} t|g'(t)|dt < \infty \qquad (11)$$

and that

$$\sum_{n=2}^{\infty} (\log n)^{-\frac{3}{2}} |b_n| = \infty. \qquad (12)$$

The proof of Theorem 3 is based on the following lemmas.

Lemma 1. *Suppose that*

$$n = n_k = 10^k \quad (k = 3, 4, 5, \cdots),$$
$$l = [\log k], \qquad (13)$$

$$\Delta_{r,\nu} = \sin(\nu\pi \cdot 10^{-r}) - - \sin(\nu\pi \cdot 10^{-r-1}), \qquad (14)$$

then there are positive constants E and n_0 such that

$$\sum_{\nu=10^{j-1}+1}^{10^j} \sqrt{\sum_{r=l}^{k-1} \Delta_{r,\nu}^2} \geqslant E\sqrt{j} \cdot 10^j$$
$$(k \geqslant j > 2l, \quad k \geqslant 3) \qquad (15)$$

for $n > n_0$.

Lemma 2. *Suppose that n, k, l satisfy* (13). *There are positive constants A, B and a series of even function $f_n(t)$ possessing the following properties:*

1°.

$$f_n(t) = \begin{cases} \varepsilon_r & t \in (\pi \cdot 10^{-r-1}, \pi \cdot 10^{-r}), \\ & l \leqslant r < k, \quad \varepsilon_r = \pm 1 \\ 0 & t \in [0, \pi \cdot 10^{-k}] \cup [\pi \cdot 10^{-l}, \pi], \end{cases}$$

2°.

$$\int_0^t u|df_n(u)| \leqslant At, \quad (0 \leqslant t \leqslant \pi)$$

$$\int_0^t u|df_n(u)| = 0, \quad (0 \leqslant t \leqslant \pi \cdot 10^{-k})$$

3°. *the Fourier coefficients $C_\nu = C_\nu(f_n)$ $(\nu = 1, 2, \cdots)$ satisfy*

$$\sum_{\nu=2}^{n} (\log \nu)^{-\frac{3}{2}} |C_\nu(f_n)| \geqslant B \log\log n.$$

Lemma 3. *For every integer $n = n_k = 10^k$ $(k > 3)$, there is a series of continuous even functions $g_n(t)$ with the following properties:*

a) *$g(t)$ is differentiable except for a finite number of points,*

b)

$$\int_0^t u|g_n'(u)|du \leqslant Ht,$$

$$(H \text{ being const.}, \ 0 \leqslant t \leqslant \pi)$$

$$\int_0^t u|g_n'(u)|du = 0,$$

$$(0 \leqslant t \leqslant 10^{-k}\pi)$$

c) *the Fourier coefficients $C_\nu(g_n)$ $(\nu = 1, 2, \cdots)$ satisfy*

$$\sum_{\nu=2}^{n} (\log \nu)^{-\frac{3}{2}} |C_\nu(g_n)| > K \log\log n.$$

$$(K \textit{ being const.}, \; k \geqslant 3)$$

Now we write

$$m_s = 10^{10^{m_{s-1}}}, \quad m_1 = 10^3,$$

$$m_0 = 1, \quad \zeta_s = m_{s-1}^{-\frac{1}{2}},$$

it can then be shown that the function

$$g(t) = \sum \zeta_s g m_s(t)$$

satisfies all the conditions given in Theorem 3.

References

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PHYSICS

ДИАГРАММА СОСТОЯНИЯ СПЛАВОВ ТРОЙНОЙ СИСТЕМЫ Al-Cd-Cu ПРИ КОМНАТНОЙ ТЕМПЕРАТУРЕ

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В этой статье вкратце представлена работа, выполненная нами при построении диаграммы фазового состояния системы Al-Cd-Cu путём рентгенодифракционного метода. Работа разделена на два этапа. Сперва определяем обогащённо медный угол этой диаграммы[1], а затем исследуем всю диаграмму, внося поправки в работу первого хода. Результаты нашей работы показаны на рис. 1.

Этот фазовый разряд при комнатной температуре состоит из следующих фазовых областей: 10 однофазных областей (т.е. α, β, γ, γ_2, δ, δ', ζ_2, η_2, θ и ε); 18 двухфазных областей (т.е. $\alpha+\beta$, $\alpha+\gamma$, $\alpha+\gamma_2$, $\alpha+\delta'$, $\gamma_2+\delta'$, $\gamma+\beta$, $\gamma+\delta'$, $\gamma_2+\delta$, $\delta+\varepsilon$, $\varepsilon+\gamma_2$, $Cd+\varepsilon$, $Cd+\eta_2$, $Cd+\zeta_2$, $Cd+\theta$, $\delta'+\varepsilon$, $\eta_2+\theta$, $\zeta_2+\eta_2$ и $\theta+Al$); 10 трёхфазных областей (т.е. $\alpha+\beta+\gamma$, $\alpha+\gamma+\delta'$, $\alpha+\gamma_2+\delta'$, $\delta'+\varepsilon+\gamma_2$, $\delta+\gamma_2+\varepsilon$, $Cd+\varepsilon+\delta$, $Cd+\zeta_2+\delta$, $Cd+\zeta_2+\eta_2$, $Cd+\eta_2+\theta$ и $Cd+\theta+Al$). Все монофазы совпадают с монофазами трёх двойных систем, и никакой новой фазы не обнаруживается.

Приводя эту работу, мы приняли спектрочистый металл кадмия; чистота алюминия составляет 99,994%, а чистота меди — свыше 99,9%; большинство из них достигают 99,999%. Сплавы взвешиваются чувствительными весами с чувствительностью 1/10000 *г*.

Учтя, что температура кипения кадмия слишком низкая, мы не воспользовались,