THE REPRESENTATIVE THEOREM OF A CLASS OF HARMONIC FUNCTIONS ON RIEMANNIAN MANIFOLDS

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I. Introduction

Let M be a complete noncompact Riemannian manifold, R^+ be non-negative semi-real axis, $H_M(x, y, t)$ be the heat kernel on M, $P_t(x, y) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-s} s^{-1/2} H_M(x, y, t^2/4s)\,ds$ be the Poisson kernel on M. The Poisson integral of $f(x) \in L^p(M)$ $(1 \leqslant p \leqslant +\infty)$ is defined by $u(x, t) = \int_M P_t(x, y) f(y)\,dy$. In this note by establishing some "Liouville" type theorems of harmonic functions on product Riemannian manifolds, we prove that a class of harmonic function of $M \times R^+$ must be Poisson integral of some function. When $M = R^n$, the corresponding results are known in [1].

II. Lemmas and Their Proofs

M and N denote the complete noncompact Riemannian manifolds.

Lemma 1. *If* $\mathrm{Ric}_N(\alpha) \geqslant 0$ *and* $\int_1^{+\infty} V_N(B_{\alpha_0}(\sqrt{t}))^{-1} dt = +\infty$, *where* $\mathrm{Ric}_N(\alpha)$ *denotes the Ricci curvature of* N, $V_N(B_{\alpha_0}(\sqrt{t}))$ *denotes the volume of the geodesic ball centered at* α_0 *with radius* $\sqrt{t}$, *and if* $u(x, \alpha)$ *is a harmonic function on* $M \times N$ *satisfying* $\sup_{\alpha \in N} \|u(\cdot, \alpha)\|_{L^p(M)} < +\infty$ $(1 < p < +\infty)$, *then* $u(x, \alpha) = const.$

Lemma 2. *If* $\mathrm{Ric}_M(x) \geqslant -C\{r^2(x, x_0) + 1\}$ *and* $\mathrm{Ric}_N(\alpha) \geqslant 0$, *where* C *is a positive constant,* $r(x, x_0)$ *denotes the geodesic distance from* x *to some fixed point* x_0, *and if* $u(x, \alpha)$ *is a harmonic function on* $M \times N$ *satisfying* $\sup_{\alpha \in N} \|u(\cdot, \alpha)\|_{L^p(M)} < +\infty$ $(1 \leqslant p < +\infty)$, *then* $u(x, \alpha) = const.$

Lemma 3. *If* $\mathrm{Ric}_M(x) \geqslant -C\{r^2(x, x_0) + 1\}$ *and* $\mathrm{Ric}_N(\alpha) \geqslant 0$, *then the bounded non-*

constant harmonic function on $M \times N$ *must be the bounded nonconstant harmonic function on* M.

We shall show the sharpness of our curvature assumption in Lemma 3. Some other "Liouville" type theorems can be seen in footnote[1)].

Proof of Lemma 1. Let $\Phi_R(x)$ be a cut-off function with the property that

$$\Phi_R(x) = \begin{cases} 1, & \text{on } B_{x_0}(R); \\ 0, & \text{on } M \setminus B_{x_0}(2R), \end{cases}$$

with $0 \leqslant \Phi_R(x) \leqslant 1$ and $|\nabla_M \Phi_R| \leqslant C/R$, where ∇_M denotes the gradient of M, C denotes a positive constant. Consider

$$\int_M \Delta_N |u(x, \alpha)|^p \Phi_R^2(x) dx$$

$$= \int_M \{p|u(x, \alpha)|^{p-1} \Delta_N |u(x, \alpha)| + p(p-1)|u(x, \alpha)|^{p-2} |\nabla_N |u(x, \alpha)||^2\} \varnothing_R^2(x) dx$$

$$\geqslant 2p \int_M |u(x, \alpha)|^{p-1} < \nabla_M \Phi_R, \nabla_M |u| > \Phi_M dx$$

$$+ p(p-1) \int_M |u(x, \alpha)|^{p-2} |\nabla_M |u(x, \alpha)||^2 \Phi_R^2(x)\, dx$$

$$\geqslant \frac{p(p-1)}{2} \int_M |\nabla_M |u(x, \alpha)||^2\, |u(x, \alpha)|^{p-2} \Phi_R^2(x) dx$$

$$- \frac{2p}{p-1} \int_M |\nabla_M \Phi_R^2(x)|^2\, |u(x, \alpha)|^p dx.$$

Letting $R \to +\infty$, we obtain the result that $\int_M |u(x, \alpha)|^p dx$ is a bounded subharmonic function on N and then $u(x, \alpha) = \text{const.}$ by the results of [2] and [3].

Before the proof of Lemma 2 and Lemma 3, we prove the following

Lemma 4. *If* M, N *and* $u(x, \alpha)$ *satisfy the assumption of Lemma* 2 *or Lemma* 3, *then*

$$u(x, \alpha) = \int_M \int_N H_M(x, y, t) H_N(\alpha, \beta, t) u(y, \beta) d\beta dy. \tag{2.1}$$

Proof. Let $p_0 = (x_0, \alpha_0) \in M \times N$, and $\overline{B}_{p_0}(R)$, $B^1_{x_0}(R)$ and $B^2_{\alpha_0}(R)$ denote the geodesic balls of $M \times N$, M and N respectively, their volumes are denoted by $\overline{V}_{p_0}(R)$, $V_{x_0}(R)$ and $V_{\alpha_0}(R)$ respectively. The heat kernel of $M \times N$ is denoted by $H(p, q, t)$, then $H(p, q, t) = H_M(x, y, t) H_N(\alpha, \beta, t)$, where $p = (x, \alpha)$, $q = (y, \beta)$. Δ and ∇ denote the Laplace operator and gradient on $M \times N$ respectively. Partial differentiating with respect to t on the right side of (2.1), we have

1) Yu Ze, Harmonic functions on product Riemannian manifolds, Doctoral dissertation, Hangzhou University, 1989.

$$\frac{\partial}{\partial t}\int_M\int_N \overline{H}(p,q,t)u(y,\beta)d\beta dy=\int_M\int_N \Delta\overline{H}(p,q,t)u(y,\beta)d\beta dy. \tag{2.2}$$

We claim that integration by parts is valid and then we obtain the result that the right side of (2.1) is independent of t, which implies (2.1).

To justify this, we consider

$$|\int_{\overline{B}_{p_0}(R)}\Delta\overline{H}(p,q,t)u(y,\beta)d\beta dy-\int_{\overline{B}_{p_0}(R)}\overline{H}(p,q,t)\Delta u(y,\beta)d\beta dy|$$

$$=|\int_{\partial\overline{B}_{p_0}(R)}\frac{\partial\overline{H}}{\partial\gamma}(p,q,t)u(y,\beta)d\beta dy-\int_{\partial\overline{B}_{p_0}(R)}\overline{H}(p,q,t)\frac{\partial u}{\partial\gamma}(y,\beta)d\beta dy|$$

$$\leqslant\int_{\partial\overline{B}_{p_0}(R)}|\nabla\overline{H}(p,q,t)||u(y,\beta)|d\beta dy+\int_{\partial\overline{B}_{p_0}(R)}\overline{H}(p,q,t)|\nabla u(y,\beta)|d\beta dy, \tag{2.3}$$

where γ denotes the external normal direction of $\overline{B}_{p_0}(R)$.

By applying the mean value inequality of harmonic function and the estimates of Lemma 7 and (4.18) in [4] and Theorem 3.1 in [5], we have

$$\int_{\overline{B}_{p_0}(R+1)\setminus\overline{B}_{p_0}(R)}|\nabla\overline{H}(p,q,t)||u(y,\beta)|d\beta dy$$

$$\leqslant\sup_{\overline{B}_{p_0}(R+1)}|u(y,\beta)|\int_{\overline{B}_{p_0}(R+1)\setminus\overline{B}_{p_0}(R)}|\nabla\overline{H}(p,q,t)|d\beta dy$$

$$\leqslant C\exp\{CR^2t+CR^{3/2}\}\overline{V}_{p_0}^{-1/p}(2R+2)V_{\alpha_0}^{1/p}(2R+2)\overline{V}_{p_0}^{1/2}(R+1)\overline{V}_p^{-1/2}(\sqrt{t})\times$$

$$(R^{-2}+t^{-1})^{1/2}\exp\left\{-\frac{(R-2r(p,p_0))^2}{80t}\right\}\sup_\beta\|u(\cdot,\beta)\|_{L^p(M)}. \tag{2.4}$$

Under the assumption of Ricci curvature, we know that for sufficiently small $t>0$, (2.4) goes to 0 as R goes to infinity.

By applying the inequality

$$\int_{M\times N}\Phi^2(y,\beta)|u(y,\beta)|\Delta|u(y,\beta)|d\beta dy\geqslant 0,$$

we have

$$\int_{\overline{B}_{p_0}(R+1)\setminus\overline{B}_{p_0}(R)}|\nabla u(y,\beta)|^2d\beta dy\leqslant 12\int_{\overline{B}_{p_0}(R+2)}|u(y,\beta)|^2d\beta dy, \tag{2.5}$$

where

$$\Phi(y,\beta)=\begin{cases}1, & \text{on } \overline{B}_{p_0}(R+1)\setminus\overline{B}_{p_0}(R);\\ 0, & \text{on } \overline{B}_{p_0}(R-1)U\{M\times N-\overline{B}_{p_0}(R+2)\},\end{cases}$$

with

$$0\leqslant\Phi\leqslant 1,|\nabla\Phi|\leqslant 3.$$

By applying (2.5) and the method of heat estimate applied in [5], we can estimate the term

$$\int_{\overline{B}_{p_0}(R+1)\setminus\overline{B}_{p_0}(R)}\overline{H}(p,q,t)|\nabla u(y,\beta)|d\beta dy$$

$$\leqslant \sup_{q\in \overline{B}_{p_0}(R+1)\setminus \overline{B}_{p_0}(R)} \overline{H}(p,q,t)\left\{\overline{V}_{p_0}(R+1)12\int_{\overline{B}_{p_0}(R+2)} |u(y,\beta)|^2 d\beta dy\right\}^{1/2}$$

$$\leqslant\left\{C\overline{V}_p^{-1}(\sqrt{t}\ )t^{-n/4}\exp\left\{-\frac{(R-r(p,p_0))^2}{5t}+C(R^2+R^{-2})t+CR(r(p,p_0)+\sqrt{t}\ )\right\}\times\right.$$
$$\left.\left\{C\overline{V}_{p_0}^{1/2}(R+1)\overline{V}_{p_0}^{1/2}(R+2)\overline{V}_{p_0}^{-1/p}(2R+4)V_{\alpha_0}^{1/p}(2R+4)\sup_\beta\|u(\cdot,\beta)\|_{L^p(M)}\right\}\right. . \quad (2.6)$$

Also for sufficiently small $t>0$, (2.6) goes to 0 as R goes to infinity. The estimates (2.4) and (2.6) are obtained under the assumption of Lemma 2, similar estimates can also be obtained under the assumption of Lemma 3.

By applying the mean value theorem to (2.4) and (2.6), we know that there exists $R_i \to +\infty$ such that (2.3) goes to 0. This implies that (2.1) is valid for sufficiently small $t>0$, the semi-group property of heat kernel implies that (2.1) is valid for all $t>0$.

Proof of Lemma 2. By applying (2.1), we have

$$\nabla_N u(x,\alpha)=\int_{M\times N} H_M(x,y,t)\nabla_N H_N(\alpha,\beta,t)u(y,\beta)d\beta dy,$$

By applying the Hölder inequality, $\int_M H_M(x,y,t)\,dy=1$, the volume comparison theorem and the following gradient estimate:

$$|\nabla_N H_N(\alpha,\beta,t)|\leqslant \frac{c}{\sqrt{t}\ V_\alpha(\sqrt{t}\ )}\exp\left\{-\frac{r^2(\alpha,\beta)}{5t}\right\},$$

we have

$$\|\nabla_N u(\cdot,\alpha)\|_{L^p(M)}\leqslant \frac{2C}{5\sqrt{t}}\int_0^{+\infty}(1+\tau)^n\tau\exp\left(-\frac{\tau^2}{5}\right)d\tau\ \sup_\beta\|u(\cdot,\beta)\|_{L^p(M)}.$$

Letting $t\to+\infty$, we have $\nabla_N u(\cdot,\alpha)=0$, which implies that $u(x,\alpha)$ is an L^p-harmonic function on M, then $u(x,\alpha)=$constant.

The proof of Lemma 3 is similar to that of Lemma 2, see footnote 1).

Now we show that the assumption of Lemma 3 is best in some sense: let $M=\mathbb{R}^2$, the metric tensor is defined by $ds^2=dr^2+f(r)\,d\theta^2$, where $f(r)=r$ if $r\in(0,1]$ and $f(r)=e^{-Cr^{2+\varepsilon}}$ if $r\in[2,+\infty)$. Then $K(r)\approx -C^2(2+\varepsilon)^2 r^{2+2\varepsilon}$, where $K(r)$ is the sectional curvature. By the result of Azencott[6], we know that there exists a fundamental solution of heat equation $H(x,y,t)$ satisfying $\int_M H(x,y,t)dy<1$, set $u_1=1$, $u_2=\frac{1}{\sqrt{\pi}}\int_M\int_0^{+\infty}e^{-s}s^{-1/2}\times H(x,y,\alpha^2/4s)dsdy$ $\alpha>0$. Letting $u^*(x,\alpha)=u_1(x,\alpha)-u_2(x,\alpha)$, if $\alpha\geqslant0$; $u^*(x,\alpha)=-\{u_1(x,-\alpha)-u_2(x,-\alpha)\}$, if $\alpha<0$, then u^* is a bounded nonconstant harmonic function of $M\times R$

1) Yu Ze, Harmonic functions on product Riemannian manifolds, Doctoral dissertation, Hangzhou University, 1989.

and is related to α.

III. Main Theorem and the Proof

Theorem. *If* $u(x, t)$ *is a harmonic function on* $M\times R^+$ *satisfying* $\sup_{t>0}\|u(x, t)\|_{L^p(M)} < +\infty$, $(1\leqslant p\leqslant +\infty)$, *then*

(i) $1<p<+\infty$, *there exists* $f(x)\in L^p(M)$ *such that*

$$u(x, t)=\int_M P_t(x, y)f(y)\,dy;$$

(ii) $p=1$ *or* $p=+\infty$, *there exists a finite Borel measure* μ *on* M *or* $f(x)\in L^{+\infty}(M)$ *such that*

$$u(x, t)=\int_M P_t(x, y)\,d\mu(y),$$

or

$$u(x, t)=\int_M P_t(x, y)f(y)\,dy,$$

provided that M *satisfies the assumption that* $\mathrm{Ric}_M(x)\geqslant -C\{r^2(x, x_0)+1\}$. *The assumption on Ricci curvature of* M *is the best in some sense.*

Combining Lemma 1, Lemma 2, Lemma 3 and the method used to treat the case of $\mathbb{R}^n\times R^+$ with the L^p convergence of Poisson integral, we can prove the theorem. The L^p convergence of the Poisson integral for $1\leqslant p<+\infty$ can be seen in [7], the $L^{+\infty}$ convergence is not difficult to prove, see footnote 1).

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