Wavelets associated with Hankel transform and their Weyl transforms

PENG Lizhong & MA Ruiqin

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China (email: lzpeng@pku.edu.cn, rqma@water.pku.edu.cn)

Received May 9, 2003

Abstract The Hankel transform is an important transform. In this paper, we study the wavelets associated with the Hankel transform, then define the Weyl transform of the wavelets. We give criteria of its boundedness and compactness on the L^p- spaces.

Keywords: Bessel function, Hankel transform, wavelets, Weyl transform, boundedness, compactness.

DOI: 10.1360/02ys0371

1 Introduction

The Weyl transform was introduced by Weyl in ref. [1], then studied by many people. Remarkable papers include refs. [2,3]. Weyl proved that the Weyl transform is a Hilbert-Schmidt operator when the symbol is square integrable. In 1966[2] Pool considered the situation for the symbol belonging to $L^p(1 \leqslant p \leqslant 2)$, in this case the Weyl correspondence is bounded, moreover the Weyl transform is also compact. For the case of $p > 2$, Simon, in 1992[3], obtained the result that the Weyl correspondence is not bounded. All these results may be found in ref. [4], in which Wong also studied some other important properties of the Weyl transform. Nowadays, the Weyl correspondence has found many applications to time-frequency analysis, the theory of differential equations, linear system theory and etc. And the Weyl transforms under other background have been studied. In ref. [5] Jiang studied the rotational invariance of the Weyl correspondence. In ref. [6], Rachdi and Trimeche defined the Weyl transforms associated with the spherical mean operator, then studied its properties. In ref. [7] Peng and Zhao studied the wavelet and Weyl transforms associated with the spherical mean operator. We defined the Weyl transforms of wavelets and discussed the problem of their boundedness in ref. [8]. As well known, the Fourier transform restricted on the radical function is the Hankel transform. In this paper we study the wavelets associated with the Hankel transform and the Weyl transform of the wavelets.

An important transform relating with the classical Weyl transform is the Wigner transform. In order to get the unboundedness of the Weyl correspondence, Simon changed the problem into the unboundedness of the Wigner transform. In fact, the boundedness of the

Weyl correspondence is equivalent to the boundedness of the Winger transform. In our case, the problem is transferred into the boundedness of the wavelet transform. We obtain the following result: in the case that the symbol belongs to $L^p(1 \leqslant p \leqslant 2)$, the Weyl correspondence is bounded and the Weyl transform is also compact; in the case of $p > 2$, the Weyl correspondence is not bounded, i. e. there exists a function $\sigma \in L^p$ such that the Weyl transform W_σ is not a bounded operator.

For the function f defined on $\mathbb{R}^n, n \geqslant 2$, as in ref. [10], the spherical mean operator $\mathcal{R}$ is defined by

$$\mathcal{R}f(r) = \int_{S^{n-1}} f(r\omega) d\sigma(\omega),$$

where S^{n-1} is the unite sphere in $\mathbb{R}^n$ and σ is the normalized surface measure on S^{n-1}. And we know

$$\int_{S^{n-1}} e^{-\mathrm{i}rx\cdot\omega} d\sigma(\omega) = \int_{S^{n-1}} e^{\mathrm{i}r|x|\omega_1} d\sigma(\omega) \qquad \forall r > 0, x \in \mathbb{R}^n.$$

Denote $\phi_\lambda(r) = \int_{S^{n-1}} e^{\mathrm{i}r\lambda\omega_1} d\sigma(\omega),\ \lambda, r > 0$. Then

$$\phi_\lambda(r) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\pi e^{\mathrm{i}r\lambda\cos\theta} \sin^{n-2}\theta d\theta = \Gamma\left(\frac{n}{2}\right) J_{\frac{n-2}{2}}(r\lambda) \left(\frac{r\lambda}{2}\right)^{\frac{2-n}{2}}, \tag{1.1}$$

where J_ν is the Bessel function (see ref. [11] or ref. [12]) defined by

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu + 1/2)} \int_0^\pi e^{\mathrm{i}z\cos\theta} \sin^{2\nu}\theta d\theta, \qquad \nu > -\frac{1}{2}.$$

By expression (1.1), we can generalize the half integer or integer index $\nu = \frac{n-2}{2}$ to all the real number $\nu > -\frac{1}{2}$. Then we obtain the generalized function and denote it by the same symbol.

$$\phi_\lambda(r) = \frac{\Gamma(\nu + 1)}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\pi e^{\mathrm{i}r\lambda\cos\theta} \sin^{2\nu}\theta d\theta = \Gamma(\nu + 1) J_\nu(r\lambda) \left(\frac{r\lambda}{2}\right)^{-\nu}.$$

Define the measure $d\nu(r) = \frac{1}{2^\nu\Gamma(\nu+1)} r^{2\nu+1} dr$ on $\mathbb{R}_+ = (0, \infty)$. Let $L^p(d\nu)$ be the space of measurable function on $\mathbb{R}_+ = (0, \infty)$ satisfying

$$\|f\|_p = \left(\int_0^\infty |f(r)|^p d\nu(r)\right)^{1/p} < \infty, \qquad 1 \leqslant p < \infty,$$
$$\|f\|_\infty = \operatorname{ess} \sup_{r\in(0,\infty)} |f(r)| < \infty, \qquad p = \infty.$$

For $r \in (0, \infty)$ the translation operator τ_r is defined on $L^1(d\nu)$ by

$$\tau_r f(s) = \frac{\Gamma(\nu + 1)}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\pi f(\sqrt{r^2 + s^2 - 2rs\cos\theta}) \sin^{2\nu}\theta d\theta.$$

Then, we define the convolution of $f, g \in L^1(d\nu)$ by

$$f * g(r) = \int_0^\infty f(s)\tau_r g(s) d\nu(s).$$

The translation operator is commutative and it has the following properties.

Proposition 1.1[9]**.**

1. $\tau_r \phi_\lambda(s) = \phi_\lambda(r)\phi_\lambda(s)$.

2. $\int_0^\infty \tau_r f(s) d\nu(s) = \int_0^\infty f(s) d\nu(s)$, for all $r \in [0, \infty)$.

3. If $f \in L^p(d\nu), 1 \leqslant p \leqslant \infty$, then for all $r \in [0, \infty)$, the function $\tau_r f$ belongs to $L^p(d\nu)$, and

$$||\tau_r f||_p \leqslant ||f||_p.$$

4. For $f \in L^1(d\nu), g \in L^p(d\nu), 1 \leqslant p \leqslant \infty$, the function $f * g \in L^p(d\nu)$, and

$$||f * g||_p \leqslant ||f||_1 ||g||_p.$$

Now we can introduce the definition of the Hankel transform on $L^1(d\nu)$.

$$\mathcal{H}f(\lambda) = \int_0^\infty f(r)\phi_\lambda(r) d\nu(r).$$

In fact, it is the Fourier transform restricted on the radical functions for $\nu = \frac{n-2}{2}$, $n > 2$, and $n \in N$. Similar to the Fourier transform, the Hankel transform has the following properties.

Proposition 1.2[9]**.**

1. For $f \in L^1(d\nu)$ such that $\mathcal{H}f \in L^1(d\nu)$, we have the Hankel inversion formula

$$f(r) = \int_0^\infty \mathcal{H}f(\lambda)\overline{\phi_\lambda(r)} d\nu(\lambda), \quad \text{a.e. } r \in [0, \infty).$$

2. For $f, g \in L^2(d\nu)$, we have the Parseval equality

$$< f, g > = < \mathcal{H}f, \mathcal{H}g > .$$

3. Given $f \in L^1(d\nu)$, then for all $s \in [0, \infty)$, we have

$$\mathcal{H}(\tau_s f)(\lambda) = \phi_s(\lambda)\mathcal{H}f(\lambda).$$

4. If $f, g \in L^1(d\nu)$, then

$$\mathcal{H}(f * g)(\lambda) = \mathcal{H}f(\lambda)\mathcal{H}g(\lambda).$$

5. If f belongs to $L^p(d\nu), 1 \leqslant p \leqslant 2$, then $\mathcal{H}f$ belongs to $L^{p'}(d\nu)$, where p' is the dual index of p, and

$$||\mathcal{H}f||_{p'} \leqslant ||f||_p.$$

In fact, the Hankel inversion formula is essentially a Fourier-Bessel integral, which is described in detail in ref. [11].

2 Admissible wavelet transform

Let $f \in L^2(d\nu)$. The dilation $D_a \, (a > 0)$ is defined by $D_a f(r) = a^{\nu+1} f(ar)$. Now we can introduce the conception of the admissible wavelet (compared with ref. [13], pages 24 and 25).

Definition 2.1. $\psi \in L^2(d\nu)$, not identically zero, is called an admissible wavelet if it satisfies

$$\int_0^\infty \int_0^\infty | < \psi, \psi^{a,r} > |^2 a^{2\nu+1} da d\nu(r) < \infty,$$

where $\psi^{a,r}(s) = \tau_r D_a \psi(s)$.

In fact, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty | < \psi, \psi^{a,r} > |^2 a^{2\nu+1} da d\nu(r) \\ &= \int_0^\infty \int_0^\infty |\psi * D_a \overline{\psi}(r)|^2 d\nu(r) a^{2\nu+1} da \\ &= \int_0^\infty \int_0^\infty |\mathcal{H}\psi(\lambda)\mathcal{H}D_a\overline{\psi}(\lambda)|^2 d\nu(\lambda) a^{2\nu+1} da \\ &= \int_0^\infty \int_0^\infty |\mathcal{H}\psi(\lambda)|^2 \left|\mathcal{H}\psi\left(\frac{\lambda}{a}\right)\right|^2 a^{-2(\nu+1)} d\nu(\lambda) a^{2\nu+1} da \\ &= \|\psi\|^2 \int_0^\infty \frac{|\mathcal{H}\psi(\lambda)|^2}{a} da. \end{aligned}$$

So the admissible condition in the above definition is equivalent to

$$C_\psi = \int_0^\infty \frac{|\mathcal{H}\psi(\lambda)|^2}{a} da < \infty.$$

Let AW denote the space of all admissible wavelets, define the normal on AW by

$$||\psi||_{AW} = \left(\int_0^\infty |\psi(r)|^2 d\nu(r) + \int_0^\infty |\mathcal{H}\psi(a)|^2 \frac{da}{a}\right)^{\frac{1}{2}}.$$

The wavelet transform associated with an admissible wavelet ψ is defined by

$$T_\psi f(a, r) =< f, \psi^{a,r} >, \qquad f \in L^2(d\nu).$$

Easily, we obtain the Moyal's formula

$$\int_0^\infty f(r)\overline{g}(r) d\nu(r) = \frac{1}{C_\psi} \int_0^\infty \int_0^\infty T_\psi f(a, r)\overline{T_\psi g(a, r)} a^{2\nu+1} da d\nu(r), \qquad (2.1)$$

for all $f, g \in L^2(d\nu)$. Formula (2.1) can be read as

$$f(s) = \frac{1}{C_\psi} \int_0^\infty \int_0^\infty T_\psi f(a, r)\psi^{a,r}(s) a^{2\nu+1} da d\nu(r)$$

with convergence of the integral in a weak sense. A special case of (2.1) is

$$\int_0^\infty |f(r)|^2 d\nu(r) = \frac{1}{C_\psi} \int_0^\infty \int_0^\infty |T_\psi f(a, r)|^2 a^{2\nu+1} da d\nu(r), \qquad f \in L^2(d\nu).$$

This means the wavelet transform T_ψ maps $L^2(d\nu)$ into $L^2(R_+^2, a^{2\nu+1} da d\nu(r))$. But the image space $\mathcal{H}_\psi$ is a subspace, not all of $L^2(R_+^2, a^{2\nu+1} da d\nu(r))$. For any $F \in \mathcal{H}_\psi$, there exists a function $f \in L^2(d\nu)$ such that $F = T_\psi f$. By (2.1), we have

$$F(a,r) =< f,\psi^{a,r} > = \frac{1}{C_\psi}\int_0^\infty\int_0^\infty T_\psi f(a',r')\overline{T_\psi\psi^{a,r}(a',r')}{a'}^{2\nu+1}da'd\nu(r')$$
$$= \frac{1}{C_\psi}\int_0^\infty\int_0^\infty F(a',r')K(a,r,a',r'){a'}^{2\nu+1}da'd\nu(r')$$

with $K(a,r,a',r') = \overline{T_\psi\psi^{a,r}(a',r')} =< \psi^{a,r},\psi^{a',r'} >$. The above shows that $\mathcal{H}_\psi$ is a reproducing kernel Hilbert space.

Denote $L^p(da\otimes d\nu)$ as the space of measurable functions on R_+^2 satisfying

$$\|F\|_p = \left(\int_0^\infty\int_0^\infty |F(a,r)|^p a^{2\nu+1}dad\nu(r)\right)^{1/p} < \infty, \qquad 1\leqslant p<\infty,$$
$$\|F\|_p = \text{ess}\sup_{a,r\in(0,\infty)} |F(a,r)| < \infty, \qquad p=\infty.$$

By the definition of the wavelet transform and (2.1), we have the following theorem.

Theorem 2.1. Let $\psi\in AW$. Then for any $f\in L^2(d\nu)$, $T_\psi f\in L^p(da\otimes d\nu)$, $2\leqslant p\leqslant\infty$ and

$$\|T_\psi f\|_p \leqslant \|\psi\|_{AW}\|f\|_2.$$

3 Weyl transform of wavelet

3.1 Weyl transform with symbols in $L^p(da\otimes d\nu)$, for $1\leqslant p\leqslant 2$

The rapidly decreasing function space $\mathcal{S}(R_+^2)$ is the space of infinite differential function satisfying

$$\sup_{a,r\in R_+} |(1+a^2+r^2)^m \frac{\partial^{\alpha+\beta}}{\partial a^\alpha\partial r^\beta}\sigma| \leqslant C_{m,\alpha,\beta},$$

where $m,\alpha,\beta\in N$, and $C_{m,\alpha,\beta}$ is a constant only depending on m,α,β. It is easy to see that $\mathcal{S}(R_+^2)$ is dense in $L^p(da\otimes d\nu)$ for $1\leqslant p<\infty$.

Definition 3.1. Let $\sigma\in\mathcal{S}(R_+^2)$. We define the Weyl transform W_σ by

$$< f, W_\sigma\psi >= \int_{R_+}\int_{R_+} \bar\sigma(a,r) < f,\psi^{a,r} > a^{2\nu+1}dad\nu(r), \tag{3.1}$$

where $f\in L^2(d\nu),\psi\in AW$.

From (3.1), we get the expression of the Weyl transform

$$W_\sigma\psi(s) = \int_{R_+}\int_{R_+}\sigma(a,r)\psi^{a,r}(s)a^{2\nu+1}dad\nu(r). \tag{3.2}$$

Using Theorem 2.1, we have the following proposition.

Proposition 3.1. If $\sigma\in\mathcal{S}(R_+^2)$, then $W_\sigma : AW\to L^2(d\nu)$ is a bounded operator, and for $1\leqslant p\leqslant 2$

$$\|W_\sigma\| \leqslant \|\sigma\|_p.$$

By the density of the rapidly decreasing functions, we can extend the Weyl correspondence $\sigma\to W_\sigma$ to $L^p(d\rho\otimes d\nu)$.

Theorem 3.1. For $p \in [1, 2]$, there exists a unique bounded operator W from $L^p(d\rho \otimes d\nu)$ into $\mathcal{B}(AW, L^2(d\nu)) : \sigma \to W_\sigma$, such that for all $\psi \in AW, f \in L^2(d\nu)$, we have

$$< f, W_\sigma \psi >= \int_{R_+} \int_{R_+} \bar{\sigma}(a, r) < f, \psi^{a,r} > a^{2\nu+1} da d\nu(r),$$

and

$$\|W_\sigma\| \leqslant \|\sigma\|_p.$$

Theorem 3.2. For $\sigma \in L^p(da \otimes d\nu)$, $p \in [1, 2]$, the operator W_σ from AW into $L^2(d\nu)$ is a compact operator.

Proof. We only need to prove the conclusion for $\sigma \in \mathcal{S}(R_+^2)$. Let the space $S = \{\psi \in L^2(d\nu) : ||\psi||_S = (\int_{R_+} |\mathcal{H}\psi(a)|^2 \frac{da}{a})^{1/2} < \infty\}$. We know $AW = S$ and $\|\psi\|_S \leqslant \|\psi\|_{AW}$, so if we prove that the operator W_σ is compact according to the normal of S, then it is also compact according to the normal of AW. In the following, we will prove the operator is compact on S. By (3.2), we have

$$\begin{aligned}\mathcal{H}(W_\sigma\psi)(\lambda) &= \int_{R_+} \int_{R_+} \sigma(a, r)\phi_r(\lambda)\mathcal{H}\psi\left(\frac{\lambda}{a}\right) a^{-\nu-1} a^{2\nu+1} da d\nu(r)\\ &= \int_{R_+} \mathcal{H}\sigma_2(a, \lambda)\mathcal{H}\psi\left(\frac{\lambda}{a}\right) a^\nu da\\ &= \int_{R_+} \left(\frac{\lambda}{a}\right)^{\nu+1} \mathcal{H}\sigma_2\left(\frac{\lambda}{a}, \lambda\right) \mathcal{H}\psi(a)\frac{da}{a},\end{aligned}$$

where $\mathcal{H}\sigma_2(a, \lambda)$ denotes the Hankel transform of σ with respect to the second variable. Then

$$\begin{aligned}\|W_\sigma\|^2_{HS(S,L^2)} &= \int_{R_+} \int_{R_+} \left|\left(\frac{\lambda}{a}\right)^{\nu+1} \mathcal{H}\sigma_2\left(\frac{\lambda}{a}, \lambda\right)\right|^2 \frac{da}{|a|} d\nu(\lambda)\\ &= \int_{R_+} \int_{R_+} |\mathcal{H}\sigma_2(a, \lambda)|^2 a^{2\nu+1} da d\nu(\lambda)\\ &= \int_{R_+} \int_{R_+} |\sigma(a, r)|^2 a^{2\nu+1} da d\nu(r) < \infty.\end{aligned}$$

This means that the operator is Hilbert-schmidt operator. So, the operator is compact.

3.2 Weyl transform with symbols in $L^p(da \otimes d\nu)$, for $2 < p < \infty$

In order to consider the property of the Weyl transform with the symbol in $L^p, 2 < p < \infty$, we first define the Weyl transform for $\sigma \in \mathcal{S}'(R_+^2)$.

For $\sigma \in \mathcal{S}'(R_+^2)$, and $\psi \in AW \cap \mathcal{S}(R_+)$, we define the operator $W_\sigma(\psi)$ on $\mathcal{S}(R_+)$ by

$$[W_\sigma(\psi)](\overline{f}) = \sigma(< \psi^{a,r}, f >) \qquad f \in \mathcal{S}(R_+). \tag{3.3}$$

Obviously, $W_\sigma(\psi)$ belongs to $\mathcal{S}'(R_+)$.

Theorem 3.3. For $2 < p < \infty$, there exists a function σ in $L^p(da \otimes d\nu)$ such that W_σ, defined by (3.3), is not a bounded linear operator from AW to $L^2(d\nu)$.

The theorem is a consequence of the following two lemmas.

Lemma 3.1. Suppose that for all $\sigma \in L^p(da \otimes d\nu)$, $2 < p < \infty$, the Weyl transform W_σ, defined by (3.3), is a bounded linear operator from AW to $L^2(d\nu)$. Then there exists a positive constant C such that

$$\|W_\sigma\| \leqslant C\|\sigma\|_p, \qquad \sigma \in L^p(d\rho \otimes d\nu). \tag{3.4}$$

Proof. Suppose that for all $\sigma \in L^p(da \otimes d\nu), 2 < p < \infty$. There exists a positive constant C_σ such that for all $\psi \in AW$

$$\|W_\sigma \psi\|_2 \leqslant C_\sigma \|\psi\|_{AW}.$$

Let $f \in \mathcal{S}(R_+), \psi \in AW \cap \mathcal{S}(R_+)$ such that $\|f\|_2 = \|\psi\|_{AW} = 1$. Consider the bounded linear functional $Q_{\psi,f} : L^p(da \otimes d\nu) \to C$ defined by

$$Q_{\psi,f}(\sigma) = < W_\sigma \psi, f > .$$

Then

$$\sup |Q_{\psi,f}(\sigma)| \leqslant C_\sigma, \qquad \sigma \in L^p(da \otimes d\nu),$$

where the supremum is taken over all functions f, ψ satisfying the previous conditions. By the uniform boundedness principle, there exists a positive constant C such that

$$\sup ||Q_{\psi,f}|| \leqslant C, \text{i.e.} \sup_{||\sigma||_p=1} | < W_\sigma \psi, f > | \leqslant C.$$

So,

$$| < W_\sigma \psi, f > | \leqslant C||\sigma||_p||\psi||_{AW}||f||_2,$$

for all $\sigma \in L^p(da \otimes d\nu)$, and $f \in \mathcal{S}(R_+), \psi \in AW \cap \mathcal{S}(R_+)$. Thus, we prove the lemma.

Lemma 3.2. For $2 < p < \infty$, there is no positive constant C such that (3.4) holds.

Proof. Suppose that there exists a positive constant C such that (3.4) holds. Then

$$\begin{aligned} \| < \psi^{a,r}, f > \|_{p'} &= \sup_{\|\sigma\|_p=1} \left| \int_{R_+} \int_{R_+} \sigma(a,r) < \psi^{a,r}, f > a^{2\nu+1} da d\nu(r) \right| \\ &= \sup_{\|\sigma\|_p=1} | < W_\sigma \psi, f > | \\ &\leqslant \sup_{\|\sigma\|_p=1} \|W_\sigma \psi\|_2 \|f\|_2 \\ &\leqslant C\|\psi\|_{AW}\|f\|_2, \end{aligned}$$

for all $f \in \mathcal{S}(R_+), \psi \in AW \cap \mathcal{S}(R_+)$.

Let $f \in L^2(d\nu), \psi \in AW$. Then, we let $\{f_k\}_{k=1}^\infty$ be a sequence of functions in $\mathcal{S}(R_+)$ and $\{\psi_k\}_{k=1}^\infty$ be a sequence of functions in $AW \cap \mathcal{S}(R_+)$ such that $f_k \to f$ in $L^2(d\nu)$ and $\psi_k \to \psi$ in AW as $k \to \infty$. It is easy to prove that $\{< \psi_k^{a,r}, f_k >\}_{k=1}^\infty$ is a Cauchy sequence in $L^{p'}(da \otimes d\nu)$ and its limit is equal to $< \psi^{a,r}, f >$. Then,

$$\| < \psi^{a,r}, f > \|_{p'} \leqslant C\|\psi\|_{AW}\|f\|_2 \tag{3.5}$$

for all $f \in L^2(d\nu), \psi \in AW$.

Take $\mathcal{H}\psi(\lambda) = \lambda^\alpha \chi_{[0,1]}(\lambda)$ with $\alpha > 0$, where $\chi_{[0,1]}(\lambda)$ is the character function of [0,1], ψ is an admissible wavelet. And we have $\| < \psi^{a,r}, \psi > \|_{p'} < \infty$. But

$$\begin{aligned}
&\int_{R_+}\int_{R_+} | < \psi^{a,r}, \psi > |^{p'} a^{2\nu+1} da d\nu(r) \\
\geqslant &\int_{R_+}\left(\int_{R_+} |\mathcal{H} < \psi^{a,\cdot}, \psi > (\lambda)|^p d\nu(\lambda)\right)^{\frac{p'}{p}} a^{2\nu+1} da \\
= &\int_{R_+}\left(\int_{R_+} |\mathcal{H}(D_a\psi)(\lambda)\overline{\mathcal{H}\psi(\lambda)}|^p d\nu(\lambda)\right)^{\frac{p'}{p}} a^{2\nu+1} da \\
\geqslant &\int_1^\infty \left(\int_0^1 |a^{-\nu-1-\alpha}\lambda^{2\alpha}|^p d\nu(\lambda)\right)^{\frac{p'}{p}} a^{2\nu+1} da \\
= &\int_0^1 \lambda^{2\alpha p} d\nu(\lambda)^{\frac{p'}{p}} \int_1^\infty a^{-(\nu+1+\alpha)p'} a^{2\nu+1} da \\
= &\int_0^1 \lambda^{2\alpha p} d\nu(\lambda)^{\frac{p'}{p}} \int_0^1 a^{(\nu+1+\alpha)p'} a^{-2\nu-3} da.
\end{aligned}$$

Therefore, if we choose α such that $(\nu+1)(p'-2) + \alpha p' - 1 < -1$, then

$$\int_0^1 a^{(\nu+1+\alpha)p'} a^{-2\nu-3} da = \infty,$$

which is contract to $< \psi^{a,r}, \psi > \in L^{p'}(da \otimes d\nu)$.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant No. 90104004) and 973 Project of China (Grant No. G1999075105).

References

1. Weyl, H., The Theory of Groups and Quantum Mechanics, New York: Dover, 1931.
2. Pool, J. C. T., Mathematical aspect of the Weyl correspondence, J. Math. Phys., 1966, 7: 66—76.
3. Simon, B., The Weyl transform and L^p functions on phase space, Proc. Amer. Math. Soc., 1992, 116: 1045—1047.
4. Wong, M. W., Weyl Transform, New York: Springer-Verlag, 1998.
5. Jiang, Q., Rotation Invariant ambiguity functions, Proc. Amer. Math. Soc., 1996, 126(2): 561—567.
6. Rachdi, L. T., Trimeche, K., Weyl transforms associated with the spherical mean operator, Anal. & Applic., (to appear).
7. Peng, L. Z., Zhao, J. M., Wavelet and Weyl transforms associated with spherical mean operator, Integral Equations and Operator Theory, (to appear).
8. Ma, R. Q., Peng, L. Z., Weyl transforms of wavelets, Progress on Natural Science, 2003, 13(7): 489—496.
9. Azza Dachraoui, Weyl-Bessel transforms, J. Comput. Appl. Math., 2001, 133: 263—276.
10. Stein, E. M., Weiss, G., Introduction to Fourier Analysis on Euclidean Spaces, Princeton: Princeton University Press, 1971.
11. Watson, G. N., A Treatise on the Theory of Bessel Functions, London and New-York: Cambridge University Press, 1966.
12. Vilenkin, N. Ja., Klimyk, A. U., Representation of Lie Groups and Special Functions, Dordrecht: Kluwer Academic Publishers, 1991.
13. Daubechies, I., Ten lectures on wavelet, CBMS-NSF Series in Applied Math, 61, SIAM Publ., 1992.