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The Gleason's problem for some polyharmonic and hyperbolic harmonic function spaces

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Abstract Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain with C^2 boundary. For $0 < p, q \leqslant \infty$ and a normal weight φ, the mixed norm space $H_k^{p,q,\varphi}(\Omega)$ consists of all polyharmonic functions f of order k for which the mixed norm $\|\cdot\|_{p,q,\varphi} < \infty$. In this paper, we prove that the Gleason's problem $(\Omega, a, H_k^{p,q,\varphi})$ is always solvable for any reference point $a \in \Omega$. Also, the Gleason's problem for the polyharmonic φ-Bloch (little φ-Bloch) space is solvable. The parallel results for the hyperbolic harmonic mixed norm space are obtained.

Keywords: Gleason's problem, mixed norm space, Bloch-type space.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary and $\lambda(x)$ be a defining function of Ω. That is, λ is a C^2 real valued function and $\Omega = \{x \in \mathbb{R}^n : \lambda(x) < 0\}$ is bounded, $|\nabla\lambda(x)| \neq 0$ on the boundary $\partial\Omega$ of Ω (see ref. [1]). Here $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)$. For $r > 0$ small enough, let $\Omega_r = \{x \in \mathbb{R}^n : \lambda(x) < -r\}$, Ω_r is also a C^2 domain with the defining function $\lambda(x) + r$ and $\partial\Omega_r = \{x \in \mathbb{R}^n : \lambda(x) = -r\}$. We denote by $d\sigma_r$ the induced surface measure on $\partial\Omega_r$. Of course, there are infinitely many defining functions of Ω and two different defining functions yield two different systems of $\{\partial\Omega_r\}$ and $\{d\sigma_r\}$.

A positive continuous function φ on the interval $(0, \varepsilon]$ is called normal if there exist two positive constants $0 < a < b$ such that

$$\frac{\varphi(r)}{r^a} \quad \text{is} \quad \text{increasing} \quad \text{and} \lim_{r\to+0} \frac{\varphi(r)}{r^a} = 0, \tag{1}$$

$$\frac{\varphi(r)}{r^b} \quad \text{is} \quad \text{decreasing} \quad \text{and} \quad \lim_{r\to+0} \frac{\varphi(r)}{r^b} = \infty. \tag{2}$$

Throughout this paper, φ will always be normal on $(0, \varepsilon]$ and k will be a fixed positive integer. We denote by $H_k(\Omega)$ the family of all polyharmonic functions of order k on Ω. That is, $H_k(\Omega) = \{f \in C^\infty(\Omega) : \Delta^k f \equiv 0\}$, where Δ^k is the k-th power of the Laplacian. Then, $H_1(\Omega)$ is the class of all harmonic functions on Ω. For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $x = (x_1, x_2, \cdots, x_n) \in \Omega$ and $f \in C^\infty(\Omega)$, we write $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $D^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$. For $m = 0, 1, 2, \cdots$, let $\nabla^m f = (D^\alpha f)_{|\alpha|=m}$ be the m-th gradient of f with $|\nabla^m f(x)| = \sum_{|\alpha|=m} |D^\alpha f(x)|$.

Given a defining function λ, $0 < p \leqslant \infty$ and r small enough, for $f \in C(\Omega)$ we write

$$M_p(f, r, \lambda) = \left\{ \int_{\partial\Omega_r} |f(\xi)|^p d\sigma_r(\xi) \right\}^{\frac{1}{p}}, \ 0 < p < \infty; \ M_\infty(f, r, \lambda) = \sup_{\xi \in \partial\Omega_r} |f(\xi)|.$$

For $\varepsilon > 0$ small and fixed, $0 < p, q \leqslant \infty$ and φ normal on $(0, \varepsilon]$, the mixed norm $\|f\|_{p,q,\varphi,\lambda,\varepsilon}$ for $f \in C(\Omega)$ is defined as

$$\|f\|_{p,q,\varphi,\lambda,\varepsilon} = \left\{ \int_0^\varepsilon M_p^q(f, r, \lambda) \frac{\varphi^q(r)}{r} dr \right\}^{\frac{1}{q}}, \ 0 < q < \infty;$$
$$\|f\|_{p,\infty,\varphi,\lambda,\varepsilon} = \sup_{0 < r < \varepsilon} M_p(f, r, \lambda) \varphi(r).$$

We will simply write $\|f\|_{p,q,\varphi}$ for $\|f\|_{p,q,\varphi,\lambda,\varepsilon}$ and $M_p(f,r)$ for $M_p(f, r, \lambda)$ if no confusion occurs. Then, the polyharmonic mixed norm space $H_k^{p,q,\varphi}(\Omega)$ is defined to be

$$H_k^{p,q,\varphi}(\Omega) = \{f \in H_k(\Omega) : \|f\|_{p,q,\varphi} < \infty\}.$$

In the case $0 < p = q < \infty$, we will see that this space is just the polyharmonic Bergman space.

For $\varepsilon > 0$ small and fixed, φ normal on $(0, \varepsilon]$ and $f \in C^1(\Omega)$, set

$$\|f\|_{\mathcal{B}(\varphi)} = \sup_{x \in \Omega, d(x) \leqslant \varepsilon} \varphi(d(x)) |\nabla f(x)|,$$

where $d(x)$ is the Euclidean distance from x to $\partial\Omega$. The polyharmonic φ-Bloch space $\mathcal{B}(\varphi, k)$ and the polyharmonic little φ-Bloch space $\mathcal{B}_0(\varphi, k)$ are defined respectively by

$$\mathcal{B}(\varphi, k) = \left\{ f \in H_k(\Omega) : \|f\|_{\mathcal{B}(\varphi)} < \infty \right\},$$
$$\mathcal{B}_0(\varphi, k) = \{ f \in H_k(\Omega) : \lim_{x \to \partial\Omega} \varphi(d(x)) |\nabla f(x)| = 0 \}.$$

Give any fixed point $a \in \Omega$, set $\|f\| = |f(a)| + \|f\|_{\mathcal{B}(\varphi)}$. It is easy to check that $\|\cdot\|$ is a norm on both $\mathcal{B}(\varphi, k)$ and $\mathcal{B}_0(\varphi, k)$. The φ-Bloch space $\mathcal{B}(\varphi, k)$ is a Banach space and $\mathcal{B}_0(\varphi, k)$ is a closed subspace of $\mathcal{B}(\varphi, k)$ under the norm $\|\cdot\|$.

These spaces are originally studied in the complex variable setting, see refs. [2–5] for reference. In the real variable case, the harmonic Bergman space and harmonic mixed norm space were discussed in refs. [6–8]. Pavlović and Stević respectively studied the polyharmonic functions on the unit ball of $\mathbb{R}^n$, see refs. [9, 10].

Let X be a space consisting of some functions in the domain Ω and $a \in \Omega$ fixed. The Gleason's problem for X with the reference point a, denoted by (Ω, a, X), is as follows.

If $f \in X$ with $f(a) = 0$, do there exist functions $g_1, g_2, \cdots, g_n \in X$ such that

$$f(x) = \sum_{k=1}^n (x_k - a_k) g_k(x) \tag{3}$$

for all $x \in \Omega$?

Whether the Gleason's problem is solvable depends on Ω and the function space X. In the holomorphic functions setting, Gleason originally asked the problem for $\Omega = \mathbf{B} \subset \mathbf{C}^n$, $a = 0$, and $X = A(\mathbf{B})$, the ball algebra. This problem was solved by Leibenson, see ref. [11]. Subsequently, many authors studied the Gleason's problem for various function spaces in $\mathbf{B}$, see refs. [5,11–13]. In smoothly strongly pseudoconvex domain Ω, Kerzman-Nagel, Ahern-Schneider discussed the problem in Lipschitz space and C^k space, see refs. [14,15]. In ref. [16], Ren and Shi investigated the Gleason's problem for weighted Bergman space in the egg domain. The Gleason's problem for the harmonic Bergman space and Bloch space in the unit ball of $\mathbb{R}^n$ was considered in ref. [7] and ref. [17]. In ref. [8], the Gleason's problem (Ω, a, b_k^p), $(\Omega, a, \mathcal{B}(\varphi, 1))$ and $(\Omega, a, \mathcal{B}_0(\varphi, 1))$ were solved, where b_k^p is the harmonic Bergman-Sobolev space of order k, $1 \leqslant p < \infty$ and Ω is a star-shaped domain with a strong reference point $a \in \Omega$. And also, Ren and Kahler studied the Gleason's problem for the hyperbolic harmonic weighted Bergman space in the unit ball of $\mathbb{R}^n$ in ref. [18]. But in these references people can find that: (i) The domain Ω has a certain nice geometrical property, which is either smoothly strongly pseudoconvex or symmetric in some sense with the point $0 \in \Omega$. (ii) The main tool used in these mentioned work is the Forelli-Rudin type projection introduced in ref. [19]. (iii) In refs. [5,7,13,16], only the very special point $a = 0$ is considered in (Ω, a, X).

The purpose of this paper is to solve the Gleason's problem $(\Omega, a, H_k^{p,q,\varphi})$, $(\Omega, a, \mathcal{B}(\varphi, k))$ and $(\Omega, a, \mathcal{B}_0(\varphi, k))$ for any reference point $a \in \Omega$ and all possible $0 < p, q \leqslant \infty$, where Ω is the bounded convex domain with C^2 boundary. And also we will discuss the Gleason's problem for the hyperbolic harmonic mixed norm space. As an application of our approach, the analogous problems for the holomorphic functions will be studied. Our work will extend those results in refs. [7,8,13,16–18,20].

In what follows, C will stand for positive constants whose value may change from line to line but not depend on the functions being considered.

2 Some preliminary results

2.1 Independence on λ and ε

For $a \in \mathbb{R}^n$ and $r > 0$, set $B(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$. Let dm be the Lebesgue volume measure on $\mathbb{R}^n$.

Theorem 2.1. Let X be a space of certain continuous functions on $\Omega \subseteq \mathbb{R}^n$ satisfying

(**I**) For $0 < p < \infty$, there exists a positive constant A_1 such that for all $f \in X$ and $B(a, r) \subseteq \Omega$,

$$|f(a)|^p \leqslant A_1 r^{-n} \int_{B(a,r)} |f(x)|^p dm(x).$$

(**II**) For any compact set K and open set G, $K \subset G \subset \Omega$, there is a positive constant

A_2 such that for all $f \in X$,

$$\sup_{x\in K} |f(x)| \leqslant A_2 \sup_{x\in G\setminus K} |f(x)|.$$

Then the mixed norm $\|\cdot\|_{p,q,\varphi}$ on X is independent of the defining function λ and the parameter ε. That is, for any two defining functions λ_1, λ_2 and any two positive parameters ε_1, ε_2, there exist some positive constants C_1 and C_2 such that for each $f \in X$,

$$C_1\|f\|_{p,q,\varphi,\lambda_1,\varepsilon_1} \leqslant \|f\|_{p,q,\varphi,\lambda_2,\varepsilon_2} \leqslant C_2\|f\|_{p,q,\varphi,\lambda_1,\varepsilon_1}.$$

Proof. First, we prove that $\|\cdot\|_{p,q,\varphi}$ is independent of ε. It is sufficient to prove that, given $0 < p, q \leqslant \infty$ and $0 < \varepsilon_1 < \varepsilon_2 \leqslant \varepsilon$, there exists a constant C such that for all $f \in X$,

$$\max_{\varepsilon_2\leqslant r\leqslant\varepsilon} M_\infty(f,r) \leqslant C\left\{\int_{\varepsilon_1}^{\varepsilon_2} M_p^q(f,t)dt\right\}^{\frac{1}{q}}. \tag{4}$$

To prove (4), we fix α and β, $\varepsilon_1 < \alpha < \beta < \varepsilon_2$. By (**II**) we have some x_0 such that $-\beta \leqslant \lambda(x_0) \leqslant -\alpha$ and $\max_{\varepsilon_2\leqslant r\leqslant\varepsilon} M_\infty(f,r) \leqslant A_2|f(x_0)|$. As in ref. [21] (or directly by ref. [1]), we have some $s > 0$,

$$B(x_0,s) \subseteq S(\varepsilon_1,\varepsilon_2) = \{u \in \mathbb{R}^n : -\varepsilon_2 \leqslant \lambda(u) \leqslant -\varepsilon_1\}.$$

By (**I**), we get

$$\begin{aligned}
|f(x_0)| &\leqslant \left\{\frac{A_1}{s^n}\int_{B(x_0,s)} |f(x)|^p dm(x)\right\}^{\frac{1}{p}} \\
&\leqslant C\left\{\int_{S(\varepsilon_1,\varepsilon_2)} |f(x)|^p dm(x)\right\}^{\frac{1}{p}} \\
&\leqslant C\left\{\int_{\varepsilon_1}^{\varepsilon_2} M_p^p(f,t)dt\right\}^{\frac{1}{p}}.
\end{aligned} \tag{5}$$

It gives

$$\max_{\varepsilon_2\leqslant r\leqslant\varepsilon} M_\infty(f,r) \leqslant C \sup_{\varepsilon_1\leqslant t\leqslant\varepsilon_2} M_p(f,t).$$

This is (4) in the case $q = \infty$ and $0 < p \leqslant \infty$. For $p \leqslant q < \infty$, applying Hölder inequality to (5), we obtain

$$\max_{\varepsilon_2\leqslant r\leqslant\varepsilon} M_\infty^q(f,r) \leqslant C\left\{\int_{\varepsilon_1}^{\varepsilon_2} M_p^p(f,t)dt\right\}^{\frac{q}{p}} \leqslant C\int_{\varepsilon_1}^{\varepsilon_2} M_p^q(f,t)dt.$$

For $0 < q < p \leqslant \infty$, by the inequality $M_q(f,r) \leqslant CM_p(f,r)$, we have

$$\max_{\varepsilon_2\leqslant r\leqslant\varepsilon} M_\infty^q(f,r) \leqslant C\int_{\varepsilon_1}^{\varepsilon_2} M_q^q(f,t)dt \leqslant C\int_{\varepsilon_1}^{\varepsilon_2} M_p^q(f,t)dt.$$

Thus, the inequality (4) follows.

Given two defining functions λ_1 and λ of Ω and ε small enough, keeping the estimates (**I**) and (**II**) in mind and with the same approach as that of Lemma 3 in ref. [6], we can prove, if $q < \infty$, there are two positive constants c_1 and c_2 such that

$$M_p^q(f,r,\lambda_1) \leqslant \frac{C}{r}\int_{c_1 r}^{c_2 r} M_p^q(f,t,\lambda)dt \tag{6}$$

for all $0 < r \leqslant \varepsilon$ and $f \in X$. Since φ is normal, for the above constants c_1 and c_2, there are two positive constants C_1 and C_2 such that, for $0 < r, t < \varepsilon$ satisfying $c_1 \leqslant \frac{t}{r} \leqslant c_2$,

$$C_1 \leqslant \frac{\varphi(t)}{\varphi(r)} \leqslant C_2. \tag{7}$$

Then

$$\begin{aligned}
&\int_0^{\frac{\varepsilon}{c_1}} M_p^q(f, r, \lambda_1)\frac{\varphi^q(r)}{r}dr \\
&\leqslant C\int_0^{\frac{\varepsilon}{c_1}} \frac{1}{r}\frac{\varphi^q(r)}{r}\left[\int_{c_1 r}^{c_2 r} M_p^q(f, t, \lambda)dt\right]dr \\
&= C\left\{\int_0^{\varepsilon} M_p^q(f, t, \lambda)dt\int_{\frac{t}{c_2}}^{\frac{t}{c_1}} \frac{\varphi^q(r)}{r^2}dr + \sup_{\lambda(x)\leqslant -\varepsilon} |f(x)|^q\right\} \\
&\leqslant C\left\{\int_0^{\varepsilon} M_p^q(f, t, \lambda)\frac{\varphi^q(t)}{t}dt + \sup_{\lambda(x)\leqslant -\varepsilon} |f(x)|^q\right\} \\
&\leqslant C\int_0^{\varepsilon} M_p^q(f, t, \lambda)\frac{\varphi^q(t)}{t}dt.
\end{aligned} \tag{8}$$

Thus, together with (4), we get

$$\int_0^{\varepsilon} M_p^q(f, r, \lambda_1)\frac{\varphi^q(r)}{r}dr \leqslant C\int_0^{\varepsilon} M_p^q(f, r, \lambda)\frac{\varphi^q(r)}{r}dr. \tag{9}$$

Now for $q = \infty$, if $0 < r \leqslant \min\{\frac{\varepsilon}{c_2}, \varepsilon\}$, by (6) and (7) again, we obtain that

$$\begin{aligned}
M_p(f, r, \lambda_1)\varphi(r) &\leqslant C\frac{\varphi(r)}{r}\int_{c_1 r}^{c_2 r} M_p(f, t, \lambda)dt \\
&\leqslant Cr^{-1}\int_{c_1 r}^{c_2 r} M_p(f, t, \lambda)\varphi(t)dt \\
&\leqslant C\sup_{c_1 r\leqslant t\leqslant c_2 r} M_p(f, t, \lambda)\varphi(t) \\
&\leqslant C\sup_{0\leqslant t\leqslant \varepsilon} M_p(f, t, \lambda)\varphi(t).
\end{aligned} \tag{10}$$

This and (9) gives $||f||_{p,q,\varphi,\lambda_1} \leqslant C||f||_{p,q,\varphi,\lambda}$. Changing the positions of λ and λ_1, we have the domination in the other direction. The proof is completed.

2.2 Some norm estimates of the gradient

Theorem 2.2. Let X be a space of some C^1 functions on $\Omega \subseteq \mathbb{R}^n$ satisfying (**I**), (**II**) and

(**III**) For $0 < p < \infty$, there is a positive constant A_3 such that for all $f \in X$,

$$|\nabla f(a)|^p r^p \leqslant A_3 r^{-n}\int_{B(a,r)} |f(x)|^p dm(x).$$

Then there exists a constant C such that for all $f \in X$,

$$\|\nabla f\|_{p,q,r\varphi} \leqslant C\|f\|_{p,q,\varphi}. \tag{11}$$

Proof. We have a positive constant $c > 0$ such that $B(x, cr) \subset \Omega_{\frac{r}{2}}$ for any $x \in \partial\Omega_r$ and $0 < r \leqslant \varepsilon$. By (**III**),

$$|\nabla f(x)|^q \leqslant A_3\frac{1}{r^{n+q}}\int_{B(x,cr)} |f(u)|^q dm(u).$$

For $0 < p \leqslant \infty$, $0 < q < \infty$ and $0 < r \leqslant \varepsilon$, similar to (6), we have c_1 and c_2 such that

$$M_p^q(\nabla f, r) \leqslant \frac{C}{r^{q+1}} \int_{c_1 r}^{c_2 r} M_p^q(f, t) dt.$$

Therefore, similar to (8), apply Theorem 2.1 and (7) to get

$$\begin{aligned}\int_0^\varepsilon M_p^q(\nabla f, r) \frac{r^q \varphi^q(r)}{r} dr &\leqslant C \int_0^\varepsilon \frac{\varphi^q(r)}{r^2} dr \int_{c_1 r}^{c_2 r} M_p^q(f, t) dt \\ &\leqslant C \int_0^\varepsilon M_p^q(f, t) \frac{\varphi^q(t)}{t} dt. \qquad (12)\end{aligned}$$

For $q = \infty$, just doing the implication as (10), we have

$$\|\nabla f\|_{p,\infty,r\varphi} \leqslant C \|f\|_{p,\infty,\varphi}.$$

This and (12) imply the estimate (11). The proof is completed.

For any $f \in H_k(\Omega)$ and any multi-index α, $D^\alpha f$ is still a polyharmonic function of order k on Ω. And furthermore, $\{D^\alpha f\colon\ f \in H_k(\Omega), |\alpha| \leqslant m\}$ satisfy the conditions (**I**), (**II**) and (**III**), see refs. [9] and footnote 1). Therefore, from Theorem 2.1 and Theorem 2.2 we have the following corollaries.

Corollary 2.1. The spaces $H_k^{p,q,\varphi}(\Omega)$ and $\mathcal{B}(\varphi, k)$, $\mathcal{B}_0(\varphi, k)$ are independent of the defining function λ and the parameter ε.

Corollary 2.2. For $0 < p = q < \infty$, the mixed norm spaces are exactly the Bergman spaces. More precisely,

$$H_k^{p,p,\varphi}(\Omega) = \left\{ f \in H_k(\Omega);\ \ \left\{ \int_\Omega |f(x)|^p \frac{\varphi^p(d(x))}{d(x)} dm(x) \right\}^{\frac{1}{p}} < \infty \right\}.$$

Corollary 2.3. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary, $0 < p, q \leqslant \infty$ and let m be a positive integer. Then there is a constant C such that for any $f \in H_k(\Omega)$,

$$\|\nabla^m f\|_{p,q,r^m\varphi} \leqslant C \|f\|_{p,q,\varphi}.$$

Corollary 2.4. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary, $0 < p, q \leqslant \infty$ and let m be a positive integer. Then there is a constant C such that for any $f \in H_k(\Omega)$,

$$\|\nabla^m f\|_{\infty,\infty,r^{m-1}\varphi} \leqslant C \|f\|_{\mathcal{B}(\varphi)}.$$

Proof. Since $|\nabla^m f(x)| = \sum_{|\alpha|=m} |D^\alpha f(x)|$, take $p = q = \infty$ in Corollary 2.3, we obtain

$$\begin{aligned}\|\nabla^m f\|_{\infty,\infty,r^{m-1}\varphi} &\leqslant C \sum_{|\alpha|=m} \left\| \frac{\partial^m f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right\|_{\infty,\infty,r^{m-1}\varphi} \\ &\leqslant C \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i}(x) \right\|_{\infty,\infty,\varphi} \leqslant C \|\nabla f\|_{\infty,\infty,\varphi} \\ &\leqslant C \sup_{0 < -\lambda(x) < \varepsilon} \varphi(d(x)) |\nabla f(x)| = C \|f\|_{\mathcal{B}(\varphi)}.\end{aligned}$$

The proof is completed.

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2.3 The behavior of an integral operator

From now on, we suppose further that Ω is convex. Given $a \in \Omega$, we define an integral operator S on $C(\Omega)$ as

$$Sf(x) = \int_0^1 f(a + t(x-a))dt, \quad f \in C(\Omega) \quad \text{and} \quad x \in \Omega.$$

Our interest is to understand the behavior of this operator. For this purpose we need some more lemmas.

First, we adopt a special defining function ρ of Ω as in ref. [20]. Given $a \in \Omega$, set

$$\mathbf{U} = \Big\{ x \in \mathbb{R}^n : x = a + (1-t)(\xi - a), t \in \Big(-\frac{1}{2}, \frac{1}{2}\Big), \xi \in \partial\Omega \Big\}.$$

For $x = a + (1-t)(\xi - a) \in \mathbf{U}$, define the special defining function $\rho(x)$ on $\mathbf{U}$ just as

$$\rho(x) = -t, \quad \text{if} \quad x = a + (1-t)(\xi - a) \in \mathbf{U}. \tag{13}$$

By ref. [20], ρ is of C^2 and $\nabla\rho \neq 0$ on $\partial\Omega$. Therefore, as on Page 292 of ref. [21], we can extend ρ to whole $\mathbb{R}^n$ to be a C^2 defining function of Ω. Then for each $x \in \Omega$ sufficiently near $\partial\Omega$, saying $0 < -\rho(x) \leqslant \varepsilon$, we have unique $\xi \in \partial\Omega$ and $r = -\rho(x) \in (0, \varepsilon]$ such that

$$x = a + (1-r)(\xi - a) \in \partial\Omega_r.$$

Now for f continuous on Ω, we set f^+ to be

$$f^+(a + (1-r)(\xi - a)) = \sup_{\frac{r}{2} \leqslant t \leqslant r} |f(a + (1-t)(\xi - a))|.$$

Lemma 2.1. Let X be a space of some continuous functions on Ω satisfying conditions (**I**) and (**II**). Then there exists a constant C such that for all $f \in X$,

$$||f^+||_{p,q,\varphi} \leqslant C||f||_{p,q,\varphi}.$$

Proof. Without loss of generality, we may assume $a = 0$. From (**I**), there exists a constant C such that for all $\frac{r}{2} \leqslant t \leqslant r$, $r \in (0, \varepsilon]$ and $f \in X$,

$$|f((1-t)\xi)|^p \leqslant \frac{C}{r^n} \int_{B((1-t)\xi, \frac{r}{4})} |f(u)|^p dm(u) \leqslant \frac{C}{r^n} \int_{B((1-r)\xi, \frac{3}{4}r)} |f(u)|^p dm(u).$$

So

$$|f^+((1-r)\xi)|^p \leqslant \frac{C}{r^n} \int_{B((1-r)\xi, \frac{3}{4}r)} |f(u)|^p dm(u). \tag{14}$$

From this, as in the proof of Lemma 3 in ref. [6], we get

$$M_p^q(f^+, r) \leqslant \frac{C}{r} \int_{c_1 r}^{c_2 r} M_p^q(f, t)dt$$

for all $0 < p \leqslant \infty$ and $0 < q < \infty$, where c_1, c_2 are positive constants not depending on f. Thus, similar to (8) and (10), we get

$$\|f^+\|_{p,q,\varphi} \leqslant C\|f\|_{p,q,\varphi} \quad \text{when} \quad 0 < p \leqslant \infty, \quad 0 < q \leq \infty.$$

The proof is completed.

Lemma 2.2. Let φ be normal on $(0, \varepsilon]$ and $s > 0$. Then there exists some constant C such that for all nonnegative continuous function g on $(0, \varepsilon]$,

$$\int_0^\varepsilon g^s(r) \frac{\varphi(r)}{r} dr \leqslant C \left\{ \sup_{\frac{\varepsilon}{2} \leqslant r \leqslant \varepsilon} g^s(r) + \int_0^\varepsilon \left| g\Big(\frac{r}{2}\Big) - g(r) \right|^s \frac{\varphi(r)}{r} dr \right\}. \tag{15}$$

Proof. The proof goes as that of Lemma 7 in ref. [9]. Without loss of generality, we may assume that g is bounded on $(0,\varepsilon]$. Note that

$$I_1 = \int_0^\varepsilon g^s(r)\frac{\varphi(r)}{r}dr < \infty.$$

By (1) we get $\varphi(\frac{r}{2}) \leqslant \left(\frac{1}{2}\right)^a \varphi(r)$. Then

$$\begin{aligned} I_1 &= \int_0^{2\varepsilon} g^s\left(\frac{r}{2}\right)\frac{\varphi\left(\frac{r}{2}\right)}{r}dr \\ &= \int_0^{\varepsilon} g^s\left(\frac{r}{2}\right)\frac{\varphi\left(\frac{r}{2}\right)}{r}dr + \int_\varepsilon^{2\varepsilon} g^s\left(\frac{r}{2}\right)\frac{\varphi\left(\frac{r}{2}\right)}{r}dr \\ &\leqslant \left(\frac{1}{2}\right)^a \left[\int_0^\varepsilon g^s\left(\frac{r}{2}\right)\frac{\varphi(r)}{r}dr + C\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g^s(r)\right]. \end{aligned} \tag{16}$$

If $s \geqslant 1$, then by the Minkowski's inequality we have

$$\left\{\int_0^\varepsilon g^s\left(\frac{r}{2}\right)\frac{\varphi(r)}{r}dr\right\}^{\frac{1}{s}} \leqslant \left\{\int_0^\varepsilon \left|g\left(\frac{r}{2}\right) - g(r)\right|^s \frac{\varphi(r)}{r}dr\right\}^{\frac{1}{s}} + \left\{\int_0^\varepsilon g^s(r)\frac{\varphi(r)}{r}dr\right\}^{\frac{1}{s}}.$$

Thus,

$$\begin{aligned} I_1^{\frac{1}{s}} &\leqslant \left(\frac{1}{2}\right)^{\frac{a}{s}} \left\{\left[\int_0^\varepsilon g^s\left(\frac{r}{2}\right)\frac{\varphi(r)}{r}dr\right]^{\frac{1}{s}} + C^{\frac{1}{s}}\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g(r)\right\} \\ &\leqslant \left(\frac{1}{2}\right)^{\frac{a}{s}} \left\{I_1^{\frac{1}{s}} + \left[\int_0^\varepsilon \left|g\left(\frac{r}{2}\right) - g(r)\right|^s \frac{\varphi(r)}{r}dr\right]^{\frac{1}{s}} + C^{\frac{1}{s}}\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g(r)\right\}. \end{aligned}$$

This implies

$$I_1^{\frac{1}{s}} \leqslant \frac{\left(\frac{1}{2}\right)^{\frac{a}{s}}}{1-\left(\frac{1}{2}\right)^{\frac{a}{s}}} \left\{\left[\int_0^\varepsilon \left|g\left(\frac{r}{2}\right) - g(r)\right|^s \frac{\varphi(r)}{r}dr\right]^{\frac{1}{s}} + C^{\frac{1}{s}}\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g(r)\right\}. \tag{17}$$

If $0 < s < 1$, based on the inequality $(a+b)^s \leqslant a^s + b^s (a, b > 0)$, from (16) we get

$$I_1 \leqslant \left(\frac{1}{2}\right)^a \left\{\int_0^\varepsilon \left|g\left(\frac{r}{2}\right) - g(r)\right|^s \frac{\varphi(r)}{r}dr + I_1 + C\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g^s(r)\right\}.$$

This yields

$$I_1 \leqslant \frac{\left(\frac{1}{2}\right)^a}{1-\left(\frac{1}{2}\right)^a} \left\{\int_0^\varepsilon \left|g\left(\frac{r}{2}\right) - g(r)\right|^s \frac{\varphi(r)}{r}dr + C\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g^s(r)\right\}. \tag{18}$$

Hence (17) and (18) give the estimate (15). The proof is completed.

Lemma 2.3. Let φ be normal on $(0,\varepsilon]$ and $s > 0$. Then there exists some constant C such that for all nonnegative continuous functions g on $(0,\varepsilon]$,

$$\sup_{0<r\leqslant\varepsilon} g^s(r)\varphi(r) \leqslant C\left\{\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g^s(r) + \sup_{0<r\leqslant\varepsilon}\left|g\left(\frac{r}{2}\right) - g(r)\right|^s \varphi(r)\right\}. \tag{19}$$

Proof. Similar to Lemma 2.2, we can assume that g is bounded on $(0,\varepsilon]$. We consider the case that $s = 1$ first. Then

$$
\begin{aligned}
&\sup_{0<r\leqslant\varepsilon} g(r)\varphi(r)\\
=\;& \sup_{0<r\leqslant 2\varepsilon} g\left(\frac{r}{2}\right)\varphi\left(\frac{r}{2}\right)\\
\leqslant\;& \sup_{0<r\leqslant\varepsilon} g\left(\frac{r}{2}\right)\varphi\left(\frac{r}{2}\right)+\sup_{\varepsilon<r\leqslant 2\varepsilon} g\left(\frac{r}{2}\right)\varphi\left(\frac{r}{2}\right)\\
\leqslant\;& \left(\frac{1}{2}\right)^a\left\{\sup_{0<r\leqslant\varepsilon} g\left(\frac{r}{2}\right)\varphi(r)+C\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g(r)\varphi(r)\right\}\\
\leqslant\;& \left(\frac{1}{2}\right)^a\left\{\sup_{0<r\leqslant\varepsilon}\left|g\left(\frac{r}{2}\right)-g(r)\right|\varphi(r)+\sup_{0<r\leqslant\varepsilon} g(r)\varphi(r)+C\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g(r)\right\}.
\end{aligned}
$$

This gives the estimate (19) for $s = 1$. The general case that $0 < s < \infty$ still holds because the weight $\varphi^{\frac{1}{s}}$ is still normal. The proof is completed.

Theorem 2.3. Let X be a space of some continuous functions on Ω satisfying conditions (**I**) and (**II**). Then there exists a constant C such that for any $f \in X$,

$$\|Sf\|_{p,q,\varphi} \leqslant C\|f\|_{p,q,r\varphi}.$$

Proof. We may assume that $a = 0$ and prove the above estimate with the special defining function ρ as (13). Then $Sf(x) = \int_0^1 f(tx)dt$. For $x = (1-r)\xi \in \partial\Omega_r$, $r \in (0, \varepsilon]$, we have

$$
\begin{aligned}
|Sf(x)| &\leqslant \int_0^1 |f(tx)|dt\\
&\leqslant C\left\{M_\infty(f,\varepsilon)+\int_0^{\frac{\varepsilon-r}{1-r}}|f((1-t)(1-r)\xi)|dt\right\}\\
&\leqslant C\left\{M_\infty(f,\varepsilon)+\int_r^{\varepsilon}|f((1-u)\xi)|du\right\}.
\end{aligned}
$$

Writing

$$h(x) = \int_r^\varepsilon |f((1-u)\xi)|du, \tag{20}$$

we only need to prove that

$$||h||_{p,q,\varphi} \leqslant C||f||_{p,q,r\varphi}.$$

First, we claim that, for $0 < p, q \leqslant \infty$,

$$||h||_{p,q,\varphi} \leqslant C\left\{\|f^+\|_{p,q,r\varphi}+\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} M_p(h,r)\right\}. \tag{21}$$

In fact, by (20)

$$h\left(\left(1-\frac{r}{2}\right)\xi\right)-h\left((1-r)\xi\right) = \int_{\frac{r}{2}}^{r}|f((1-u)\xi)|du. \tag{22}$$

If $0 < p \leqslant 1$, then by (22)

$$
\begin{aligned}
h^p\left(\left(1-\frac{r}{2}\right)\xi\right)-h^p((1-r)\xi) &= \left\{\int_{\frac{r}{2}}^{r}|f((1-u)\xi)|\,du\right\}^p\\
&\leqslant \left(\frac{r}{2}\right)^p\sup_{\frac{r}{2}\leqslant u\leqslant r}|f((1-u)\xi)|^p.
\end{aligned}\tag{23}
$$

Note $|f^+((1-r)\xi)| = \sup_{\frac{r}{2}\leqslant u\leqslant r} |f((1-u)\xi)|$. Denoting by $d\sigma$ the surface measure on $\partial\Omega$ and integrating both sides of (23) on $\partial\Omega$, we obtain

$$\begin{aligned}&\int_{\partial\Omega} h^p\left(\left(1-\frac{r}{2}\right)\xi\right)d\sigma(\xi) - \int_{\partial\Omega} h^p((1-r)\xi)d\sigma(\xi)\\ &\leqslant Cr^p\int_{\partial\Omega}|f^+((1-r)\xi)|^p d\sigma(\xi). \qquad (24)\end{aligned}$$

Setting $g(r) = \int_{\partial\Omega} h^p((1-r)\xi)d\sigma(\xi)$ and applying Lemma 2.2 with $s=\frac{q}{p}$, we get

$$\begin{aligned}&\int_0^\varepsilon g^s(r)\frac{\varphi^q(r)}{r}dr\\ &\leqslant C\left\{\int_0^\varepsilon\left[g\left(\frac{r}{2}\right)-g(r)\right]^s\frac{\varphi^q(r)}{r}dr + \sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g^s(r)\right\}\\ &\leqslant C\left\{\int_0^\varepsilon\left[\int_{\partial\Omega}|f^+((1-r)\xi)|^p d\sigma(\xi)\right]^s\frac{[r\varphi(r)]^q}{r}dr + \sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g^s(r)\right\}.\end{aligned}$$

Because Ω is of C^2, we have two positive constants C_1 and C_2 such that for all continuous functions F on $\{x\in\Omega : 0 < -\rho(x)\leqslant\varepsilon\}$,

$$C_1\int_{\partial\Omega_r}|F(x)|d\sigma_r(x) \leqslant \int_{\partial\Omega}|F((1-r)\xi)|d\sigma(\xi) \leqslant C_2\int_{\partial\Omega_r}|F(x)|d\sigma_r(x). \qquad (25)$$

Hence,

$$\|h\|_{p,q,\varphi} \leqslant C\left\{\|f^+\|_{p,q,r\varphi} + \sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} M_p(h,r)\right\}. \qquad (26)$$

This is (21) for $0<p\leqslant 1$ and $0<q<\infty$. For $1<p\leqslant\infty$, applying Minkowski's inequality to (22), we have

$$\begin{aligned}&\left\{\int_{\partial\Omega} h^p((1-\frac{r}{2})\xi)d\sigma(\xi)\right\}^{\frac{1}{p}} - \left\{\int_{\partial\Omega} h^p((1-r)\xi)d\sigma(\xi)\right\}^{\frac{1}{p}}\\ &\leqslant \left\{\int_{\partial\Omega}\left[\int_{\frac{r}{2}}^r |f((1-u)\xi)|du\right]^p d\sigma(\xi)\right\}^{\frac{1}{p}}\\ &\leqslant \frac{r}{2}\left\{\int_{\partial\Omega}|f^+((1-r)\xi)|^p d\sigma(\xi)\right\}^{\frac{1}{p}}. \qquad (27)\end{aligned}$$

Applying Lemma 2.2 to $g(r) = \left\{\int_{\partial\Omega} h^p((1-r)\xi)d\sigma(\xi)\right\}^{\frac{1}{p}}$ with $s=q>0$, we have

$$\int_0^\varepsilon g^s(r)\frac{\varphi^q(r)}{r}dr \leqslant C\left\{\int_0^\varepsilon\left[\int_{\partial\Omega}|f^+((1-r)\xi)|^p d\sigma(\xi)\right]^{\frac{q}{p}}\frac{[r\varphi(r)]^q}{r}dr + \sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} g^q(r)\right\}.$$

From this and (25), the estimate (21) holds for $1<p\leqslant\infty$ and $0<q<\infty$. Now for the case that $q=\infty$, instead of using Lemma 2.2 we apply Lemma 2.3 on (24) and (27) respectively to get (21). From Lemma 2.1, there is a constant C such that for each $f\in X$,

$$\|f^+\|_{p,q,r\varphi} \leqslant C||f||_{p,q,r\varphi}.$$

Meanwhile, by the proof of Theorem 2.1 we obtain

$$\sup_{\frac{\varepsilon}{2}\leqslant r\leqslant\varepsilon} M_p(h,r) \leqslant C\sup_{\frac{\varepsilon}{2}\leqslant -\rho(x)\leqslant\varepsilon}|f(x)| \leqslant C||f||_{p,q,r\varphi}.$$

This and (26) imply the conclusion of the theorem. The proof is completed.

3 The Gleason's problem

In this section, we will study the Gleason's problem.

3.1 The case of polyharmonic function spaces

Notice that $H_k(\Omega)$ satisfies the conditions (**I**), (**II**) and (**III**). Hence, Theorem 2.2 and Theorem 2.3 hold for $f \in H_k(\Omega)$.

Theorem 3.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain with C^2 boundary. Suppose φ is normal and $0 < p, q \leqslant \infty$, then the Gleason's problem $(\Omega, a, H_k^{p,q,\varphi})$ can be solved. More precisely, for any $a \in \Omega$ and integer $m \geqslant 1$, there exist bounded linear operators A_α on $H_k^{p,q,\varphi}(\Omega)$, $|\alpha| = m$, such that if $f \in H_k^{p,q,\varphi}(\Omega)$ with $D^\alpha f(a) = 0$ $(|\alpha| \leqslant m-1)$, then

$$f(x) = \sum_{|\alpha|=m} (x-a)^\alpha A_\alpha f(x). \tag{28}$$

Proof. Without loss of generality, we may assume $a = 0$. For $j = 1, 2, \cdots, n$, define the operator A_j on $H_k(\Omega)$ as

$$A_j(f)(x) = \int_0^1 \frac{\partial f}{\partial x_j}(tx)dt \quad \text{for} \quad f \in H_k(\Omega).$$

Then $A_j(f) \in H_k(\Omega)$. And for $f \in H_k(\Omega)$ with $f(0) = 0$, we have

$$f(x) = \sum_{j=1}^{n} x_j A_j(f)(x). \tag{29}$$

To prove (28), we suppose $m = 1$ first. By Theorem 2.2, each operator $f \longmapsto \frac{\partial f}{\partial x_j}$ is bounded from $H_k^{p,q,\varphi}(\Omega)$ to $H_k^{p,q,r\varphi}(\Omega)$. Then, Theorem 2.3 implies that the operator $f \longmapsto A_j(f) = S(\frac{\partial f}{\partial x_j})$ is bounded on $H_k^{p,q,\varphi}(\Omega)$. This yields the conclusion (28) for $m = 1$. The equality (28) for $m \geqslant 2$ can be proved by induction. The proof is completed.

Theorem 3.2. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain with C^2 boundary. Suppose φ is normal and $a \in \Omega$. Then for any integer $m \geqslant 1$ there exist bounded linear operators A_α on $\mathcal{B}(\varphi, k)$ (or on $\mathcal{B}_0(\varphi, k)$), $|\alpha| = m$, such that if $f \in \mathcal{B}(\varphi, k)$ (or $f \in \mathcal{B}_0(\varphi, k)$) with $D^\alpha f(a) = 0$ $(|\alpha| \leqslant m-1)$, then

$$f(x) = \sum_{|\alpha|=m} (x-a)^\alpha A_\alpha f(x). \tag{30}$$

Proof. Similarly, we may assume $a = 0 \in \Omega$. And we need only to prove, for integer $m \geqslant 1$, there exist bounded linear operators A_α on $\mathcal{B}(\varphi, k)$ (and on $\mathcal{B}_0(\varphi, k)$), $|\alpha| = m$, such that if $f \in \mathcal{B}(\varphi, k)$ (or $f \in \mathcal{B}_0(\varphi, k)$) with $D^\alpha f(0) = 0$ $(|\alpha| \leqslant m-1)$, then

$$f(x) = \sum_{|\alpha|=m} x^\alpha A_\alpha f(x).$$

For this purpose, we consider the case $m = 1$ first. By Theorem 2.3,

$$||Sf||_{\infty,\infty,\varphi} \leqslant C||f||_{\infty,\infty,r\varphi}.$$

From Theorem 2.3, Corollary 2.4 and the fact that

$$\frac{\partial}{\partial x_k}\left(A_j f\right)(x) = S\left(\frac{\partial^2 f}{\partial x_j \partial x_k}\right)(x),$$

we get

$$\begin{aligned}||A_j(f)||_{\mathcal{B}(\varphi)} &= \sup_{x\in\Omega, d(x)\leqslant\varepsilon} \varphi(d(x))|\nabla A_j(f)(x)| \\ &\leqslant C\sum_{k=1}^{n}\left\|S\left(\frac{\partial^2 f}{\partial x_j\partial x_k}\right)\right\|_{\infty,\infty,\varphi} \\ &\leqslant C\sum_{k=1}^{n}\left\|\frac{\partial^2 f}{\partial x_j\partial x_k}\right\|_{\infty,\infty,r\varphi} \\ &\leqslant C||f||_{\mathcal{B}(\varphi)}. \end{aligned} \tag{31}$$

If $f \in \mathcal{B}_0(\varphi, k)$, we claim that $A_j(f) \in \mathcal{B}_0(\varphi, k)$. In fact, for any $\varepsilon > 0$, we have some $\delta > 0$ such that for $0 < r < \delta$,

$$\sup_{\xi\in\partial\Omega} \varphi(r)\sum_{j=1}^{n}\left|\frac{\partial f}{\partial x_j}((1-r)\xi)\right| < \varepsilon. \tag{32}$$

Because $|\nabla\rho(x)| \neq 0$ on $\partial\Omega$, we have some positive constant c such that

$$B((1-r)\xi, cr) \subset \Omega_{\frac{r}{2}} \setminus \Omega_{2r} \qquad \text{when} \qquad 0 < r < \varepsilon, \quad \xi \in \partial\Omega.$$

Then from the condition (**III**),

$$\sum_{k=1}^{n}\left|\frac{\partial^2 f}{\partial x_j\partial x_k}((1-r)\xi)\right| \leqslant C\frac{1}{r^{n+1}}\int_{B((1-r)\xi,cr)}\left|\frac{\partial f}{\partial x_j}(y)\right| dm(y),$$

where the constant C is independent of r and ξ. From (1), (2) and (32), we get some $\delta_1 > 0$ such that for $0 < r < \delta_1$

$$\sup_{\xi\in\partial\Omega} r\varphi(r)\sum_{j,k=1}^{n}\left|\frac{\partial^2 f}{\partial x_j\partial x_k}((1-r)\xi)\right| < \varepsilon. \tag{33}$$

We know that

$$|Sf(x)| \leqslant \int_0^1 |f(tx)|dt \leqslant C\left\{M_\infty(f,\delta_1) + \int_r^{\delta_1}|f((1-u)\xi)|du\right\}.$$

By (33),

$$\begin{aligned} &\varphi(r)\left|\frac{\partial A_j(f)}{\partial x_k}((1-r)\xi)\right| \\ &= \varphi(r)\left|S\left(\frac{\partial^2 f}{\partial x_j\partial x_k}\right)((1-r)\xi)\right| \\ &\leqslant C\varphi(r)\left\{M_\infty\left(\frac{\partial^2 f}{\partial x_j\partial x_k},\delta_1\right) + \int_r^{\delta_1}\left|\left(\frac{\partial^2 f}{\partial x_j\partial x_k}\right)((1-u)\xi)\right|du\right\} \\ &\leqslant C\left\{\varphi(r)M_\infty\left(\frac{\partial^2 f}{\partial x_j\partial x_k},\delta_1\right) + r^a\int_r^{\delta_1}\frac{\varphi(u)}{u^a}\left|\left(\frac{\partial^2 f}{\partial x_j\partial x_k}\right)((1-u)\xi)\right|du\right\} \\ &\qquad\leqslant C\left\{\varphi(r)M_\infty\left(\frac{\partial^2 f}{\partial x_j\partial x_k},\delta_1\right) + \varepsilon r^a\int_r^{\delta_1}\frac{1}{u^{a+1}}du\right\} \\ &\qquad\leqslant C\left\{\varphi(r)M_\infty\left(\frac{\partial^2 f}{\partial x_j\partial x_k},\delta_1\right) + \varepsilon\right\}. \end{aligned}$$

The constant C in the above inequalities are also independent of r and ξ. Therefore,

$$\sup_{\xi\in\partial\Omega}\varphi(r)\left|\frac{\partial A_j f}{\partial x_k}((1-r)\xi)\right|<\varepsilon, \tag{34}$$

if r is sufficiently small. That means $A_j(f)\in\mathcal{B}_0(\varphi,k)$. Furthermore, combining (29), (31) and (34) we conclude the proof for $m=1$. The general case can also be proved by induction. The proof is completed.

Remark 3.1. In ref. [7], the authors have considered the Gleason's problem for the harmonic weighted Bergman space b^p with $1\leqslant p<\infty$ and harmonic Bloch space with $\varphi(r)=r$ in the unit ball $\mathbf{B}$ of $\mathbb{R}^n$. In ref. [17], Ren and Kahler have solved the Gleason's problem $(\mathbf{B},a,\mathcal{B}^s)$, where $\mathcal{B}^s$ is the harmonic Bloch-type space with $\varphi(r)=r^s$ and $0<s<\infty$. Our Theorem 3.1 and 3.2 generalize those in refs. [7,17].

3.2 The case of hyperbolic harmonic function spaces

Denote by I_n the identity matrix. The Poincaré metric on $\mathbf{B}=\{x\in\mathbb{R}^n:|x|<1\}$ deduced from the positive-definite matrix $(g_{ij})=(1-|x|^2)^{-2}I_n$ is $ds=\frac{|dx|}{1-|x|^2}$. And the corresponding Laplace-Beltrami operator is given by

$$\Delta_h=(1-|x|^2)^2\Delta+2(n-2)(1-|x|^2)\sum_{j=1}^n x_j\frac{\partial}{\partial x_j}.$$

Originally, the hyperbolic harmonic functions f are only defined on $\mathbf{B}$ for which $\Delta_h f\equiv 0$. So, the hyperbolic harmonic functions are closely connected with the Poincaré metric and the Lorenz group $SO(n,1)$ on $\mathbf{B}$, see refs. [18,22] for a small part of references. Similar to the fact that people are often interested in harmonic functions in a domain Ω in $\mathbb{R}^n$, although the Laplacian Δ is for the Riemannian manifold $\mathbb{R}^n$ with the Euclidean metric, we are going to be interested in those functions $f\in C^2(\Omega)$ with $\Delta_h f(x)=0$ for $x\in\Omega$. These functions f are real analytic at any $x\in\Omega\setminus\partial\mathbf{B}$ by the regular theorem of elliptic PDE, see ref. [23]. On the other hand, the C^2 function f defined by

$$f(x)=\begin{cases}|x|^2-\frac{1}{|x|^2}-4\log|x|, & |x|\geqslant 1,\\ 0\ , & |x|\leqslant 1\end{cases}$$

satisfies $\Delta_h f\equiv 0$ on $\mathbb{R}^4$, but f is not infinitely differentiable at any $x\in\partial\mathbf{B}$. Therefore, it is natural to define the set $h(\Omega)$ of all hyperbolic harmonic functions on Ω, provided $\Omega\subseteq\mathbb{R}^n\setminus\partial\mathbf{B}$, to be

$$h(\Omega)=\left\{f\in C^2(\Omega):\Delta_h f(x)=0,\ x\in\Omega\right\};$$

and define the hyperbolic harmonic mixed norm space $h^{p,q,\varphi}(\Omega)$ to be

$$h^{p,q,\varphi}(\Omega)=\{f\in h(\Omega):\|f\|_{p,q,\varphi}<\infty\}.$$

The hyperbolic harmonic Bergman space can be defined in the usual way and it coincides with $h^{p,p,\varphi}(\Omega)$.

To study the Gleason's problem on some hyperbolic harmonic function spaces, one could not hope that the function g_k in (3) is still hyperbolic harmonic as pointed out in ref. [18]. We adopt the adjustment for this problem as Ren and Kahler, which is as follows.

Let X be a space of some hyperbolic harmonic functions in the domain $\Omega \subseteq \mathbb{R}^n \setminus \partial \mathbf{B}$ and denote by Y a certain function space associated to X without the condition of hyperbolic harmonicity. The Gleason's problem could be turned out in this way:

if $a \in \Omega$ and $f \in X$ with $f(a) = 0$, do there exist functions $g_1, g_2, \cdots, g_n \in Y$ such that

$$f(x) = \sum_{k=1}^{n} (x_k - a_k) g_k(x)$$

for all $x \in \Omega$?

Before formulating our solution to the Gleason's problem for the hyperbolic harmonic mixed norm space, we need some more lemmas.

Lemma 3.1. Let $\Omega \subseteq \mathbb{R}^n \setminus \partial \mathbf{B}$ be a domain. Then for any compact subset $K \subset \Omega$, multi-index α and $f \in h(\Omega)$,

$$\sup_{x \in K} |D^\alpha f(x)| = \sup_{x \in \partial K} |D^\alpha f(x)| .$$

Proof. By ref. [23], the function $f \in h(\Omega)$ has the maximal property. That is, for each compact subset $K \subset \Omega$,

$$\sup_{x \in K} |f(x)| \leqslant \sup_{x \in \partial K} |f(x)|.$$

Because $h(\Omega)$ is linear, with the above estimate and the definition of partial derivatives, we obtain

$$\sup_{x \in K} \left| \frac{\partial f}{\partial x_j}(x) \right| \leqslant \sup_{x \in \partial K} \left| \frac{\partial f}{\partial x_j}(x) \right|, \quad j = 1, \cdots, n.$$

This implies the conclusion for $|\alpha| = 1$. The general case can also be proved by induction. The proof is completed.

Lemma 3.2. Let $\Omega \subseteq \mathbb{R}^n \setminus \partial \mathbf{B}$ be a domain. Then for $0 < p < \infty$, there exists a constant C such that for each $f \in h(\Omega)$ and $B(a, r) \subset \Omega$,

$$|f(a)| + r|\nabla f(a)| + r^2 \sum_{|\alpha|=2} |D^\alpha f(a)| \leqslant C \left\{ \frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dm(x) \right\}^{\frac{1}{p}}. \quad (35)$$

Proof. For $x \in \Omega$, recall that $d(x)$ is the distance from x to $\partial\Omega$. Clearly, $d(x) \leqslant |1 - |x||$ since $\Omega \subseteq \mathbb{R}^n \setminus \partial \mathbf{B}$. For each $f \in h(\Omega)$, f is a solution of the equation

$$\Delta f + \frac{2(n-2)}{1-|x|^2} \sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j} = 0. \quad (36)$$

Applying Proposition 13.3 on Page 225 of ref. [23], for $f \in h(\Omega)$ and $B(a, r) \subset \Omega$, $0 < p < \infty$, we have

$$\sup_{x \in B(a, \frac{r}{4})} |f(x)| \leqslant C_1 \left\{ \frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dm(x) \right\}^{\frac{1}{p}}. \quad (37)$$

And applying Theorem 6.2 in ref. [24] to (36),

$$r|\nabla f(a)| + r^2 \sum_{|\alpha|=2} |D^\alpha f(a)| \leqslant C_2 \sup_{x \in B(a, \frac{r}{4})} |f(x)|, \quad (38)$$

where C_1 and C_2 are both independent of f and a, r. Now, the estimate (35) follows from (37) and (38). The proof is ended.

By Theorem 2.1, we have the following two corollaries.

Corollary 3.1. Let $\Omega \subseteq \mathbb{R}^n \setminus \partial\mathbf{B}$ be a bounded convex domain with C^2 boundary, and φ be normal, $0 < p, q \leqslant \infty$. Then the spaces $h^{p,q,\varphi}(\Omega)$ are independent of the defining function λ and the parameter ε.

Corollary 3.2. If $0 < p = q < \infty$, then $h^{p,q,\varphi}(\Omega) = a^{p,\varphi}(\Omega)$, where $a^{p,\varphi}$ is the p-th Bergman space

$$a^{p,\varphi} = \left\{ f \in h(\Omega);\ \ \left\{ \int_\Omega |f(x)|^p \frac{\varphi^p(d(x))}{d(x)} dm(x) \right\}^{\frac{1}{p}} < \infty \right\}.$$

Now we are ready to state our solution to $(\Omega, a, h^{p,q,\varphi}(\Omega))$.

Theorem 3.3. Let $\Omega \subseteq \mathbb{R}^n \setminus \partial\mathbf{B}$ be a bounded convex domain with C^2 boundary. Suppose that φ is normal and $0 < p, q \leqslant \infty$, then for any $a \in \Omega$, there exist bounded linear operators A_j from $h^{p,q,\varphi}(\Omega)$ to $X = \{f \in C(\Omega) : \|f\|_{p,q,\varphi} < \infty\}$, $j = 1, 2, \cdots, n$, such that if $f \in h^{p,q,\varphi}(\Omega)$ with $f(a) = 0$, then

$$f(x) = \sum_{j=1}^{n} (x_j - a_j) A_j f(x).$$

Proof. We may also assume $a = 0$. Similar to the proof of Theorem 3.1, for $f \in h(\Omega)$ with $f(a) = 0$, we have

$$f(x) = \sum_{j=1}^{n} x_j A_j(f)(x),$$

where $A_j(f)(x) = \int_0^1 \frac{\partial f}{\partial x_j}(tx)dt$. By Lemma 3.2 and Theorem 2.2, for $f \in h(\Omega)$

$$\left\| \frac{\partial f}{\partial x_j} \right\|_{p,q,r\varphi} \leqslant C \|f\|_{p,q,\varphi}.$$

By Lemma 3.2, similar to (14), we have

$$\left| \left(\frac{\partial f}{\partial x_j} \right)^+ ((1-r)\xi) \right|^p \leqslant \frac{C}{r^{n+p}} \int_{B((1-r)\xi, \frac{3}{4}r)} |f(u)|^p dm(u).$$

Then as the proof of Lemma 2.1, for $0 < p, q \leqslant \infty$ we obtain

$$\left\| \left(\frac{\partial f}{\partial x_j} \right)^+ \right\|_{p,q,r\varphi} \leqslant C \|f\|_{p,q,\varphi}. \tag{39}$$

For $x = (1-r)\xi \in \partial\Omega_r$, $r \in (0, \varepsilon]$, set

$$h(x) = \int_r^\varepsilon \left| \frac{\partial f}{\partial x_j}((1-u)\xi) \right| du.$$

As (21) we get, for $0 < p, q \leqslant \infty$,

$$\|h\|_{p,q,\varphi} \leqslant C \left\{ \left\| \left(\frac{\partial f}{\partial x_j} \right)^+ \right\|_{p,q,r\varphi} + \sup_{\frac{\varepsilon}{2} \leqslant r \leqslant \varepsilon} M_p(h, r) \right\}.$$

Then, Lemma 3.2 and (39) give

$$\|h\|_{p,q,\varphi} \leqslant C\left\{\left\|\left(\frac{\partial f}{\partial x_j}\right)^+\right\|_{p,q,r\varphi} + \|f\|_{p,q,\varphi}\right\} \leqslant C\|f\|_{p,q,\varphi}. \tag{40}$$

It is trivial that

$$|A_j(f)(x)| \leqslant C\left\{M_\infty\Big(\frac{\partial f}{\partial x_j}, \varepsilon\Big) + h(x)\right\}. \tag{41}$$

Now the estimate $\|A_j(f)\|_{p,q,\varphi} \leqslant C\|f\|_{p,q,\varphi}$ comes from (40) and (41) and Lemma 3.2. The proof is finished.

Remark 3.2. For $\Omega = \mathbf{B}$, $0 < p = q < \infty$ and $\varphi(r) = r^s$, the conclusion of Theorem 3.3 is the main result in ref. [18], where the approach is strongly based on the symmetry of $\mathbf{B}$.

4 Final remarks

As in ref. [8], a domain $\Omega \subseteq \mathbb{R}^n$ is called star-shaped with a strong reference point $a \in \Omega$ if Ω is star-shaped with a and there exists an angle $\theta_0 \in [0, \frac{\pi}{2})$ such that

$$\frac{\xi - a}{|\xi - a|} \cdot \overrightarrow{n}(\xi) \geqslant \cos\theta_0 > 0$$

for all $\xi \in \partial\Omega$, where $\overrightarrow{n}$ denotes the unit outward normal vector filed on $\partial\Omega$. It is trivial that a bounded convex C^2 domain must be star-shaped with any point $a \in \Omega$, and furthermore, any $a \in \Omega$ is a strong reference point. In ref. [8], the authors solved the Gleason's problem (Ω, a, b_k^p), $(\Omega, a, \mathcal{B}(\varphi, 1))$ and $(\Omega, a, \mathcal{B}_0(\varphi, 1))$, where b_k^p is the harmonic Bergman-Sobolev space of order k, $1 \leqslant p < \infty$ and Ω is a star-shaped domain with a strong reference point $a \in \Omega$.

A careful check of our results in sec. 2 and Theorem 3.1, Theorems 3.2 and 3.3 shows that the conclusions (28) and (30) remain valid if Ω is a star-shaped bounded C^2 domain with the strong reference point $a \in \Omega$. Therefore, our theorems also extend ref. [8]. To short the length, here we only exhibit it as Theorem 3.4.

Theorem 3.4. Let $\Omega \subseteq \mathbb{R}^n$ be a star-shaped bounded C^2 domain with a strong reference point $a \in \Omega$. Suppose that φ is normal and $0 < p, q \leqslant \infty$, then for any integer $m \geqslant 1$, there exist bounded linear operators A_α on $H_k^{p,q,\varphi}(\Omega)$, $|\alpha| = m$, such that if $f \in H_k^{p,q,\varphi}(\Omega)$ with $D^\alpha f(a) = 0$ $(|\alpha| \leqslant m-1)$, then

$$f(x) = \sum_{|\alpha|=m} (x-a)^\alpha A_\alpha f(x).$$

We now go back to the complex variables. Let $\Omega \subseteq \mathbf{C}^n = \mathbb{R}^{2n}$ be a star-shaped bounded C^2 domain with a strong reference point $a \in \Omega$. Denote $\mathbf{H}(\Omega)$ the family of all holomorphic functions on Ω. The holomorphic mixed norm space $\mathbf{H}_{p,q,\varphi}(\Omega)$ is defined as

$$\mathbf{H}_{p,q,\varphi}(\Omega) = \{f \in \mathbf{H}(\Omega); \|f\|_{p,q,\varphi} < \infty\}.$$

And the holomorphic φ-Bloch (little φ-Bloch) space $\mathbf{B}(\varphi)$ $(\mathbf{B}_0(\varphi))$ is defined in the same way.

It is well known that the holomorphic functions still satisfy the conditions (**I**), (**II**) and (**III**), and so these corresponding results in sec. 2 still hold for the function $f \in \mathbf{H}(\Omega)$. Therefore, with our approach in sec. 2 and subsec. 3.1, we can solve the Gleason's problem in the holomorphic function setting. We have the following theorems.

Theorem 3.5. Let $\Omega \subseteq \mathbf{C}^n$ be a star-shaped bounded C^2 domain with a strong reference point $a \in \Omega$. Suppose that φ is normal and $0 < p, q \leqslant \infty$. Then for any integer $m \geqslant 1$, there exist bounded linear operators A_α on $\mathbf{H}_{p,q,\varphi}(\Omega)$, $|\alpha| = m$, such that if $f \in \mathbf{H}_{p,q,\varphi}(\Omega)$ with $D^\alpha f(a) = 0$ ($|\alpha| \leqslant m-1$), then

$$f(z) = \sum_{|\alpha|=m} (z-a)^\alpha A_\alpha f(z).$$

Theorem 3.6. Let $\Omega \subseteq \mathbf{C}^n$ be a star-shaped bounded C^2 domain with a strong reference point $a \in \Omega$ and φ a normal function. Then for any integer $m \geqslant 1$, there exist bounded linear operators A_α on $\mathbf{B}(\varphi)$ (or on $\mathbf{B}_0(\varphi)$), $|\alpha| = m$, such that if $f \in \mathbf{B}(\varphi)$ (or $f \in \mathbf{B}_0(\varphi)$) with $D^\alpha f(a) = 0$ ($|\alpha| \leqslant m-1$), then

$$f(z) = \sum_{|\alpha|=m} (z-a)^\alpha A_\alpha f(z).$$

We define $\mathbf{A}_j$ on $\mathbf{H}(\Omega)$ as

$$\mathbf{A}_j f(z) = \int_0^1 \frac{\partial f}{\partial z_j}(tz)dt, \quad f \in \mathbf{H}(\Omega), \quad z \in \Omega.$$

Then for $f \in \mathbf{H}(\Omega)$ with $f(0) = 0$, we have

$$f(z) = \sum_{j=1}^n z_j \mathbf{A}_j f(z). \tag{42}$$

From this, the proof of these two theorems goes as the proofs of Theorem 3.1 and Theorem 3.2 with only one adjustment that (29) should be replaced by the equality (42).

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