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The symmetric space, strong isotropy irreducibility and equigeodesic properties

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Abstract A smooth curve on a homogeneous manifold G/H is called a Riemannian equigeodesic if it is a homogeneous geodesic for any G-invariant Riemannian metric. The homogeneous manifold G/H is called Riemannian equigeodesic, if for any $x \in G/H$ and any nonzero $y \in T_x(G/H)$, there exists a Riemannian equigeodesic $c(t)$ with $c(0) = x$ and $\dot{c}(0) = y$. These two notions can be naturally transferred to the Finsler setting, which provides the definitions for Finsler equigeodesics and Finsler equigeodesic spaces. We prove two classification theorems for Riemannian equigeodesic spaces and Finsler equigeodesic spaces, respectively. Firstly, a homogeneous manifold G/H with a connected simply connected quasi compact G and a connected H is Riemannian equigeodesic if and only if it can be decomposed as a product of Euclidean factors and compact strongly isotropy irreducible factors. Secondly, a homogeneous manifold G/H with a compact semisimple G is Finsler equigeodesic if and only if it can be locally decomposed as a product, in which each factor is $\mathrm{Spin}(7)/G_2$, $G_2/SU(3)$ or a symmetric space of compact type. These results imply that the symmetric space and the strongly isotropy irreducible space of compact type can be interpreted by equigeodesic properties. As an application, we classify the homogeneous manifold G/H with a compact semisimple G such that all the G-invariant Finsler metrics on G/H are Berwald. It suggests a new project in homogeneous Finsler geometry, i.e., to systematically study the homogeneous manifold G/H on which all the G-invariant Finsler metrics satisfy a certain geometric property.

Keywords equigeodesic, equigeodesic space, flag manifold, orbit type stratification, symmetric space, strongly isotropy irreducible space

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1 Introduction

A Riemannian manifold is called a *geodesic orbit* (or simply *g.o.*) space, if each maximally extended geodesic is homogeneous, i.e., it is the orbit of a one-parameter subgroup of isometries. This notion was introduced by Kowalski and Vanhecke [24] in 1991. Since then, it has been extensively studied in homogeneous Riemannian geometry and homogeneous pseudo-Riemannian geometry [2, 3, 9, 14, 16, 17, 31,

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32] (more references can be found in [7]). Recently, Yan and Deng [46] defined the g.o. property in homogeneous Finsler geometry (see [41, 42, 47] for some recent progress).

In this paper, we are motivated by the following question.

Question. *Can we define an analog of the g.o. property with the homogeneous geodesic replaced by the equigeodesic*?

Here, an *equigeodesic* is a smooth curve on a homogeneous manifold G/H, which is a homogeneous geodesic for any G-invariant Finsler metric, or any G-invariant Finsler metric in a preferred subclass, e.g., Riemannian, Randers, (α,β), etc. As we have different types of equigeodesics, we specify them as *Finsler equigeodesics*, *Riemannian equigeodesics*, Rander equigeodesics, (α,β) *equigeodesics*, etc. The equigeodesic was firstly introduced by Cohen et al. [10] in 2011. Until now, only Riemannian equigeodesics have been studied on some special homogeneous manifolds [18,35,39]. In this paper, we are only concerned with Riemannian equigeodesics and Finsler equigeodesics (see [36] for some progress on other types of equigeodesics).

Using the pattern of the g.o. property, we define the *Riemannian* or *Finsler equigeodesic space* as follows. A homogeneous manifold G/H is called *Riemannian* or *Finsler equigeodesic*, if for any $x\in G/H$ and any nonzero $y\in T_x(G/H)$, there exists a equigeodesic $c(t)$ of the specified type satisfying $c(0)=x$ and $\dot{c}(0)=y$. When G/H has an orthogonal reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ with respect to a fixed $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{g}$, G/H is Riemannian or Finsler equigeodesic if and only if each $u\in\mathfrak{m}\backslash\{0\}$ is a Riemannian or Finsler equigeodesic vector, respectively.

Compared with the mixed nature of the homogeneous geodesic and the g.o. property, which is half algebraic and half geometric, equigeodesics and equigeodesic spaces are totally algebraic properties, and they are much stronger. So it looks more likely that equigeodesics and equigeodesic spaces can be explicitly described or completely classified, without too much calculation. This thought is justified by the main theorems of this paper, which partially classify Riemannian and Finsler equigeodesic spaces.

For the Riemannian equigeodesic space, we have the following theorem.

Theorem A. *Let G/H be a simply connected homogeneous manifold on which the connected simply connected quasi compact Lie group G acts almost effectively. Then G/H is a Riemannian equigeodesic if and only if it is a product of Euclidean factors and strongly isotropy irreducible factors.*

Here, a product decomposition $G/H=G_1/H_1\times\cdots\times G_m/H_m$ for a homogeneous manifold means that

$$G=G_1\times\cdots\times G_m,\quad H=H_1\times\cdots\times H_n$$

with $H_i=H\cap G_i$ for each i.

Theorem A is a reformulation of Theorem 3.5. Firstly, it provides a new description for compact strongly isotropy irreducible spaces and their products (see [5] or [6, Theorem 27]). Secondly, it reduces the classification for some connected simply connected Riemannian equigeodesic spaces to that for compact strongly isotropy irreducible spaces [25, 27–29, 40]. Finally, it implies that the Riemannian equigeodesic space property for G/H depends not only on Lie algebras but also on Lie groups (see Remark 3.6), so our knowledge on the Riemannian equigeodesic space which is not connected or not simply connected is still quite limited.

For the Finsler equigeodesic space, we have the following theorem.

Theorem B. *Let G/H be a homogeneous manifold on which the compact semisimple Lie group G acts almost effectively. Then G/H is Finsler equigeodesic if and only if it can be locally decomposed as*

$$G/H=G_1/H_1\times\cdots\times G_m/H_m,$$

in which each G_i/H_i is $\mathrm{Spin}(7)/G_2$, $G_2/SU(3)$ *or a symmetric space of compact type.*

Here, the local product decomposition for a homogeneous manifold G/H means a product decomposition for the universal cover for a connected component $G_0/G_0\cap H$ of G/H, or equivalently the following direct sum decompositions in the Lie algebra level:

$$\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_m,\quad \mathfrak{h}=\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_m=(\mathfrak{h}\cap\mathfrak{g}_1)\oplus\cdots\oplus(\mathfrak{h}\cap\mathfrak{g}_m).$$

Theorem B reduces the classification for Finsler equigeodesic spaces of compact type to that for symmetric spaces [19]. Compared with Theorem A, the proof of Theorem B is harder, but the classification result is much cleaner and more complete.

The strategy for proving Theorem B is the following. Firstly, we prove two criteria for the Finsler equigeodesic vector and the Finsler equigeodesic space, respectively (see Theorem 4.2 and Lemma 4.9). It turns out that the Finsler equigeodesic space is a property which only depends on the Lie algebras. So secondly, we can use Theorem A (notice that a Finsler equigeodesic space must be Riemannian equigeodesic) to locally decompose a Finsler equigeodesic space G/H to a product of compact strongly isotropy irreducible factors. By Lemma 4.10, each factor is also Finsler equigeodesic. Then we need to apply the criterion to each G/H in the classification list for compact nonsymmetric strongly isotropy irreducible spaces. This would be a terribly long journey. Fortunately, we find the short cut. Roughly speaking, if G/H is Finsler equigeodesic, then $\dim\mathfrak{g}/\dim\mathfrak{h}$ cannot be too small, and for the orthogonal reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ with respect to an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{g}$, each vector in $\mathfrak{m}$ has a relatively large centralizer in $\mathfrak{g}$ (see (1) and (3) in Lemma 5.2). It is very easy to check that these numerical properties cannot be simultaneously satisfied by most compact strongly isotropy irreducible G/H (see Theorem 5.1).

Theorems A and B provide new interpretations for the strong isotropy irreducibility and symmetric space. Though in this paper, we have only studied the equigeodesic space from the compact side, we guess that those noncompact ones are also interesting and may share some similar phenomena.

Moreover, Theorem B starts a new project in homogeneous Finsler geometry, i.e., to classify the homogeneous manifold G/H on which each G-invariant Finsler metric satisfies a certain geometric property. For example, we may consider the following problem.

Problem 1.1. *Classify all the homogeneous manifolds G/H such that all the G-invariant Finsler metrics on G/H are Berwald.*

We prove that a homogeneous manifold G/H with a compact G is Finsler equigeodesic if and only if it has a reductive decomposition for which each G-invariant Finsler metric is naturally reductive (see Lemma 4.8), and if and only if the property in Problem 1.1 is satisfied (see Theorem 6.1), so Theorem B has the following application, which helps us solve Problem 1.1 partially.

Theorem C. *Let G/H be a homogeneous manifold on which the compact semisimple Lie group G acts almost effectively. Then it satisfies the condition that each G-invariant Finsler metric on G/H is Berwald if and only if it can be locally decomposed as*

$$G/H=G_1/H_1\times\cdots\times G_m/H_m,$$

in which each G_i/H_i is $\mathrm{Spin}(7)/G_2$, $G_2/SU(3)$ or a symmetric space of compact type.

It seems promising that Theorem C might be generalized for noncompact G.

We may naturally generalize [7, Question 5.12.25] and consider the following problem.

Problem 1.2. *Classify all the homogeneous manifolds G/H such that all the G-invariant Finsler metrics on G/H are the geodesic orbits* (*or have vanishing S-curvature with respect to a G-invariant measure*)*.*

It seems that Problem 1.2 is a much harder problem.

The rest of this paper is organized as follows. In Section 2, we summarize some basic knowledge on general and homogeneous Finsler geometry and on the flag manifold, which is necessary for our later discussion. In Section 3, we introduce the Riemannian equigeodesic space and prove Theorem A (i.e., Theorem 3.5). In Section 4, we introduce the Finsler equigeodesic and the Finsler equigeodesic space, and partially prove Theorem B. In Section 5, we discuss the compact strongly isotropy irreducible Finsler equigeodesic space, and finish the proof of Theorem B. In Section 6, we prove Theorem C.

2 Preliminaries

2.1 The Minkowski norm, the Finsler metric and the geodesic

A *Minkowski norm* on a real vector space $\boldsymbol{V}$ ($\dim \boldsymbol{V} = n$) is a continuous function $F : \boldsymbol{V} \to \mathbb{R}_{\geqslant 0}$ satisfying

(1) the positiveness and smoothness, i.e., $F|_{\boldsymbol{V}\backslash\{0\}}$ is a positive smooth function;

(2) the positive 1-homogeneity, i.e., $F(\lambda y) = \lambda F(y)$ for every $\lambda \geqslant 0$;

(3) the convexity, i.e., for any $y \in \boldsymbol{V}\backslash\{0\}$,

$$\langle u, v\rangle_y^F = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\bigg|_{s=t=0} F^2(y + su + tv)$$

defines an inner product on $\mathfrak{m}$.

A *Finsler metric* on a smooth manifold M is a continuous function $F : TM \to \mathbb{R}_{\geqslant 0}$ such that $F|_{TM\backslash\{0\}}$ is smooth, and for each $x \in M$, $F(x, \cdot)$ is a Minkowski norm. We also call (M, F) a *Finsler manifold* or a *Finsler space* [4].

If each $F(x, \cdot)$ is a Euclidean norm, i.e., $F(x, y) = \langle y, y\rangle^{1/2}$ for some inner product $\langle \cdot, \cdot\rangle$ on T_xM, we say F is a *Riemannian metric*. A Minkowski norm (or a Finsler metric) F is Euclidean (or Riemannian) if its Cartan tensor

$$C_y^F(u, v, w) = \frac{1}{4}\frac{\partial^3}{\partial r \partial s \partial t}\bigg|_{r=s=t=0} F^2(y + ru + sv + tw)$$

vanishes everywhere.

The geodesic on a Finsler manifold (M, F) can be similarly defined as in Riemannian geometry, which is a smooth curve satisfying the locally minimizing principle for the arch length functional. Practically, we always assume that a geodesic has positive constant speed, i.e., $F(c(t), \dot{c}(t)) \equiv \text{const} > 0$. Then a smooth curve $c(t)$ is a geodesic if and only if its lifting $(c(t), \dot{c}(t))$ in $TM\backslash\{0\}$ is an integral curve of the geodesic spray

$$\mathrm{G} = y^i\partial_{x^i} - 2\mathrm{G}^i\partial_{y^i}, \quad \text{in which } \mathrm{G}^i = \frac{1}{4}g^{il}([F^2]_{x^k y^l}y^k - [F^2]_{x^l}).$$

Locally a geodesic $c(t)$ is a solution of the ordinary differential equation (ODE) system

$$\ddot{c}^i(t) + 2\mathrm{G}(c(t), \dot{c}(t)) = 0, \quad \forall\, i$$

(see [4, 34] for more details).

2.2 The homogeneous Finsler space and the invariant Finsler metric

A Finsler manifold (M, F) is *homogeneous* if its isometry group $I(M, F)$ acts transitively on M. Since $I(M, F)$ is a Lie transformation group, we can present M as a homogeneous manifold $M = G/H$ for any Lie subgroup $G \subset I(M, F)$ which acts transitively on M, and the homogeneous metric F is also called a *G-invariant metric* [11]. In this definition, G must act effectively on $(G/H, F)$. However, on most occasions, the *almost effectiveness*, i.e., $\mathfrak{h}$ does not contain a nonzero ideal of $\mathfrak{g}$, is enough. So we choose another way to introduce homogeneous Finsler geometry and its basic algebraic setups.

In this paper, we use G/H to denote a homogeneous manifold, set $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$, and always assume the following:

(1) G acts almost effectively on G/H;

(2) H is compactly imbedded in G, i.e., $\mathrm{Ad}_{\mathfrak{g}}(H)$ has a compact closure in $\mathrm{Aut}\mathfrak{g}$.

The assumption (2) guarantees a *reductive decomposition* for G/H, i.e., an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. With respect to the given reductive decomposition, we denote by $\mathrm{pr}_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ and $\mathrm{pr}_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$ the corresponding linear projections, and define $u_{\mathfrak{h}} = \mathrm{pr}_{\mathfrak{h}}(u)$ and $u_{\mathfrak{m}} = \mathrm{pr}_{\mathfrak{m}}(u)$ for any $u \in \mathfrak{g}$. The subspace $\mathfrak{m}$ can be naturally viewed as the tangent space $T_o(G/H)$ at the origin $o = eH$ with the $\mathrm{Ad}(H)$-action on $\mathfrak{m}$ identified with the isotropy H-action on $T_o(G/H)$.

The assumption (2) also guarantees the existence of G-invariant Riemannian and Finsler metrics on G/H. By the homogeneity, a G-invariant Riemannian metric on G/H is one-to-one determined by an

$\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m} = T_e(G/H)$. Similarly, a G-invariant Finsler metric F on G/H can be uniquely determined by $F(o,\cdot)$, which can be any arbitrary $\mathrm{Ad}(H)$-invariant Minkowski norm on $\mathfrak{m}$. For simplicity, we still use the same F to denote it. Recall that the Hessian of $\frac{1}{2}F^2$ provides a family of inner products on $\mathfrak{m}$, i.e., $\langle\cdot,\cdot\rangle_y^F$ ($y \in \mathfrak{m}\backslash\{0\}$).

In homogeneous Finsler geometry, the following fundamental result is well known.

Lemma 2.1. *For a homogeneous Finsler manifold $(G/H, F)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, we have*

$$\langle [v, w_1], w_2\rangle_y^F + \langle w_1, [v, w_2]\rangle_y^F + 2C_y^F(w_1, w_2, [v, y]) = 0, \quad \forall\, v \in \mathfrak{h}, \quad w_1, w_2 \in \mathfrak{m}, \quad y \in \mathfrak{m}\backslash\{0\}.$$

In the later discussion, we sometimes replace the assumption (2) with the following even stronger assumption: $\mathrm{Ad}_{\mathfrak{g}}(G)$ is compact. Then we can find, and then fix an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{g}$. For simplicity, we also call the pair $(G, \langle\cdot,\cdot\rangle_{\mathrm{bi}})$ a *quasi compact* Lie group. In this situation, the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is chosen to be the $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$-orthogonal one, which is simply called an *orthogonal reductive decomposition*, and any G-invariant Riemannian metric, determined by the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$, one-to-one determines the metric operator $\Lambda : \mathfrak{m} \to \mathfrak{m}$ by

$$\langle u, v\rangle = \langle u, \Lambda(v)\rangle_{\mathrm{bi}}, \quad \forall\, u, v \in \mathfrak{m},$$

which exhausts all the $\mathrm{Ad}(H)$-invariant $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$-positive definite linear endomorphisms on $\mathfrak{m}$.

2.3 The homogeneous geodesic

A geodesic $c(t)$ on a Finsler manifold (M, F) is called *homogeneous*, if it is the orbit of a one-parameter subgroup of isometries [46]. For a homogeneous Finsler space $(G/H, F)$, as the isometry subgroup G has been specified, a homogeneous geodesic is then required to have the form $c(t) = \exp tu \cdot x$ for some $u \in \mathfrak{g}$ and $x \in M$. In particular, when $c(t) = \exp tu \cdot o$ is a homogeneous geodesic, we call the vector $u \in \mathfrak{g}$ a *geodesic vector* for $(G/H, F)$. The following lemma in [26] is a well-known criterion for the geodesic vector.

Lemma 2.2. *For a homogeneous Finsler space $(G/H, F)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, $u \in \mathfrak{g}$ is a geodesic vector if and only if $u \notin \mathfrak{h}$ and it satisfies*

$$\langle u_{\mathfrak{m}}, [\mathfrak{m}, u]_{\mathfrak{m}}\rangle_{u_{\mathfrak{m}}}^F = 0. \tag{2.1}$$

When F is a G-invariant Riemannian metric, determined by the $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$, this criterion for the geodesic vector is still valid [24], and we may simplify (2.1) as

$$\langle u_{\mathfrak{m}}, [\mathfrak{m}, u]_{\mathfrak{m}}\rangle = 0.$$

2.4 The homogeneous Berwald space and the naturally reductive Finsler space

A Finsler manifold (M, F) is called *Berwald* if its geodesic spray $\mathrm{G} = y^i\partial_{x^i} - 2\mathrm{G}^i\partial_{y^i}$ is *affine*, i.e., all the coefficients $\mathrm{G}^i = \mathrm{G}^i(x, y)$ are quadratic for its y-entry [34].

For a homogeneous Finsler manifold $(G/H, F)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, its Berwald property can be described by using the *spray vector field* η introduced by Huang [21, 22], i.e., a smooth map $\eta : \mathfrak{m}\backslash\{0\} \to \mathfrak{m}$ satisfying

$$\langle \eta(y), u\rangle_y = \langle y, [u, y]_{\mathfrak{m}}\rangle_y, \quad \forall\, u \in \mathfrak{m}.$$

Lemma 2.3. *A homogeneous Finsler manifold $(G/H, F)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is Berwald if and only if its spray vector field $\eta : \mathfrak{m}\backslash\{0\} \to \mathfrak{m}$ is a quadratic map.*

Proof. It has been pointed out in [43, Section 5] that the geodesic spray G of $(G/H, F)$ can be decomposed as $\mathrm{G} = \mathrm{G}_0 - \mathrm{H}$, where G_0 is the spray structure for the Nomizu connection on G/H, with respect to the given reductive decomposition, and H is a G-invariant vector field on $T(G/H)\backslash\{0\}$

which is tangent to each $T_x(G/H)$ and $\mathrm{H}\,|_{T_o(G/H)\backslash\{0\}=\mathfrak{m}\backslash\{0\}} = \eta$. Since the Nomizu connection is a linear connection on G/H [23], its corresponding spray structure G_0 is affine. So G is affine if and only if H is quadratic when restricted to each slit tangent space $T_x(G/H)\backslash\{0\}$, and by the G-invariance of H, if and only if $\eta = \mathrm{H}\,|_{T_o(G/H)\backslash\{0\}}$ is quadratic. □

Naturally reductive Finsler manifolds are a special class of homogeneous Berwald metrics [12, 26]. A homogeneous Finsler manifold $(G/H, F)$ is called *naturally reductive* with respect to a given reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ if each curve $c(t) = \exp tu \cdot o$ with $u \in \mathfrak{m}\backslash\{0\}$ is a geodesic, or equivalently it has a vanishing spray vector field $\eta : \mathfrak{m}\backslash\{0\} \to \mathfrak{m}$ [12].

2.5 Classification for flag manifolds

Let G be a compact connected semisimple Lie group. Then for any vector $u \in \mathfrak{g}$, the adjoint orbit $\mathrm{Ad}(G)u \subset \mathfrak{g}$ is called a *flag manifold*.

A flag manifold $\mathrm{Ad}(G)u$ can be presented as a homogeneous manifold $G/C_G(u)$. To determine the Lie algebra

$$\mathrm{Lie}(C_G(u)) = \mathfrak{c}_{\mathfrak{g}}(u) = \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \oplus [\mathfrak{c}_{\mathfrak{g}}(u), \mathfrak{c}_{\mathfrak{g}}(u)],$$

we only need to determine its central summand $\mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u))$ and its semisimple summand $[\mathfrak{c}_{\mathfrak{g}}(u), \mathfrak{c}_{\mathfrak{g}}(u)]$ as follows [1].

Let $\mathfrak{t}$ be a Cartan subalgebra, for which we have the root system $\Delta_{\mathfrak{g}}$ of $\mathfrak{g}$ and a prime root system $\{\alpha_1, \ldots, \alpha_n\}$, where $n = \dim \mathfrak{t}$ is the rank. We usually use an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ on $\mathfrak{g}$ to identify $\mathfrak{t}$ with $\mathfrak{t}^*$, and then the roots are viewed as vectors in $\mathfrak{t}$. Obviously, $\mathfrak{c}_{\mathfrak{g}}(u)$ contains $\mathfrak{t}$, so $\mathfrak{c}_{\mathfrak{g}}(u)$ is a *regular subalgebra* of $\mathfrak{g}$, i.e., each root or root space of $\mathfrak{c}_{\mathfrak{g}}(u)$ is also a root or root space of $\mathfrak{g}$, respectively.

By the suitable G-conjugation or the Weyl group action, we may assume that u satisfies $\langle\alpha_i, u\rangle_{\mathrm{bi}} \geqslant 0$ for each i. Suppose that we have $\langle\alpha_i, u\rangle_{\mathrm{bi}} = 0$ for $1 \leqslant i \leqslant k$ and $\langle\alpha_i, u\rangle_{\mathrm{bi}} > 0$ for $k < i \leqslant n$. Then the central summand $\mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u))$ in $\mathfrak{c}_{\mathfrak{g}}(u)$ is $(n-k)$-dimensional, linearly spanned by α_i with $k < i \leqslant n$. For the semisimple summand $[\mathfrak{c}_{\mathfrak{g}}(u), \mathfrak{c}_{\mathfrak{g}}(u)]$, $\{\alpha_1, \ldots, \alpha_k\}$ provides a prime root system, i.e., each root of $\mathfrak{g}$ belongs to $[\mathfrak{c}_{\mathfrak{g}}(u), \mathfrak{c}_{\mathfrak{g}}(u)]$ if and only if it is a linear combination of α_i with $1 \leqslant i \leqslant k$. To get the Dynkin diagram for $[\mathfrak{c}_{\mathfrak{g}}(u), \mathfrak{c}_{\mathfrak{g}}(u)]$, we may start with the Dynkin diagram for $\mathfrak{g}$ and then delete all the dots for α_i with $k < i \leqslant n$ and all the edges connected to these dots.

3 The Riemannian equigeodesic and the Riemannian equigeodesic space

3.1 The Riemannian equigeodesic and the Riemannian equigeodesic vector

A smooth curve $c(t)$ on a homogeneous manifold G/H is called a *Riemannian equigeodesic*, if it is a homogeneous geodesic for any G-invariant Riemannian metric on G/H. A vector $u \in \mathfrak{g}$ is called a *Riemannian equigeodesic vector* if $c(t) = \exp tu \cdot o$ is a Riemannian equigeodesic.

Now suppose that $(G, \langle\cdot,\cdot\rangle_{\mathrm{bi}})$ is quasi compact and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is the corresponding orthogonal reductive decomposition for G/H. Using Lemma 2.2, we can easily deduce the following criterion for the Riemannian equigeodesic vector.

Lemma 3.1. *Let G/H be a homogeneous manifold with a quasi compact $(G, \langle\cdot,\cdot\rangle_{\mathrm{bi}})$ and the corresponding orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then $u \in \mathfrak{g}$ is an equigeodesic vector for G/H if and only if $u \notin \mathfrak{h}$ and $[\Lambda(u_{\mathfrak{m}}), u]_{\mathfrak{m}} = 0$ for every metric operator Λ.*

Since we can choose $\Lambda = \mathrm{id}$, Lemma 3.1 provides $[u_{\mathfrak{m}}, u_{\mathfrak{h}}] = [u_{\mathfrak{m}}, u]_{\mathfrak{m}} = 0$ when u is a Riemannian equigeodesic vector, i.e., u and $u_{\mathfrak{m}}$ generate the same Riemannian equigeodesic

$$c(t) = \exp tu \cdot o = \exp tu_{\mathfrak{m}} \exp tu_{\mathfrak{h}} \cdot o = \exp tu_{\mathfrak{m}} \cdot o.$$

To summarize, the discussion for Riemannian equigeodesics on G/H with a quasi compact $(G, \langle\cdot,\cdot\rangle_{\mathrm{bi}})$ and the corresponding orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ can be reduced to that for Riemannian equigeodesic vectors in $\mathfrak{m}\backslash\{0\}$, i.e., $c(t) = \exp tv \cdot x$ is a Riemannian equigeodesic passing $x = g \cdot o$ if and only if $(\mathrm{Ad}(g^{-1})v)_{\mathfrak{m}}$ is a Riemannian equigeodesic vector.

3.2 The Riemannian equigeodesic space

Now we define the Riemannian equigeodesic space.

Definition 3.2. We call a homogeneous manifold G/H Riemannian equigeodesic or a Riemannian equigeodesic space if for each $x \in G/H$ and each nonzero $y \in T_x(G/H)$, there exists a Riemannian equigeodesic $c(t)$ with $c(0) = x$ and $\dot{c}(0) = y$.

When G is quasi compact, we can use Lemma 3.1 and the observation in the last paragraph of Subsection 3.1 to give the following equivalent description for the Riemannian equigeodesic space (the proof is easy and skipped).

Lemma 3.3. *Let G/H be a homogeneous manifold with the quasi compact $(G, \langle\cdot,\cdot\rangle_{\mathrm{bi}})$ and the corresponding orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then G/H is Riemannian equigeodesic if and only if each nonzero $u \in \mathfrak{m}$ is a Riemannian equigeodesic vector, i.e., $[\Lambda(u), u]_{\mathfrak{m}} = 0$ for each $u \in \mathfrak{m}\backslash\{0\}$ and every metric operator Λ.*

For example, an isotropy irreducible G/H with a compact G is Riemannian equigeodesic. A connected Abelian Lie group $G = G/\{e\}$ is Riemannian equigeodesic. More generally, we have the following example.

Example 3.4. Suppose that

$$G/H = (G_0 \times \cdots \times G_k)/(H_0 \times \cdots \times H_k) = G_0/H_0 \times G_1/H_1 \times \cdots \times G_k/H_k,$$

in which G_0 is a connected Abelian Lie group, $H_0 = \{e\}$, and for each $i > 0$, G_i and H_i are compact and G_i/H_i is isotropy irreducible. Applying the Schur lemma, we see that any metric operator for G/H has the form $\Lambda = \Lambda_0 \oplus c_1 \mathrm{id}|_{\mathfrak{m}_1} \oplus \cdots \oplus c_k \mathrm{id}|_{\mathfrak{m}_k}$, in which Λ_0 is an endomorphism on $\mathfrak{g}_0$. Using Lemma 3.3, we can easily see that G/H is a Riemannian equigeodesic space.

The following theorem indicates that the Riemannian equigeodesic space is such a strong condition that Example 3.4 becomes a typical model.

Theorem 3.5. *Let G/H be a homogeneous manifold on which G acts almost effectively. Suppose that G is connected, simply connected and quasi compact, and H is a connected. Then G/H is Riemannian equigeodesic if and only if we have the decompositions $G = G_1 \times \cdots \times G_m$ and $H = H_1 \times \cdots \times H_m$ such that each G_i/H_i is one of the following*:

(1) *a real line, i.e., $G_i = \mathbb{R}$ and $H_i = \{0\}$*;

(2) *a strongly isotropy irreducible G_i/H_i with compact connected semisimple G_i and connected H_i.*

Recall that a homogeneous manifold G/H is *isotropy irreducible* if the isotropic H-action is irreducible, and it is *strongly isotropy irreducible* if the isotropy action is irreducible when restricted to H_0. Compact strongly isotropy irreducible spaces and compact isotropy irreducible spaces have been classified in [25, 27–29, 40] and [38], respectively. There exist many examples of isotropy irreducible G/H, which is not strongly isotropy irreducible.

Proof of Theorem 3.5. Firstly, we assume that G/H is Riemannian equigeodesic and prove the decompositions.

Let $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the corresponding orthogonal reductive decomposition. We further $\mathrm{Ad}(H)$-equivariantly decompose $\mathfrak{m}$ as $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_m$ such that each $\mathfrak{m}_i$ is irreducible.

Claim 1. $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{h}$ when $i \neq j$.

For any $u_i \in \mathfrak{m}_i$ and $u_j \in \mathfrak{m}_j$, we apply Lemma 3.3 to $u = u_i + u_j$ and $\Lambda = \mathrm{id}|_{\mathfrak{m}_i} \oplus 2\mathrm{id}|_{\sum_{l\neq i}\mathfrak{m}_l}$, and get $[u_i, u_j]_{\mathfrak{m}} = [\Lambda(u), u]_{\mathfrak{m}} = 0$, which proves Claim 1.

Claim 2. $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h} + \mathfrak{m}_i$ for each i.

By the $\mathrm{Ad}(G)$-invariance of $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$,

$$\langle[\mathfrak{m}_i, \mathfrak{m}_i], \mathfrak{m}_j\rangle_{\mathrm{bi}} \subset \langle[\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_i\rangle_{\mathrm{bi}} \subset \langle\mathfrak{h}, \mathfrak{m}_i\rangle_{\mathrm{bi}} = 0, \quad \forall\, j \neq i.$$

So we have $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h} + \mathfrak{m}_i$, which proves Claim 2.

Claim 3. $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$ when $i \neq j$.

By the almost effectiveness of the G-action on G/H, we only need to prove that $[\mathfrak{m}_i, \mathfrak{m}_j]$ with $i \neq j$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$.

It is obvious that $[\mathfrak{m}_i, \mathfrak{m}_j]$ is an ideal of $\mathfrak{h}$, i.e., $[[\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{h}] \subset [\mathfrak{m}_i, \mathfrak{m}_j]$. When $i \neq j \neq k \neq i$, we have $[[\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_k] = 0$ because by Claim 1, $[[\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_k] \subset [\mathfrak{h}, \mathfrak{m}_k] \subset \mathfrak{m}_k$ on the one hand, and

$$[[\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_k] \subset [[\mathfrak{m}_i, \mathfrak{m}_k], \mathfrak{m}_j] + [\mathfrak{m}_i, [\mathfrak{m}_j, \mathfrak{m}_k]] \subset \mathfrak{m}_j + \mathfrak{m}_i$$

on the other hand. The subspace $[[\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_i] \subset [\mathfrak{h}, \mathfrak{m}_i] \subset \mathfrak{m}_i$ vanishes because

$$\begin{aligned}\langle[[\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_i], \mathfrak{m}_i\rangle_{\mathrm{bi}} &= \langle[\mathfrak{m}_i, \mathfrak{m}_j], [\mathfrak{m}_i, \mathfrak{m}_i]\rangle_{\mathrm{bi}} = \langle[\mathfrak{m}_j, [\mathfrak{m}_i, [\mathfrak{m}_i, \mathfrak{m}_i]]]\rangle_{\mathrm{bi}} \\ &\subset \langle\mathfrak{m}_j, [\mathfrak{m}_i, \mathfrak{h} + \mathfrak{m}_i]\rangle_{\mathrm{bi}} \subset \langle\mathfrak{m}_j, \mathfrak{h} + \mathfrak{m}_i\rangle_{\mathrm{bi}} = 0,\end{aligned}$$

where we have used Claim 2. For the same reason, $[[\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_j]$ also vanishes. Summarizing the above argument, we see that $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{h}$ with $i \neq j$ is an ideal of $\mathfrak{g}$. Then Claim 3 is proved.

Claim 4. We have Lie algebra direct sum decompositions $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ and $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_m$, in which $\mathfrak{g}_i = [\mathfrak{m}_i, \mathfrak{m}_i] + \mathfrak{m}_i$ and $\mathfrak{h}_i = [\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{h}}$.

By Claims 2 and 3, each $\mathfrak{g}_i \subset \mathfrak{g}$ is an ideal of $\mathfrak{g}$ with $\mathfrak{g}_i \cap \mathfrak{m} = \mathfrak{m}_i$. So we have a Lie algebra direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathfrak{g}'$, in which the ideal $\mathfrak{g}'$ is the $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$-orthogonal complement of $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$. Obviously, $\mathfrak{m} \subset \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$, so the ideal $\mathfrak{g}'$ of $\mathfrak{g}$ is contained in $\mathfrak{h}$, which must vanish by the almost effectiveness. So we get the decomposition for $\mathfrak{g}$. Meanwhile Claim 2 implies that each $\mathfrak{g}_i$ is compatible with the orthogonal reductive decomposition, i.e., $\mathfrak{g}_i = \mathfrak{h}_i + \mathfrak{m}_i$, in which $\mathfrak{h}_i = \mathfrak{g}_i \cap \mathfrak{h} = [\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{h}}$ and $\mathfrak{m}_i = \mathfrak{g}_i \cap \mathfrak{g}$. The decomposition for $\mathfrak{h}$ follows immediately. Now Claim 4 is proved.

Finally, we consider the corresponding decompositions for G, H and G/H.

Since G is connected and simply connected, Claim 4 provides the Lie group product decomposition $G = G_1 \times \cdots \times G_m$, in which each G_i is a connected simply connected quasi compact Lie subgroup generated by $\mathfrak{g}_i$. In each G_i, we have a connected Lie subgroup H_i with $\mathrm{Lie}(H_i) = \mathfrak{h}_i$. Then $H = H_1 \times \cdots \times H_n$ because both sides are connected Lie groups generated by the same Lie subalgebra. Obviously, $H_i = H \cap G_i$. By the closeness and connectedness of H, each H_i is a closed connected subgroup of G_i. The almost effectiveness of the G-action on G/H implies the almost effectiveness of the G_i-action on G_i/H_i. In the decomposition $G/H = G_1/H_1 \times \cdots \times G_m/H_m$, each G_i/H_i has a strongly irreducible isotropy representation. The classification in [40] indicates that G_i must be compact and semisimple, unless $G_i = \mathbb{R}$ and $H_i = \{0\}$.

To summarize, the above argument proves Theorem 3.5 in the one direction. The other direction is obvious by the discussion for Example 3.4. □

This proof is self-contained. It can be simplified by using some results in [33, Section 5]. Another possible proof is to verify that each G-invariant Riemannian metric on G/H is normal and then use an analog of the main theorem in [5].

Remark 3.6. Theorem 3.5 only classifies some simply connected Riemannian equigeodesic spaces. The universal cover of a Riemannian equigeodesic space may not be Riemannian equigeodesic any more. For example, $G/H = SU(3)/T^2\mathbb{Z}_3$ is isotropy irreducible, so it is Riemannian equigeodesic. Its universal cover $SU(3)/T^2$ is not Riemannian equigeodesic by Theorem 3.5, because it is not strongly isotropy irreducible. Using the classification work in [38], we can find many other similar examples.

4 The Finsler equigeodesic and the Finsler equigeodesic space

4.1 The Finsler equigeodesic and the Finsler equigeodesic vector

The definitions for the *Finsler equigeodesic* and the *Finsler equigeodesic vector* were proposed in [36].

Definition 4.1. A smooth curve on G/H is called a *Finsler equigeodesic* if it is a homogeneous geodesic for any G-invariant Finsler metric on G/H. A vector $u \in \mathfrak{g}$ is called a *Finsler equigeodesic vector* if it generates an equigeodesic $c(t) = \exp tu \cdot o$ on G/H.

Equivalently speaking, u is a Finsler equigeodesic vector if and only if it is a geodesic vector for any G-invariant Finsler metric on G/H. Obviously, any Finsler equigeodesic is also a Riemannian equigeodesic. So the Finsler equigeodesic is a stronger algebraic property than the Riemannian equigeodesic. For the Finsler equigeodesic vector, the observation is similar.

4.2 A criterion and some examples

Let G/H be a homogeneous manifold with a compact $(G, \langle\cdot,\cdot\rangle_{\mathrm{bi}})$ and the corresponding orthogonal irreducible decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. By the observation in Subsection 3.1, the discussion for Finsler equigeodesics can be reduced to that for Finsler equigeodesic vectors in $\mathfrak{m}\backslash\{0\}$.

For any vector $u \in \mathfrak{m}\backslash\{0\}$, we set $H_u = \{g \in H \mid \mathrm{Ad}(g)u = u\}$, and denote by $\boldsymbol{V}_u$ the $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$-orthogonal complement of $[\mathfrak{h}, u]$ in $\mathfrak{m}$. Then the $\mathrm{Ad}(H_u)$-action preserves $\boldsymbol{V}_u$. So we can further decompose $\boldsymbol{V}_u$ as $\boldsymbol{V}_u = \boldsymbol{V}_{u,0} + \boldsymbol{V}_{u,1}$, in which $\boldsymbol{V}_{u,0}$ is the fixed point set of the $\mathrm{Ad}(H_u)$-action and $\boldsymbol{V}_{u,1}$ is the $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$-orthogonal complement of $\boldsymbol{V}_{u,0}$ in $\boldsymbol{V}_u$.

With the above settings, the criterion for u to be a Finsler equigeodesic vector is as follows.

Theorem 4.2. *Keep all the above assumptions and notations, and then any vector $u \in \mathfrak{m}\backslash\{0\}$ is a Finsler equigeodesic vector if and only if it satisfies*

$$[u, \mathfrak{m}]_{\mathfrak{m}} \subset [u, \mathfrak{h}] + \boldsymbol{V}_{u,1}. \tag{4.1}$$

Proof. We first assume that $u \in \mathfrak{m}\backslash\{0\}$ satisfies (4.1) and prove that it is a Finsler equigeodesic.

Let F be any G-invariant Finsler metric on G/H. Denote by $\boldsymbol{W}_{u,F} = \{w \in \mathfrak{m} \mid \langle u, w\rangle_u^F = 0\}$ the subspace of all the directions in which the derivative of F vanishes at u.

By Lemma 2.1 and the property $C_u^F(u,\cdot,\cdot) \equiv 0$ for the Cartan tensor, we immediately get $\langle u, [\mathfrak{h}, u]\rangle_u^F = 0$, i.e., $[\mathfrak{h}, u] \subset \boldsymbol{W}_u^F$.

Let w be any vector in $\boldsymbol{V}_{u,1}$. Since the Minkowski norm $F = F(o,\cdot)$ on $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant, for any $g \in H_u$, we have

$$\langle \mathrm{Ad}(g)w, u\rangle_u^F = \langle \mathrm{Ad}(g)w, \mathrm{Ad}(g)u\rangle_{\mathrm{Ad}(g)u}^F = \langle w, u\rangle_u^F. \tag{4.2}$$

Since G is compact, H and H_u are also compact. So we can integrate (4.2) over H_u and get

$$\left\langle \int_{g\in H_u} \mathrm{Ad}(g) w d\mathrm{vol}_{H_u}, u \right\rangle_u^F = \mathrm{vol}(H_u) \cdot \langle w, u\rangle_u^F,$$

where $d\mathrm{vol}$ is a bi-invariant measure on H_u and $\mathrm{vol}(H_u) = \int_{H_u} d\mathrm{vol}_{H_u} \in (0, +\infty)$. Since

$$\int_{g\in H_u} \mathrm{Ad}(g) w d\mathrm{vol}_{H_u} \in \boldsymbol{V}_{u,1},$$

and it is fixed by all the $\mathrm{Ad}(H_u)$-actions, i.e., it is also contained in $\boldsymbol{V}_{u,0}$, we get

$$\int_{g\in H_u} \mathrm{Ad}(g) w d\mathrm{vol}_{H_u} = 0$$

and then $\langle w, u\rangle_u^F = 0$.

To summarize, we combine the above arguments and (4.1), and then we see that

$$[\mathfrak{m}, u]_{\mathfrak{m}} \subset [\mathfrak{h}, u] + \boldsymbol{V}_{u,1} \subset \boldsymbol{W}_u^F.$$

It implies that $\langle u, [\mathfrak{m}, u]_{\mathfrak{m}}\rangle_u^F = 0$, i.e., u is a geodesic vector for $(G/H, F)$. Since F is chosen arbitrarily, u is a Finsler equigeodesic vector. This ends the proof of Theorem 4.2 in the one direction.

We then prove the other direction, i.e., a Finsler equigeodesic u must satisfy (4.1).

Assume conversely that the Finsler equigeodesic vector $u \in \mathfrak{m}\backslash\{0\}$ does not satisfy (4.1). Lemma 2.2 implies $[\mathfrak{m}, u]_{\mathfrak{m}} \subset \bigcap_F \boldsymbol{W}_u^F$, where the intersection is taken for all the G-invariant Finsler metrics on G/H. So we only need to prove that $[u, \mathfrak{h}] + \boldsymbol{V}_{u,1} = \bigcap_F \boldsymbol{W}_u^F$.

In the above argument, we have already proved $[u,\mathfrak{h}]+\boldsymbol{V}_{u,1}\subset\bigcap_F \boldsymbol{W}_u^F$. To prove the inverse inclusion, we consider any nonzero vector $w\in\boldsymbol{V}_{u,0}$ and look for a G-invariant Finsler metric on G/H with $w\notin\boldsymbol{W}_u^F$. The construction of F is the following.

Without loss of generality, we may assume $\langle u,u\rangle_{\mathrm{bi}}=1$. By the slice theorem [30], the compact $\mathrm{Ad}(H)$-action on the unit sphere $S=\{u\mid\langle u,u\rangle_{\mathrm{bi}}=1,u\in\mathfrak{m}\}\subset\mathfrak{m}$ provides an orbit type stratification $S=S_1\coprod\cdots\coprod S_N$. For each i, S_i is an imbedded submanifold in S with a smooth fiber bundle structure, in which each fiber is an $\mathrm{Ad}(H)$-orbit of the same type. We may assume $\mathrm{Ad}(H)u\subset S_1$, and then we can find a sufficiently small $\mathrm{Ad}(H)$-invariant open neighborhood $\mathcal{U}$ of $\mathrm{Ad}(H)u$ in S_1 such that the quotient map $\pi:\mathcal{U}\to\mathcal{U}/H\cong\mathbb{R}^k$ is a smooth fiber bundle, and the fibers are the $\mathrm{Ad}(H)$-orbits parametrized as $\mathcal{O}_{x_1,\ldots,x_k}$ for $(x_1,\ldots,x_k)\in\mathbb{R}^k$ with $\mathcal{O}_{0,\ldots,0}=\mathrm{Ad}(H)u$. Since $\boldsymbol{V}_{u,0}$ is tangent to $\mathcal{U}$ and $\boldsymbol{V}_{u,0}\cap T_u(\mathrm{Ad}(H)u)=0$, we may adjust the parameter space $\mathbb{R}^k$ such that $\pi_*(w)$ coincides with $\frac{\partial}{\partial x_1}$ at the origin. We can find a smooth real function φ on $\mathbb{R}^k$ with $\frac{\partial}{\partial x_1}\varphi(0,\ldots,0)\neq 0$ and a sufficiently small compact support. The function $\varphi\circ\pi$ is viewed as an $\mathrm{Ad}(H)$-invariant smooth function S_1 which only takes zero value outside $\mathcal{U}$.

Next, we thicken $\mathcal{U}$ to an $\mathrm{Ad}(H)$-invariant neighborhood $\mathcal{U}'$ of $\mathrm{Ad}(H)u$ in S. The function $\varphi\circ\pi$ can be further extended to a compactly supported smooth function ψ on $\mathcal{U}'$. By the averaging process for the $\mathrm{Ad}(H)$-action, the $\mathrm{Ad}(H)$-invariance of ψ can be achieved.

Finally, with the above preparations, we are ready to construct the G-invariant Finsler metric. For any sufficiently small $\epsilon>0$,

$$F(y)=\langle y,y\rangle_{\mathrm{bi}}^{1/2}\cdot\left(1+\epsilon\cdot\psi\left(\frac{y}{\langle y,y\rangle_{\mathrm{bi}}^{1/2}}\right)\right)\tag{4.3}$$

induces an $\mathrm{Ad}(H)$-invariant Minkowski norm on $\mathfrak{m}$. The derivative of F at u does not vanish in the direction of w. So for the G-invariant Finsler metric F determined by this Minkowski norm, we have $w\notin\boldsymbol{W}_u^F$.

This ends the proof of Theorem 4.2 in the other direction. □

As immediate applications of Theorem 4.2, we have the following examples.

Example 4.3. Let G/H be a symmetric space of compact type with the Cartan decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. Then any vector $u\in\mathfrak{m}\backslash\{0\}$ is a Finsler equigeodesic vector.

Here, we call the homogeneous manifold G/H a *symmetric space of compact type* if G is compact semisimple, and G/H has a *Cartan decomposition*, i.e., a reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ satisfying $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$. Notice that the Cartan decomposition is orthogonal with respect to the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$ and we may take $\langle\cdot,\cdot\rangle_{\mathrm{bi}}=-B_{\mathfrak{g}}(\cdot,\cdot)$. So the Cartan decomposition is the corresponding orthogonal reductive decomposition. By Theorem 4.2, each $u\in\mathfrak{m}\backslash\{0\}$ is a Finsler equigeodesic vector for G/H because $[\mathfrak{m},u]_{\mathfrak{m}}=0$ in this situation. This example motivates us to study Finsler equigeodesic spaces (see Subsection 4.3).

Example 4.4. Let G/H be a homogeneous manifold with a compact semisimple G and $\mathrm{rk}G=\mathrm{rk}H$ (here rk is the dimension of the maximal torus subgroup). Then the set of Finsler equigeodesic vectors is nonempty.

More precisely, [44, Lemma 5.3] provides the following Finsler equigeodesic vectors. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ which is contained in $\mathfrak{h}$. Then the reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ for G/H is unique, and we have the following root plane decompositions:

$$\mathfrak{g}=\mathfrak{t}+\sum_{\alpha\in\Delta_{\mathfrak{g}}}\mathfrak{g}_{\pm\alpha},\quad\mathfrak{h}=\mathfrak{t}+\sum_{\alpha\in\Delta_{\mathfrak{h}}}\mathfrak{g}_{\pm\alpha},\quad\mathfrak{m}=\sum_{\alpha\in\Delta_{\mathfrak{g}}\backslash\Delta_{\mathfrak{h}}}\mathfrak{g}_{\pm\alpha},$$

where $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{h}}$ are the root systems of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then for any $\alpha\in\Delta_{\mathfrak{g}}\backslash\Delta_{\mathfrak{h}}$, any vector $u\in\mathfrak{g}_{\pm\alpha}\backslash\{0\}\subset\mathfrak{m}$ is a Finsler equigeodesic vector. This fact can be explained by Theorem 4.2, because on this occasion we have

$$[\mathfrak{g}_{\pm\alpha},\mathfrak{m}]_{\mathfrak{m}}\subset\sum_{\alpha\neq\beta\in\Delta_{\mathfrak{g}}\backslash\Delta_{\mathfrak{h}}}\mathfrak{g}_{\pm\beta}\subset\boldsymbol{V}_{u,1}.$$

Example 4.5. For a compact Lie group G, $u \in \mathfrak{g}\backslash\{0\}$ is a Finsler equigeodesic vector for $G = G/\{e\}$ if and only if $u \in \mathfrak{c}(\mathfrak{g})$. So in this case, the Riemannian equigeodesics (resp. Riemannian equigeodesic vectors) and the Finsler equigeodesics (resp. Finsler equigeodesic vectors) are the same.

4.3 The Finsler equigeodesic space and a criterion

As an analog of Definition 3.2, we define a *Finsler equigeodesic space* as follows.

Definition 4.6. We call a homogeneous manifold G/H Finsler equigeodesic or a Finsler equigeodesic space if for each $x \in G/H$ and each nonzero $y \in T_x(G/H)$, there exists a Finsler equigeodesic $c(t)$ with $c(0) = x$ and $\dot{c}(0) = y$.

Let G/H be a homogeneous manifold with a compact $(G, \langle\cdot,\cdot\rangle_{\rm bi})$ and the corresponding orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then by the observation in Subsection 3.1, the Finsler equigeodesic space can be equivalently described as follows.

Lemma 4.7. *A homogeneous manifold G/H with a compact $(G, \langle\cdot,\cdot\rangle_{\rm bi})$ and a corresponding orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is Finsler equigeodesic if and only if each $u \in \mathfrak{m}\backslash\{0\}$ is a Finsler equigeodesic vector.*

The Finsler equigeodesic property for G/H with a compact G can be described by natural reductiveness as follows.

Lemma 4.8. *Let G/H be a homogeneous manifold with a compact $(G, \langle\cdot,\cdot\rangle_{\rm bi})$. Then we have the following*:

(1) *if it has a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}'$, with respect to which all the G-invariant Finsler metrics on G/H are naturally reductive, then G/H is a Finsler equigeodesic space*;

(2) *if G/H is a Finsler equigeodesic space, then any G-invariant Finsler metric on G/H is naturally reductive with respect to the orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.*

Proof. (1) By the description for Finsler natural reductiveness in [12] or Subsection 2.4, each nonzero vector in $\mathfrak{m}'$ is a Finsler equigeodesic vector for G/H. The projection $\mathrm{pr}_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$ is a linear isomorphism when restricted to $\mathfrak{m}'$. By the discussion after Lemma 3.1, each nonzero vector in $\mathfrak{m}$ is Finsler equigeodesic. So G/H is Finsler equigeodesic by Lemma 4.7.

(2) Let F be any G-invariant Finsler metric on G/H. Lemma 4.7 indicates that each nonzero vector $u \in \mathfrak{m}$ generates a geodesic $c(t) = \exp tu \cdot o$ on $(G/H, F)$. By the description in [12] or Subsection 2.4, $(G/H, F)$ is naturally reductive with respect to the orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. □

By Theorem 4.2 and Lemma 4.7, we have the following criterion for Finsler equigeodesic spaces.

Lemma 4.9. *A homogeneous manifold G/H with a compact $(G, \langle\cdot,\cdot\rangle_{\rm bi})$ and the corresponding orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is Finsler equigeodesic if and only if*

$$\text{there exists a conic open dense subset } \mathcal{U} \text{ of } \mathfrak{m}\backslash\{0\} \text{ satisfying } [\mathfrak{m}, u]_{\mathfrak{m}} \subset [\mathfrak{h}, u],\ \forall\, u \in \mathcal{U}. \tag{4.4}$$

Proof. Assume that G/H is Finsler equigeodesic. Then we can take $\mathcal{U}$ to be the union of all the principal $\mathrm{Ad}(H)$-orbits in $\mathfrak{m}$. By the slice theorem for a linear group action [20,30], $\mathcal{U}$ is a conic open dense subset of $\mathfrak{m}$. For any $u \in \mathcal{U}$, $\mathrm{Ad}(H)$ acts trivially on the $\langle\cdot,\cdot\rangle_{\rm bi}$-orthogonal complement of $T_u(\mathrm{Ad}(H)u) = [\mathfrak{h}, u]$ in $\mathfrak{m}$, i.e., $\boldsymbol{V}_{u,1} = 0$. So Theorem 4.2 provides $[\mathfrak{m}, u]_{\mathfrak{m}} \subset [\mathfrak{h}, u]$.

Assume $[\mathfrak{m}, u]_{\mathfrak{m}} \subset [\mathfrak{h}, u]$ for each u in a conic dense open subset $\mathcal{U} \subset \mathfrak{m}\backslash\{0\}$. Then for any G-invariant Finsler metric F on G/H, we have (2.1), i.e., $\langle u, [v, u]_{\mathfrak{m}}\rangle_u^F = 0$, $\forall\, v \in \mathfrak{m}$. By the continuity, (2.1) is satisfied for all $u \in \mathfrak{m}\backslash\{0\}$. Since F is arbitrarily chosen, we see that G/H is Finsler equigeodesic. □

The criterion Lemma 4.9 reveals an interesting phenomenon for the Finsler equigeodesic property, i.e., it is only relevant to Lie algebras. So we have the following immediate consequences.

Lemma 4.10. *A homogeneous manifold $G/H = (G_1 \times G_2)/(H_1 \times H_2) = G_1/H_1 \times G_2/H_2$ with a compact G is Finsler equigeodesic if and only if each G_i/H_i is Finsler equigeodesic.*

Proof. The proof repeatedly uses Lemma 4.9. Assume that G/H is Finsler equigeodesic, which provides $\mathcal{U} \subset \mathfrak{m}\backslash\{0\}$. Denote by $\mathrm{pr}_i : \mathfrak{m} \to \mathfrak{m}_i$ the linear projection according to $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$. Then we can take $\mathcal{U}_i = \mathrm{pr}_i(\mathcal{U}\backslash(\mathfrak{m}_1 \cup \mathfrak{m}_2))$ for G_i/H_i. Assume that each G_i/H_i is Finsler equigeodesic, providing $\mathcal{U}_i \subset \mathfrak{m}_i\backslash\{0\}$. Then we can choose $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ for G/H. □

Lemma 4.11. *Let G/H be a homogeneous manifold with a compact semisimple G. Then the following statements are equivalent*:

(1) *G/H is Finsler equigeodesic*;

(2) *$G_0/(G_0 \cap H)$ is Finsler equigeodesic, in which G_0 is the identity component of G*;

(3) *G/H_0 is Finsler equigeodesic, in which H_0 is the identity component of H*;

(4) *$\widetilde{G}_0/\widetilde{H}_0$ is Finsler equigeodesic, in which $\widetilde{G}_0$ is the universal cover of G_0 and $\widetilde{H}_0$ is the connected subgroup in $\widetilde{G}_0$ covering H_0.*

Proof. Since G is compact, G_0 and $\widetilde{G}_0$ are also compact, i.e., Lemma 4.9 is applicable for each in (1)–(4). The homogeneous manifolds in (1)–(4) of Lemma 4.11 share the same $\mathfrak{g}$ and $\mathfrak{h}$, so they also share the same statement (4.4) in Lemma 4.9. □

4.4 The proof of Theorem B

Now we prove Theorem B by the following steps, in which the details for Step 3 are postponed to Section 5.

Step 1. We can use Lemma 4.11 to replace G/H by $\widetilde{G}_0/\widetilde{H}_0$, i.e., we may assume that G is compact, connected and simply connected, and H is connected.

Step 2. The Finsler equigeodesic space G/H is also Riemannian equigeodesic, so Theorem 3.5 can be applied to decomposing G/H as $G/H = G_1/H_1 \times \cdots \times G_m/H_m$, in which each G_i/H_i is strongly isotropy irreducible. By Lemma 4.10, G/H can be replaced by each G_i/H_i, i.e., we may further assume that G/H is a compact strongly isotropy irreducible space on which the compact semisimple G acts almost effectively.

Step 3. In Section 5, we classify the strongly isotropy irreducible compact Finsler equigeodesic space G/H in the Lie algebra level (see Theorem 5.1), i.e., locally, G/H must be one of the following:

$$\mathrm{Spin}(7)/G_2, \quad \text{or } G_2/SU(3), \quad \text{or a symmetric space of compact type.} \tag{4.5}$$

To summarize, the above steps provide a local decomposition $G/H = G_1/H_1 \times \cdots \times G_m/H_m$, in which each G_i/H_i satisfies (4.5).

Step 4. We prove that if a homogeneous manifold G/H with a compact semisimple G has a local decomposition $G/H = G_1/H_1 \times \cdots \times G_m/H_m$ in which each G_i/H_i satisfies (4.5), then G/H is Finsler equigeodesic. Obviously, $S^7 = \mathrm{Spin}(7)/G_2$ and $S^6 = G_2/SU(3)$ are Finsler equigeodesic because the invariant Finsler metric on each of them is unique up to a scalar, i.e., a Riemannian metric with positive constant curvature. In Example 4.3, we see that a symmetric space of compact type is Finsler equigeodesic. Using Lemmas 4.10 and 4.11, we see that G/H is Finsler equigeodesic.

This ends the proof of Theorem B.

5 The strongly isotropy irreducible compact Finsler equigeodesic space

The goal of this section is to prove the following classification result.

Theorem 5.1. *Let G/H be a Finsler equigeodesic space on which the compact connected semisimple G acts almost effectively with a strongly irreducible isotropy representation. Then the pair $(\mathfrak{g}, \mathfrak{h})$ is $(so(7), G_2)$, $(G_2, su(3))$, or a symmetric pair.*

To prove this theorem, we need two preparations. Firstly, the numerical properties of a Finsler equigeodesic space in the following lemma are crucial for the later case-by-case discussion.

Lemma 5.2. *Let G/H be a homogeneous manifold with a compact connected semisimple $(G, \langle\cdot,\cdot\rangle_{\mathrm{bi}})$ and the corresponding orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Suppose that G/H is Finsler equigeodesic. Then we have the following*:

(1) $\dim \mathfrak{c}_{\mathfrak{g}}(u) \geqslant \dim \mathfrak{g} - 2\dim \mathfrak{h}$ *for any each* $u \in \mathfrak{m}$;

(2) $\dim \mathrm{Ad}(G)u + \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \geqslant \dim \mathfrak{m}$, *in which* $u \in \mathfrak{m}$ *satisfies*

$$\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) = \max_{v\in\mathfrak{m}} \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(v));$$

(3) $2\dim \mathfrak{h} + \mathrm{rk}\mathfrak{g} > \dim \mathfrak{m}$.

Proof. (1) Lemma 4.9 provides a conic dense open subset $\mathcal{U} \subset \mathfrak{m}\backslash\{0\}$. Let u be any vector in $\mathcal{U}$. Then $[\mathfrak{m}, u]_{\mathfrak{m}} \subset [\mathfrak{h}, u]$.

On the one hand, we claim that the image of $\mathrm{pr}_{\mathfrak{m}}|_{\mathfrak{c}_{\mathfrak{g}}(u)} : \mathfrak{c}_{\mathfrak{g}}(u) \to \mathfrak{m}$ is the kernel of the linear map $l(\cdot) = [\cdot, u]_{\mathfrak{h}}|_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{h}$.

For any $w' \in \mathfrak{h}$ and $w \in \mathfrak{m}$, we have

$$[w' + w, u] = [w', u] + [w, u]_{\mathfrak{m}} + [w, u]_{\mathfrak{h}},$$

where the first two summands on the right-hand side are contained in $\mathfrak{m}$ and the third is contained in $\mathfrak{h}$. So $[w' + w, u] = 0$ implies $[w, u]_{\mathfrak{h}} = 0$, i.e., $w \in \ker l$.

For any $w \in \ker l$, the property of u provides a vector $w' \in \mathfrak{h}$ such that $[w, u]_{\mathfrak{m}} = -[w', u]$. Then $v = w' + w$ satisfies

$$[v, u] = [w', u] + [w, u]_{\mathfrak{h}} + [w, u]_{\mathfrak{m}} = -[w, u]_{\mathfrak{m}} + [w, u]_{\mathfrak{m}} = 0,$$

i.e., there exists a vector $v \in \mathfrak{c}_{\mathfrak{g}}(u)$ with $v_{\mathfrak{m}} = w$. This ends the proof of our claim.

On the other hand, we see the obvious fact that the kernel of $\mathrm{pr}_{\mathfrak{m}}|_{\mathfrak{c}_{\mathfrak{g}}(u)}$ is $\mathfrak{c}_{\mathfrak{h}}(u)$.

Summarizing the above two observations, we get

$$\dim \mathfrak{c}_{\mathfrak{g}}(u) = \dim \mathfrak{c}_{\mathfrak{h}}(u) + \dim \ker l, \tag{5.1}$$

which implies

$$\dim \mathfrak{c}_{\mathfrak{g}}(u) \geqslant \dim \ker l \geqslant \dim \mathfrak{m} - \dim \mathfrak{h} = \dim \mathfrak{g} - 2\dim \mathfrak{h}.$$

This proves (1) for $u \in \mathcal{U}$.

Notice that $\dim \mathfrak{c}_{\mathfrak{g}}(u)$ depends semicontinuously on $u \in \mathfrak{m}$, i.e.,

$$\dim \mathfrak{c}_{\mathfrak{g}}\Big(\lim_{n\to\infty} u_n\Big) \geqslant \overline{\lim}_{n\to\infty} \dim \mathfrak{c}_{\mathfrak{g}}(u_n).$$

This semicontinuity implies that (1) is valid on $\overline{\mathcal{U}} = \mathfrak{m}$.

(2) Suppose that $\max_{v\in\mathfrak{m}} \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(v))$ is achieved at $u \in \mathfrak{m}$. Denote by $\mathfrak{t}$ any Cartan subalgebra containing u, and by $\mathfrak{g} = S_1 \coprod \cdots \coprod S_N$ the orbit type stratification for the $\mathrm{Ad}(G)$-action. We assume that S_1 contains u. Then the quotient map $\pi : S_1 \to S_1/G$ is a smooth fiber bundle satisfying the following:

(1) each fiber is an $\mathrm{Ad}(G)$-orbit with same orbit type as $\mathrm{Ad}(G)u$;

(2) locally around u, $S_1 \cap \mathfrak{t}$ is the section for this fiber bundle and it is a linear subspace of dimension $\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u))$, which is the intersection of some Weyl walls.

On the other hand, by the classification of flag manifolds (see [1] or Subsection 2.5), for any $w \in \mathfrak{g}\backslash S_1$ which is sufficiently close to u, we have

$$\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(w)) > \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)).$$

So the assumption $\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) = \max_{v\in\mathfrak{m}} \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(v))$ implies that there exists a neighborhood of u in $\mathfrak{m}$ which is contained in S_1.

To summarize, we have

$$\dim S_1 = \dim \mathrm{Ad}(G)u + \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \geqslant \dim \mathfrak{m},$$

which proves (2).

(3) For the vector $u \in \mathfrak{m}$ provided by (2), we have

$$\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \leqslant \dim \mathfrak{t} = \mathrm{rk}\mathfrak{g} \tag{5.2}$$

which is obvious, and

$$\dim(\mathrm{Ad}(g)u) = \dim \mathfrak{g} - \dim \mathfrak{c}_{\mathfrak{g}}(u) \leqslant 2\dim \mathfrak{h} \tag{5.3}$$

by Lemma 5.2(1). Inputting (5.2) and (5.3) into the equality in Lemma 5.2(2), we get

$$2\dim \mathfrak{h} + \mathrm{rk}\mathfrak{g} \geqslant \dim \mathfrak{m}. \tag{5.4}$$

So to prove (3), we only need to verify that the equality in (5.4) cannot happen.

Assume conversely $2\dim \mathfrak{h} + \mathrm{rk}\mathfrak{g} = \dim \mathfrak{m}$. Then u is a regular vector, i.e.,

$$\dim \mathfrak{c}_{\mathfrak{g}}(u) = \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) = \mathrm{rk}\mathfrak{g}. \tag{5.5}$$

On the other hand, Lemma 5.2(1) indicates

$$\dim \mathfrak{c}_{\mathfrak{g}}(u) \geqslant \dim \mathfrak{g} - 2\dim \mathfrak{h} = \dim \mathfrak{h} + \mathrm{rk}\mathfrak{g}. \tag{5.6}$$

Comparing (5.5) and (5.6), we get

$$\dim \mathfrak{h} = 0 \quad \text{and} \quad \mathrm{rk}\mathfrak{g} = \dim \mathfrak{g},$$

i.e., $\mathfrak{g}$ is Abelian. This leads to a contradiction. □

Secondly, we need the classification list in [40] for a nonsymmetric strongly isotropy irreducible G/H on which a compact connected G acts almost effectively. We list their Lie algebra pairs in Table 1.

Remark 5.3. The pair $(so(4n), sp(1) \oplus sp(n))$ in Table 1 is a symmetric pair when $n = 2$. Using a graph automorphism for $so(8)$, we can change it to the standard symmetric pair $(so(8), so(3) \oplus so(5))$ for a real Grassmannian.

Proof of Theorem 5.1. Let G/H be a nonsymmetric strongly isotropy irreducible homogeneous manifold on which the compact connected semisimple G acts almost effectively. Then $(\mathfrak{g}, \mathfrak{h})$ is listed in Table 1.

We check each case in Table 1 and see that Lemma 5.2(3) is only satisfied when $(\mathfrak{g}, \mathfrak{h}) = (so(7), G_2)$, $(\mathfrak{g}, \mathfrak{h}) = (G_2, su(3))$ or $(\mathfrak{g}, \mathfrak{h})$ is in Table 2.

So to prove Theorem 5.1, we only need to conversely assume that G/H is each one in Table 2, and check case by case for contradictions. In the upcoming case-by-case discussion, we apply the following conventions in [45]. We choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $\mathfrak{t} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$. Using $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$, we see that the root systems $\Delta_{\mathfrak{g}}$ for $\mathfrak{g}$ and $\Delta_{\mathfrak{h}}$ for $\mathfrak{h}$ are viewed as subsets in $\mathfrak{t}$ and $\mathfrak{t} \cap \mathfrak{h}$, respectively. The inner product $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ and the orthonormal basis $\{e_1, \ldots, e_n\}$ are suitably chosen such that $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{h}}$ can be canonically presented. All the roots and all the root planes are with respect to $\mathfrak{t}$ or its intersection with the specified subalgebras.

Case 1. $(\mathfrak{g}, \mathfrak{h}) = (su(6), su(2) \oplus su(3))$.

In this case, we have the root system

$$\Delta_{\mathfrak{g}} = \{\pm(e_i - e_j), \forall\, 1 \leqslant i < j \leqslant 6\},$$

and $\mathfrak{t} \cap \mathfrak{h}$ is spanned by $e_1+e_3+e_5-e_2-e_4-e_6$ from the $su(2)$-summand and $e_1+e_2-e_3-e_4$, $e_3+e_4-e_5-e_6$ from the $su(3)$-summand. So $\mathfrak{t}\cap\mathfrak{m}$ consists of all the vectors of the form $ae_1 - ae_2 + be_3 - be_4 + ce_5 - ce_6$, $\forall\, a, b, c \in \mathbb{R}$ with $a+b+c = 0$. A generic vector in $\mathfrak{t}\cap\mathfrak{m}$, for example, $u = e_1 - e_2 + 2e_3 - 2e_4 - 3e_5 + 3e_6$, is a regular vector in $\mathfrak{g}$. Then $\dim \mathfrak{c}_{\mathfrak{g}}(u) = 5$ does not satisfy Lemma 5.2(1), which leads to a contradiction.

Case 2. $(\mathfrak{g}, \mathfrak{h}) = (sp(2), so(3))$.

Table 1 Lie algebra pairs for compact nonsymmetric strongly isotropy irreducible spaces

$\mathfrak{g}$	$\dim \mathfrak{g}$	$\mathfrak{h}$	$\dim \mathfrak{h}$	Conditions
$su(pq)$	p^2q^2-1	$su(p) \oplus su(q)$	p^2+q^2-2	$p \geqslant q \geqslant 2$, $pq > 4$
$su(16)$	255	$so(10)$	45	
$su(27)$	728	E_6	78	
$su(\frac{n(n-1)}{2})$	$\frac{n^2(n-1)^2}{4}-1$	$su(n)$	n^2-1	$n \geqslant 5$
$su(\frac{n(n+1)}{2})$	$\frac{n^2(n+1)^2}{4}-1$	$su(n)$	n^2-1	$n \geqslant 3$
$sp(2)$	10	$so(3)$	3	
$sp(7)$	105	$sp(3)$	21	
$sp(10)$	210	$su(6)$	35	
$sp(16)$	528	$so(12)$	66	
$sp(28)$	1,596	E_7	133	
$so(20)$	190	$su(4)$	15	
$so(70)$	2,415	$su(8)$	63	
$so(n^2-1)$	$\frac{(n^2-1)(n^2-2)}{2}$	$su(n)$	n^2-1	$n \geqslant 3$
$so(16)$	120	$so(9)$	36	
$so(2n^2+n)$	$\frac{(2n^2+n)(2n^2+n-1)}{2}$	$so(2n+1)$	$n(2n+1)$	$n \geqslant 2$
$so(2n^2+3n)$	$\frac{(2n^2+3n)(2n^2+3n-1)}{2}$	$so(2n+1)$	$n(2n+1)$	$n \geqslant 2$
$so(42)$	861	$sp(4)$	36	
$so(2n^2-n-1)$	$\frac{(2n^2-n-1)(2n^2-n-2)}{2}$	$sp(n)$	$2n^2+n$	$n \geqslant 3$
$so(2n^2+n)$	$\frac{(2n^2+n)(2n^2+n-1)}{2}$	$sp(n)$	$2n^2+n$	$n \geqslant 3$
$so(128)$	8,128	$so(16)$	120	
$so(2n^2-n)$	$\frac{(2n^2-n)(2n^2-n-1)}{2}$	$so(2n)$	$n(2n-1)$	$n \geqslant 4$
$so(2n^2+n-1)$	$\frac{(2n^2+n-1)(2n^2+n-2)}{2}$	$so(2n)$	$n(2n-1)$	$n \geqslant 4$
$so(7)$	21	G_2	14	
$so(14)$	91	G_2	14	
$so(26)$	325	F_4	52	
$so(52)$	1,326	F_4	52	
$so(78)$	3,003	E_6	78	
$so(133)$	8,778	E_7	133	
$so(248)$	30,628	E_8	248	
G_2	14	$so(3)$	3	
G_2	14	$su(3)$	8	
F_4	52	$so(3) \oplus G_2$	17	
F_4	52	$su(3) \oplus su(3)$	18	
E_6	78	$su(3)$	8	
E_6	78	G_2	14	
E_6	78	$su(3) \oplus G_2$	22	
E_6	78	$su(3) \oplus su(3) \oplus su(3)$	24	
E_7	133	$su(3)$	8	
E_7	133	$sp(3) \oplus G_2$	35	
E_7	133	$su(2) \oplus F_4$	55	
E_7	133	$su(3) \oplus su(6)$	43	
E_8	248	$G_2 \oplus F_4$	66	
E_8	248	$su(9)$	80	
E_8	248	$su(3) \oplus E_6$	86	
$sp(n)$	$2n^2+n$	$sp(1) \oplus so(n)$	$\frac{n(n-1)}{2}+3$	$n \geqslant 3$
$so(4n)$	$2n(4n-1)$	$sp(1) \oplus sp(n)$	$2n^2+n+3$	$n \geqslant 3$

Table 2 Lie algebra pairs in Table 1 which satisfy Lemma 5.2(3)

No.	$\mathfrak{g}$	$\dim \mathfrak{g}$	$\mathfrak{h}$	$\dim \mathfrak{h}$
1	$su(6)$	35	$su(2) \oplus su(3)$	11
2	$sp(2)$	10	$so(3)$	3
3	F_4	52	$su(2) \oplus G_2$	17
4	F_4	52	$su(3) \oplus su(3)$	18
5	E_7	133	$so(3) \oplus F_4$	55
6	E_7	133	$su(3) \oplus su(6)$	43
7	E_8	248	$su(3) \oplus E_6$	86
8	$so(12)$	66	$sp(1) \oplus sp(3)$	24
9	$so(16)$	120	$sp(1) \oplus sp(4)$	39

This G/H is in fact the Berger space $Sp(2)/SU(2)$ in the classification for positively curved homogeneous manifolds [8, 13]. We have the root system $\Delta_{\mathfrak{g}} = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$, and $\mathfrak{t} \cap \mathfrak{m}$ is spanned by $u = (e_1 + e_2) - (-e_1) = 2e_1 + e_2$. This u is a regular vector in $\mathfrak{g}$. So $\dim \mathfrak{c}_{\mathfrak{g}}(u) = 2$ does not satisfy Lemma 5.2(1), which leads to a contradiction.

Case 3 or Case 4. $(\mathfrak{g}, \mathfrak{h}) = (F_4, su(2) \oplus G_2)$ or $(F_4, su(3) \oplus su(3))$.

By Lemma 5.2(1), $\dim \mathfrak{c}_{\mathfrak{g}}(u) \geqslant 18$ or $\dim \mathfrak{c}_{\mathfrak{g}}(u) \geqslant 16$. By the classification for flag manifolds (see [1] or Subsection 2.5, same below), we see $\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) = 1$ for any $u \in \mathfrak{m} \backslash \{0\}$ with $\mathrm{Ad}(G)u = F_4/\mathrm{Spin}(7)U(1)$ or $\mathrm{Ad}(G)u = F_4/Sp(3)U(1)$. Then by Lemma 5.2(2),

$$31 = \dim \mathrm{Ad}(G)u + \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \geqslant \dim \mathfrak{m} = 35$$

or

$$31 = \dim \mathrm{Ad}(G)u + \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \geqslant \dim \mathfrak{m} = 36$$

provides a contradiction.

Case 5. $(\mathfrak{g}, \mathfrak{h}) = (E_7, F_4 \oplus su(2))$.

Let u' be a generic vector in $\mathfrak{t} \cap \mathfrak{m}$. Then $\mathfrak{c}_{\mathfrak{g}}(u') = so(8) \oplus \mathbb{R}^3$. This observation needs more explanation, which is put in Appendix A.

We can find a vector $u \in \mathfrak{m}$ such that u is sufficiently close to u' and $\mathrm{Ad}(H)u$ is a principal orbit. Then u is contained in a conic open dense subset $\mathcal{U}$ indicated by Lemma 4.9. The centralizer $\mathfrak{c}_{\mathfrak{g}}(u)$ must be isomorphic to $\mathfrak{c}_{\mathfrak{g}}(u')$, because otherwise by the classification for flag manifolds, we have $\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \geqslant 4$ and $\dim \mathfrak{c}_{\mathfrak{g}}(u) \leqslant 19$, which contradicts Lemma 5.2(1).

As indicated by the table in [40, Theorem 11.1], the $\mathrm{Ad}(H)$-action on $\mathfrak{m}$ is the tensor product between the natural $SO(3)$-action on $\mathbb{R}^3$ and the isotropy representation for the symmetric space E_6/F_4. So the $\mathrm{Ad}(H)$-action on $\mathfrak{m}$ is faithful. On the other hand, it is not in [20, Table B], i.e., any principal $\mathrm{Ad}(H)$-orbit in $\mathfrak{m}$ has the same dimension as H. So we have $\mathfrak{c}_{\mathfrak{h}}(u) = 0$.

Now we consider the linear map $l(\cdot) = [u, \cdot]_{\mathfrak{h}}|_{\mathfrak{m}}$ from $\mathfrak{m}$ to $\mathfrak{h}$, which appears in the proof of Lemma 5.2. For any $w \in \mathfrak{m}$, we have $\langle [u, w], \mathfrak{h} \rangle_{\mathrm{bi}} = \langle w, [\mathfrak{h}, u] \rangle_{\mathrm{bi}}$, so $\ker l$ is the $\langle \cdot, \cdot \rangle_{\mathrm{bi}}$-orthogonal complement of $[\mathfrak{h}, u]$ in $\mathfrak{m}$. Because $\mathfrak{c}_{\mathfrak{h}}(u) = 0$, $\dim \ker l = \dim \mathfrak{m} - \dim \mathfrak{h} = 23$.

Finally, (5.1) can be applied to this $u \in \mathcal{U}$, which provides $\dim \mathfrak{c}_{\mathfrak{g}}(u) = 23$. This contradicts the previous observation $\mathfrak{c}_{\mathfrak{g}}(u) = so(8) \oplus \mathbb{R}^3$.

Case 6. $(\mathfrak{g}, \mathfrak{h}) = (E_7, su(3) \oplus su(6))$.

By Lemma 5.2(1), we have

$$\dim \mathfrak{c}_{\mathfrak{g}}(u) \geqslant 47 \quad \text{and} \quad \dim \mathrm{Ad}(G)u \leqslant 86 \tag{5.7}$$

for any $u \in \mathfrak{m}$. By the classification for flag manifolds, if $\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \geqslant 3$, $\dim \mathfrak{c}_{\mathfrak{g}}(u) \leqslant 31$ with the equality achieved when $\mathrm{Ad}(G)u = E_6/\mathrm{Spin}(8)T^3$. So we have $\max_{v \in \mathfrak{m}} \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(v)) \leqslant 2$, and by Lemma 5.2(2) and the second inequality in (5.7), for a generic $u \in \mathfrak{m}$,

$$88 \geqslant \dim \mathrm{Ad}(G)u + \max_{v\in\mathfrak{m}} \dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(v)) \geqslant \mathfrak{m} = 90.$$

This is a contradiction.

Case 7. $(\mathfrak{g},\mathfrak{h}) = (E_8, su(3)\oplus E_6)$.

On the one hand, by Lemma 5.2(1), we have $\dim \mathfrak{c}_{\mathfrak{g}}(u) \geqslant 76$ for each $u \in \mathfrak{m}$. By the classification of flag manifolds, we must have

$$\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \leqslant 2, \quad \forall\, u \in \mathfrak{m}, \tag{5.8}$$

and otherwise $\dim \mathfrak{c}_{\mathfrak{g}}(u) \leqslant 48$ with the equality achieved when $\mathrm{Ad}(G)u = E_8/\mathrm{Spin}(10)T^3$.

On the other hand, there is a $\mathfrak{k} = E_7$ in $\mathfrak{g}$ which contains the E_6-summand $\mathfrak{h}_1$ in $\mathfrak{h}$ and intersects the $su(3)$-summand $\mathfrak{h}_2$ in $\mathfrak{h}$ at a line. The pair $(\mathfrak{k}, \mathfrak{k}\cap\mathfrak{h}) = (E_7, E_6\oplus\mathbb{R})$ is a symmetric pair. Denote by $\mathfrak{m}'$ the $\langle\cdot,\cdot\rangle_{\mathrm{bi}}$-complement of $\mathfrak{k}\cap\mathfrak{h}$ in $\mathfrak{k}$. Then $\mathfrak{k} = (\mathfrak{k}\cap\mathfrak{h}) + \mathfrak{m}'$ is a Cartan decomposition. The rank of $E_7/E_6U(1)$ is 3 (see [19]), i.e., we have found a 3-dimensional commutative subspace $\mathfrak{t}'$ in $\mathfrak{m}'$, from which we can find a vector u with $\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \geqslant 3$. Since $\mathrm{rk}\mathfrak{g} = \mathrm{rk}\mathfrak{h}$ and $\mathfrak{k} = E_7$ is regular in $\mathfrak{g} = E_8$ (i.e., each root plane of $\mathfrak{k}$ is a root plane of $\mathfrak{g}$), $\mathfrak{m}'$ and $\mathfrak{m}$ are both sums of root planes of $\mathfrak{g}$, so we have $\mathfrak{m}' \subset \mathfrak{m}$. The previously mentioned $u \in \mathfrak{m}' \subset \mathfrak{m}$ satisfies $\dim \mathfrak{c}(\mathfrak{c}_{\mathfrak{g}}(u)) \geqslant 3$. This contradicts (5.8).

Case 8. $(\mathfrak{g},\mathfrak{h}) = (so(12), sp(1)\oplus sp(3))$.

In this case, the root system $\Delta_{\mathfrak{g}} = \{\pm e_i \pm e_j, \forall\, 1 \leqslant i < j \leqslant 6\}$, and the subspace $\mathfrak{t}\cap\mathfrak{h}$ is linearly spanned by $e_1+\cdots+e_6$ from the $sp(1)$-summand and e_1-e_2, e_3-e_4, e_5-e_6 from the $sp(3)$-summand. So $\mathfrak{t}\cap\mathfrak{m}$ consists of all the vectors of the form $ae_1+ae_2+be_3+be_4+ce_5+ce_6$, $\forall\, a,b,c\in\mathbb{R}$ with $a+b+c=0$.

For the vector $u = e_1+e_2+2e_3+2e_4-3e_5-3e_6 \in \mathfrak{t}\cap\mathfrak{m}$, the dimension of

$$\mathfrak{c}_{\mathfrak{g}}(u) = su(2)\oplus su(2)\oplus su(2)\oplus\mathbb{R}^3$$

is 12. This contradicts Lemma 5.2(1).

Case 9. $(\mathfrak{g},\mathfrak{h}) = (so(16), sp(1)\oplus sp(4))$.

In this case, the root system $\Delta_{\mathfrak{g}} = \{\pm e_i \pm e_j, \forall\, 1 \leqslant i < j \leqslant 8\}$, and the subspace $\mathfrak{t}\cap\mathfrak{h}$ is linearly spanned by $e_1+\cdots+e_8$ from the $sp(1)$-summand and e_1-e_2, e_3-e_4, e_5-e_6, e_7-e_8 from the $sp(4)$-summand. So $\mathfrak{t}\cap\mathfrak{m}$ consists of all the vectors of the form $ae_1+ae_2+be_3+be_4+ce_5+ce_6+de_7+de_8$, $\forall\, a,b,c,d\in\mathbb{R}$ with $a+b+c+d=0$.

For the vector $u = e_1+e_2+2e_3+2e_4+3e_5+3e_6-6e_7-6e_8 \in \mathfrak{t}\cap\mathfrak{m}$, the dimension of

$$\mathfrak{c}_{\mathfrak{g}}(u) = su(2)\oplus su(2)\oplus su(2)\oplus su(2)\oplus\mathbb{R}^4$$

is 16. This contradicts Lemma 5.2(1).

This ends the proof of Theorem 5.1. □

6 The homogeneous manifold on which all the invariant metrics are Berwald

In this section, we prove Theorem C, which classifies the homogeneous manifold G/H with a compact semisimple G, on which all the G-invariant metrics are Berwald. It is an immediate corollary of Theorem B and the following theorem.

Theorem 6.1. *Let G/H be a homogeneous manifold with a compact $(G,\langle\cdot,\cdot\rangle_{\mathrm{bi}})$. Then G/H is a Finsler equigeodesic space if and only if each G-invariant Finsler metric on G/H is Berwald.*

Proof. Firstly, we assume that G/H is Finsler equigeodesic and prove each G-invariant Finsler metric F on G/H is Berwald. By Lemma 4.8, F is naturally reductive with respect to the orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h}+\mathfrak{m}$. It has a vanishing spray vector field, so by Lemma 2.3, it is Berwald. This proves one side of Theorem 6.1.

Next, we assume that each G-invariant Finsler metric on G/H is Berwald and prove that G/H is Finsler equigeodesic. Denote by

$$S=\{u \mid u\in\mathfrak{m},\, |u|_{\mathrm{bi}}=\langle u,u\rangle_{\mathrm{bi}}^{1/2}=1\}$$

the unit sphere in $\mathfrak{m}$. By Lemma 4.7, we just need to prove that each $u\in S$ is a Finsler equigeodesic vector. We may assume $\mathrm{Ad}(H)u\neq S$, and otherwise G/H is a compact rank-one Riemannian symmetric space [37], which is obviously Finsler equigeodesic.

Let F be any G-invariant Finsler metric on G/H. We use the same $F=F(o,\cdot)$ to denote the $\mathrm{Ad}(H)$-invariant Minkowski norm on $\mathfrak{m}$. We can find two sufficiently small $\mathrm{Ad}(H)$-invariant open neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of the orbit $\mathrm{Ad}(H)u$ in S with $\mathcal{U}_1\subset\mathcal{U}_2$, and an $\mathrm{Ad}(H)$-invariant smooth cut-off function $f:S\to[0,1]$ satisfying $f(\mathcal{U}_1)=1$ and $f(S\backslash\mathcal{U}_2)=0$. Notice that $S\backslash\mathcal{U}_2$ contains a nonempty open subset of S. For $t\in\mathbb{R}$ which is sufficiently close to 0,

$$F_t(y)=\sqrt{\langle y,y\rangle_{\mathrm{bi}}+tf\left(\frac{y}{|y|_{\mathrm{bi}}}\right)F(y)^2}$$

defines a smooth family of $\mathrm{Ad}(H)$-invariant Minkowski norms on $\mathfrak{m}$. We use the same F_t to denote the corresponding G-invariant Finsler metrics on G/H.

Denote by $\langle\cdot,\cdot\rangle_y$ and $\langle\cdot,\cdot\rangle_y^{F_t}$ the fundamental tensors of the Minkowski norms F and F_t, respectively. Let $\eta_t:\mathfrak{m}\backslash\{0\}\to\mathfrak{m}$ be the spray vector field of F_t and $\omega=\frac{d}{dt}|_{t=0}\eta_t$. Then we have the following observations. Notice that $F_0=|\cdot|_{\mathrm{bi}}$, so $\langle\cdot,\cdot\rangle_y^{F_0}=\langle\cdot,\cdot\rangle_{\mathrm{bi}}$ and $\eta_0=0$. For $y\in\mathfrak{m}\backslash\mathbb{R}_{\geqslant0}\mathcal{U}_2$, $\langle\cdot,\cdot\rangle_y^{F_t}=\langle\cdot,\cdot\rangle_{\mathrm{bi}}$, and for $y\in\mathbb{R}_{>0}\mathcal{U}_1$, $\frac{d}{dt}\langle\cdot,\cdot\rangle_y^{F_t}=\langle\cdot,\cdot\rangle_y$. Since each F_t is Berwald, by Lemma 2.3, each η_t is quadratic, and then ω is also quadratic.

The definition of spray vector field provides

$$\langle\eta_t(y),w\rangle_y^{F_t}=\langle y,[w,y]_{\mathfrak{m}}\rangle_y^{F_t},\quad \forall\, y\in\mathfrak{m}\backslash\{0\},\quad w\in\mathfrak{m}. \tag{6.1}$$

By the observations in the previous paragraph, the derivative of (6.1) for the t-variable which is evaluated at $t=0$ can be presented as

$$\langle\omega(y),w\rangle_{\mathrm{bi}}=\frac{d}{dt}\bigg|_{t=0}\langle y,[w,y]_{\mathfrak{m}}\rangle_y^{F_t},\quad \forall\, y\in\mathfrak{m}\backslash\{0\},\quad w\in\mathfrak{m}. \tag{6.2}$$

The left-hand side of (6.2) is quadratic for y, and the right-hand side vanishes for $y\in\mathfrak{m}\backslash\mathbb{R}_{\geqslant0}\mathcal{U}_2$. So both sides of (6.2) vanish for all $y\in\mathfrak{m}\backslash\{0\}$. In particular, for $y=u\in\mathbb{R}_{>0}\mathcal{U}_1$, we have

$$\frac{d}{dt}\bigg|_{t=0}\langle u,[w,u]_{\mathfrak{m}}\rangle_u=\langle u,[w,u]_{\mathfrak{m}}\rangle_u=0,\quad \forall\, u\in\mathfrak{m}.$$

To summarize, $u\in S\subset\mathfrak{m}\backslash\{0\}$ is a geodesic vector for any G-invariant Finsler metric F, so u is a Finsler equigeodesic vector. Since u is arbitrary, G/H is Finsler equigeodesic. This proves the other side of Theorem 6.1. □

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Appendix A Some discussion for the algebraic structure of $(\mathfrak{g}, \mathfrak{h}) = (E_7, su(2) \oplus F_4)$

For simplicity, we denote by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ the $su(2)$- and F_4-summands in $\mathfrak{h}$, respectively.

Notice that $\mathfrak{t} \cap \mathfrak{h}_1$ is a line and its centralizer in $\mathfrak{g} = E_7$ has a semisimple summand $\mathfrak{g}' = E_6$. The subalgebra $\mathfrak{h}_2 = F_4$ is contained in $\mathfrak{g}' = E_6$ such that E_6/F_4 is a symmetric space. We can expand a Cartan subalgebra of $\mathfrak{h}_2 = F_4$ first to $\mathfrak{g}'$ and then to $\mathfrak{g}$, which provides a Cartan subalgebra $\mathfrak{t}$ such that $\mathfrak{t} \cap \mathfrak{h}_1$, $\mathfrak{t} \cap \mathfrak{h}_2$ and $\mathfrak{t} \cap \mathfrak{g}'$ are Cartan subalgebras for $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{g}'$, respectively. The roots and root planes with respect to these specified Cartan subalgebras can be arranged as follows. Using $\langle \cdot, \cdot \rangle_{\mathrm{bi}}$, we see that the roots are viewed as vectors in $\mathfrak{t}$ rather than $\mathfrak{t}^*$.

The root system $\Delta_{\mathfrak{g}}$ of $\mathfrak{g}$ consists of the following roots:

$$\begin{aligned} &\pm e_i \pm e_j, \quad \forall\, 1 \leqslant i < j \leqslant 6, \quad \pm\sqrt{2} e_7, \\ &\pm\frac{1}{2} e_1 \pm \cdots \pm \frac{1}{2} e_6 \pm \frac{\sqrt{2}}{2} e_7 \quad \text{with even } + \frac{1}{2}\text{-coefficients.} \end{aligned}$$

It has a prime root system

$$\begin{aligned} &\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_4 - e_5, \\ &\alpha_5 = e_5 - e_6, \quad \alpha_6 = e_5 + e_6, \quad \alpha_7 = -\frac{1}{2}(e_1 + \cdots + e_6 + \sqrt{2} e_7). \end{aligned}$$

The subset $\{\alpha_2, \ldots, \alpha_7\}$ is the prime root system of $\mathfrak{g}' = E_6$.

The subalgebra $\mathfrak{h}_2 = F_4$ is the fixed point set for the involutive automorphism σ of $\mathfrak{g}'$, which maps each α_i to α_{9-i} for $i = 2, 3, 6, 7$, and fixes α_4 and α_5. So $\mathfrak{t} \cap \mathfrak{h}_2$ is spanned by α_4, α_5, $\alpha_3 + \alpha_6 = e_3 - e_4 + e_5 + e_6$ and $\alpha_2 + \alpha_7 = \frac{1}{2}(-e_1 + e_2 - 3e_3 - e_4 - e_5 - e_6 - \sqrt{2} e_7)$.

The subspace $\mathfrak{t} \cap \mathfrak{h}_1$ in $\mathfrak{h}_1 = su(2)$ commutes with each root plane of $\mathfrak{h}_2 = F_4$, so it commutes with each root plane of $\mathfrak{g}' = E_6$. Then we see that $\mathfrak{t} \cap \mathfrak{h}_1$ is $\langle \cdot, \cdot \rangle_{\mathrm{bi}}$-orthogonal to $\mathfrak{t} \cap \mathfrak{g}'$, i.e., it is spanned by $2e_1 - \sqrt{2} e_7$.

The above description is enough for us to calculate $\mathfrak{t} \cap \mathfrak{m}$, which is linearly spanned by $e_3 - e_4 - e_5 - e_6$ and $e_1 + 3e_2 + \sqrt{2} e_7$.

Let u' be a generic vector in $\mathfrak{t} \cap \mathfrak{m}$. For example, we can choose $u' = 7(e_1 + 3e_2 + \sqrt{2} e_7) + 5(e_3 - e_4 - e_5 - e_6)$. The centralizer $\mathfrak{c}_{\mathfrak{g}}(u')$ has the following roots:

$$\begin{aligned} &\pm(e_3 + e_4),\ \pm(e_3 + e_5),\ \pm(e_3 + e_6),\ \pm(e_4 - e_5),\ \pm(e_4 - e_6),\ \pm(e_5 - e_6), \\ &\pm\frac{1}{2}(e_1 - e_2 + \sqrt{2} e_7) \pm \frac{1}{2}(e_3 - e_4 + e_5 + e_6), \\ &\pm\frac{1}{2}(e_1 - e_2 + \sqrt{2} e_7) \pm \frac{1}{2}(e_3 + e_4 - e_5 + e_6), \\ &\pm\frac{1}{2}(e_1 - e_2 + \sqrt{2} e_7) \pm \frac{1}{2}(e_3 + e_4 + e_5 - e_6), \end{aligned}$$

which provide a root system of $so(8)$. So $\mathfrak{c}_{\mathfrak{g}}(u') = so(8) \oplus \mathbb{R}^3$.

Remark A.1. Another description for $(E_7, su(2) \oplus F_4)$ can be found in [15, Table 35].