SCIENCE CHINA
Mathematics

• ARTICLES •

November 2021 Vol. 64 No. 11: 2381–2394
https://doi.org/10.1007/s11425-020-1789-5

On the first negative Hecke eigenvalue of an automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$

Yuk-Kam Lau[1], Ming Ho Ng[2], Hengcai Tang[3] & Yingnan Wang[4,*]

[1]*Department of Mathematics, The University of Hong Kong, Hong Kong, China;*
[2]*Department of Mathematics, The Chinese University of Hong Kong, Hong Kong, China;*
[3]*Institute of Modern Mathematics, School of Mathematics and Statistics, Henan University, Kaifeng 475004, China;*
[4]*Shenzhen Key Laboratory of Advanced Machine Learning and Applications, College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China*

Email: yklau@maths.hku.hk, minghong515@gmail.com, hctang@henu.edu.cn, ynwang@szu.edu.cn

Received April 27, 2020; accepted September 28, 2020; published online May 13, 2021

Abstract Let π be a self-dual irreducible cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character. Its Hecke eigenvalue $\lambda_\pi(n)$ is a real multiplicative function in n. We show that $\lambda_\pi(n) < 0$ for some $n \ll Q_\pi^{2/5}$, where Q_π denotes (a special value of) the analytic conductor. The value $\frac{2}{5}$ is the first explicit exponent for Hecke-Maass newforms.

Keywords automorphic representation, Hecke eigenvalue, Maass cusp form, sign change

MSC(2020) 11F11, 11F30

Citation: Lau Y-K, Ng M H, Tang H C, et al. On the first negative Hecke eigenvalue of an automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$. Sci China Math, 2021, 64: 2381–2394, https://doi.org/10.1007/s11425-020-1789-5

1 Introduction

In the past decade, there were delicate investigations devoted to the size of the first sign change of Hecke eigenvalues of holomorphic cusp forms (see [8, 14, 15, 21, 34]). Let g be a primitive holomorphic cusp form of weight k for the Hecke congruence group $\Gamma_0(N)$ with trivial nebentypus, and $\lambda_g(n)$ be its n-th Hecke eigenvalue. The eigenvalue $\lambda_g(n)$ is a real multiplicative function in n satisfying the Hecke relation

$$\lambda_g(m)\lambda_g(n) = \sum_{d\,|\,(m,n)} \lambda_g\left(\frac{mn}{d^2}\right) \quad \text{for } (mn, N) = 1. \tag{1.1}$$

Denote by n_g the smallest integer n such that $\lambda_g(n) < 0$ and $(n, N) = 1$. Write $Q_g = k^2N$. By using the convexity bound of $L(s, g)$ and some consequences of the Hecke relation (1.1), it is quite easy (see, for example, [8, Section 3]) to get

$$n_g \ll Q_g^{1/2+\varepsilon}, \tag{1.2}$$

regarded as the trivial bound for n_g. Apparently the bound size in (1.2) will be reduced if one has an improvement over the convexity bound—the subconvexity bound—for $L(1/2 + \mathrm{i}t, g)$.

* Corresponding author

Interestingly, Iwaniec et al. [8] sharpened (1.2) without using any subconvexity bound for $L(1/2 + \mathrm{i}t, g)$ but instead invoked some sieve techniques. Their result was refined by Kowalski et al. [15] and Matomäki [21] subsequently. Matomäki [21] showed that

$$n_g \ll Q_g^{\frac{1}{2}-\frac{1}{8}}, \tag{1.3}$$

which remains the most effective approach to date. One of the ingredients required in this approach is Deligne's bound for $\lambda_g(p)$:

$$|\lambda_g(p)| \leqslant 2 \quad \text{for any prime} \quad p \nmid N \tag{1.4}$$

when constructing the sieve function. In general this bound is not proved, though being anticipated, and known as the (generalized) Ramanujan conjecture. Hence the method in (1.3) is not directly applicable to Hecke-Maass cusp forms.

In this case (of Hecke-Maass cusp forms), one can prove the trivial bound as in (1.2) and there is an improvement due to Qu [25] by adopting a subconvexity bound of the L-function. Qu proved

$$n_f \ll Q_f^{1/2-\delta}, \tag{1.5}$$

where f is a Hecke-Maass newform for the Hecke congruence group $\Gamma_0(N)$ of the spectral parameter ν_f. Here, n_f denotes the smallest integer n such that the Hecke eigenvalue $\lambda_f(n) < 0$ and $(n, N) = 1$, $Q_f = (3 + |\nu_f|^2)N$ and δ is a positive absolute constant. However, the value of δ in (1.5) is unspecified, which comes from the uniform subconvexity bound

$$L\left(\frac{1}{2} + \mathrm{i}t, f\right) \ll Q_f^{\frac{1}{4}-\delta}$$

shown by Michel and Venkatesh [23][1)]. The sieving argument in [21] does not fit because of the absence of Deligne's bound for Maass cusp forms, i.e., (1.4) is not known for Maass cusp forms.

In this paper, we modify the sieve function (see Subsection 4.2) used in [15, 21] to control the possible exceptional Hecke eigenvalues (not fulfilling (1.4)) and work out a result in the more general context—for the Hecke eigenvalues of an automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ (see Section 2). Consequently, we improve Qu's result (1.5) in the direction of an explicit bound exponent, though $\sim 6.7\%$ bigger than the result in (1.3), without using the subconvexity bound for the L-function.

Theorem 1.1. *Let f be a Hecke-Maass primitive form (newform) on $\Gamma_0(N)$. Then there exists a positive integer n satisfying*

$$n \ll Q_f^{\frac{1}{2}-\delta}, \quad (n, N) = 1$$

such that $\lambda_f(n) < 0$, where $\delta = \frac{1}{10}$.

2 Set-up and the main result

Let $d \in \mathbb{N}$ and $\pi = \bigotimes_{p\leqslant\infty} \pi_p$ be an irreducible cuspidal automorphic representation of $GL_d(\mathbb{A}_\mathbb{Q})$ with trivial central character[2)]. The L-function associated with π is a Dirichlet series of the form

$$L(s, \pi) = \sum_{n\geqslant1} \lambda_\pi(n) n^{-s} \quad (\Re e\, s \gg 1)$$

that factors into an Euler product $\prod_{p<\infty} L_p(s, \pi)$ with

$$L_p(s, \pi) = L(s, \pi_p) := \prod_{1\leqslant i\leqslant d} (1 - \alpha_{\pi,i}(p)p^{-s})^{-1} \quad (p < \infty)$$

1) Recently, Wu [33] gave the explicit value $\frac{1-2\theta}{32}$ for δ, where $\theta = \frac{7}{64}$.

2) See [1, p. 92], [3, Definition 3.2] or [7, Definition 5.1.14]; we reserve p for (usually finite) primes/places.

for some complex numbers $\alpha_{\pi,i}(p)$. Hence the reciprocal $L_p(s,\pi)^{-1}$ is a polynomial in p^{-s} of degree at most d with constant term 1. The coefficients $\lambda_\pi(n)$'s ($n \in \mathbb{N}$) are the Hecke eigenvalues. The L-function $L(s,\pi)$ accounts for the finite part and is completed with the local factor $L_\infty(s,\pi)$. At $p=\infty$, the local representation π_∞ is given as a Langlands quotient and the local factor $L_\infty(s,\pi)$ is expressible as a product of gamma factors (see [30, Appendix A.3]),

$$L_\infty(s,\pi) = L(s,\pi_\infty) = \prod_{1\leqslant i\leqslant d} \Gamma_{\mathbb{R}}(s+\mu_{\pi,i})$$

for some $\mu_{\pi,i} \in \mathbb{C}$, where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$. Readers interested in (a broader class of) these L-functions are referred to [5] for a recent comprehensive account.

The completed L-function $L_\infty(s,\pi)L(s,\pi)$ extends to an entire function which is of finite order and bounded in vertical strips, and it satisfies the functional equation

$$q_\pi^{s/2} L_\infty(s,\pi)L(s,\pi) = w(\pi) q_\pi^{(1-s)/2} L_\infty(1-s,\widetilde{\pi})L(1-s,\widetilde{\pi}), \tag{2.1}$$

where $\widetilde{\pi}$ is the contragredient of π, $w(\pi) \in S^1$ is the root number and $q_\pi \geqslant 1$ is an integer, called the arithmetic conductor of π. Note that π is ramified at every prime p dividing q_π and unramified at any other finite primes, $q_{\widetilde{\pi}} = q_\pi$ and $L_p(s,\widetilde{\pi}) = \overline{L_p(\overline{s},\pi)}$ for all $p \leqslant \infty$ (hence $\{\alpha_{\widetilde{\pi},i}(p) : 1 \leqslant i \leqslant d\} = \{\overline{\alpha_{\pi,i}(p)} : 1 \leqslant i \leqslant d\}$ as multi-sets). If π is self-dual (i.e., $\pi \simeq \widetilde{\pi}$), then $\lambda_\pi(n)$'s are real. Define the analytic conductor $Q_\pi(t)$ of π by

$$Q_\pi(t) := q_\pi \prod_{1\leqslant i\leqslant d} (1 + |\mathrm{i}t + \mu_{\pi,i}|)$$

and set $Q_\pi := Q_\pi(0)$[3].

We confine to the case where $d=2$. The classification for π_∞ is known, and for the irreducible cuspidal automorphic representation π with trivial central character, we have[4]

- discrete series: $\{\mu_{\pi,1}, \mu_{\pi,2}\} = \{m - \frac{1}{2}, m + \frac{1}{2}\}$ for some $m \in \mathbb{N}$;
- unitary principal series: $\{\mu_{\pi,1}, \mu_{\pi,2}\} = \{\nu + \delta, -\nu + \delta\}$, where $\nu \in \mathrm{i}\mathbb{R}$ and $\delta \in \{0,1\}$;
- complementary series: $\{\mu_{\pi,1}, \mu_{\pi,2}\} = \{\nu + \delta, -\nu + \delta\}$, where $0 < \nu < \frac{1}{2}$ and $\delta \in \{0,1\}$.

At finite prime p, an infinite-dimensional irreducible admissible representation of $GL_2(\mathbb{Q}_p)$ must be either principal series, special or supercuspidal[5]. Since the central character is trivial, we infer from the known classification (see [20, p. 1182]) that

- principal series: $\prod_{i=1,2} \alpha_{\pi,i}(p) \in \{0,1\}$;
- special: $\alpha_{\pi,1}(p) \in \{\pm p^{-1/2}\}$, $\alpha_{\pi,2}(p) = 0$;
- supercuspidal: $\alpha_{\pi,1}(p) = \alpha_{\pi,2}(p) = 0$.

When π_p is unramified, it is a principal series by [7, Proposition 11.7.3] and

$$\prod_{i=1,2} \alpha_{\pi,i}(p) = 1.$$

The representation π is unramified at all but finitely many primes p (dividing q_π for finite p). The generalized Ramanujan conjecture (GRC) asserts that $|\alpha_{\pi,i}(p)| = 1$, which is only settled for the case of discrete series (see [31, (9)]). The current known result towards the GRC is

$$|\alpha_{\pi,i}(p)| \leqslant p^\theta \quad \text{with } \theta = 7/64 \tag{2.2}$$

obtained by Kim and Sarnak [12]. Under the GRC, $\theta = 0$ and $|\lambda_\pi(p)| \leqslant 2$ for $p < \infty$.

3) See [22, Lecture 1].

4) By [7, Theorems 11.14.2 and 11.16.1], π_∞ is $\mathcal{B}_\infty(\chi_1,\chi_2)$ with $\chi_1(x) = |x|^{\nu_1}\mathrm{sgn}(x)^{\delta_1}$ and $\chi_2(x) = |x|^{\nu_2}\mathrm{sgn}(x)^{\delta_2}$ ($\nu_1,\nu_2 \in \mathbb{C}$, $\delta_1,\delta_2 \in \{0,1\}$), or a discrete series representation in $\mathcal{B}_\infty(\chi_1,\chi_2)$ with $\chi_1(x) = |x|^{(\mathrm{m}+1)/2+\mathrm{i}t}\mathrm{sgn}(x)^{\mathrm{m}}$ and $\chi_2(x) = |x|^{-(\mathrm{m}+1)/2+\mathrm{i}t}$ ($\mathrm{m} \in \mathbb{N}_0$). As the central character is trivial, we have $\chi_1\chi_2 = 1$; thus $\nu_2 = -\nu_1$ and $\delta_1 = \delta_2$ in the former case, and $t = 0$ and $\mathrm{m} \in 2\mathbb{N}_0$ in the latter case. (Note $m = (\mathrm{m}-2)/2$.) By [7, Corollary 9.5.10], π is unitarizable. The unitaricity implies that $\nu = \mathrm{i}t$ ($t \in \mathbb{R}$) or $\nu \in (-\frac{1}{2}, \frac{1}{2})$ (see [7, Proposition 9.4.5]).

5) See [7, Theorem 6.16.1].

Theorem 2.1. *Let π be a self-dual irreducible cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character. Write $\lambda_\pi(n)$ for the Hecke eigenvalue and Q_π for the analytic conductor at $t=0$. Then there exists a positive integer n satisfying*

$$n \ll Q_\pi^{\frac{1}{2}-\delta}, \quad (n, q_\pi) = 1$$

such that $\lambda_\pi(n) < 0$, where $\delta = \frac{1}{10}$.

Remark 2.2. We have not used the subconvexity bound of Michel and Venkatesh [23] or Wu [33] to further refine the value of δ.

Specialization. Let $\Gamma_0(N) \subset SL_2(\mathbb{Z})$ be the Hecke congruence subgroup and $\mathbb{H}$ be the upper half-plane. A holomorphic (resp. Hecke-Maass) newform f is a modular form of weight $k \in 2\mathbb{N}$ (resp. a Maass cusp form with the eigenvalue $\lambda = \frac{1}{4} + \nu_f^2 \in \mathbb{C}^\times$) such that f is a common eigenfunction of the family of Hecke operators $\{T_n : (n, N) = 1\}$ and lies in the orthogonal complement of the space of oldforms (see [9, Subsection 14.7]). The multiplicity-one theorem implies that f is a common eigenfunction of all the Hecke operators T_n ($n \in \mathbb{N}$), and thus its associated L-function $L(s, f)$ factors into an Euler product. The analytic conductor of $L(s, f)$ is $\asymp k^2 N$ (resp. $|\lambda| N$) for holomorphic (resp. Hecke-Maass) f. The function f is lifted to a square-integrable function f in $L^2(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}}))$ and associated with a self-dual irreducible cuspidal automorphic representation π_f of $GL_2(\mathbb{A}_{\mathbb{Q}})$ with trivial central character. Their L-functions are identical, i.e., $L(s, f) = L(s, \pi_f)$, and in particular $\lambda_f(n) = \lambda_{\pi_f}(n)$, where $T_n f = \lambda_f(n) f$ for $(n, N) = 1$. Theorem 2.1 implies Theorem 1.1.

3 Preliminaries

This section is devoted to preparing some tools for the proof of Theorem 2.1.

3.1 Number-theoretic tools

We collect some results from Liu and Ye [19], Lau et al. [16] and Matomäki [21].

Lemma 3.1. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be absolutely convergent for $\sigma > \sigma_a$. Let

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma}$$

for $\sigma > \sigma_a$. Then for $b > \sigma_a$, $x \geqslant 2$, $T \geqslant 2$ and $H \geqslant 2$,

$$\sum_{n \leqslant x} a_n = \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}T}^{b+\mathrm{i}T} f(s) \frac{x^s}{s} ds + O\bigg(\sum_{x-\frac{x}{H} < n \leqslant x+\frac{x}{H}} |a_n| \bigg) + O\bigg(\frac{x^b H B(b)}{T} \bigg).$$

This is a modified version of Perron's formula in [19, Theorem 2.1].

Lemma 3.2. *Let $\kappa > 0$ be a constant, $N \in \mathbb{N}$ and write*

$$D_{\kappa,N} := (\log(\omega(N) + 3))^{\mathrm{e}\kappa+2}.$$

There is a positive constant C_κ depending on κ such that uniformly for $x \geqslant \exp(C_\kappa D_{\kappa,N})$,

$$\sum_{\substack{n \leqslant x \\ (n,N)=1}}^{\flat} \kappa^{\omega(n)} = \frac{\Pi_{N,\kappa}}{\Gamma(\kappa)} x (\log x)^{\kappa-1} \bigg\{ 1 + O_\kappa\bigg(\frac{D_{\kappa,N}}{\sqrt{\log x}} \bigg) \bigg\},$$

where $\Gamma(\cdot)$ *is the Gamma function and*

$$\Pi_{N,\kappa} := \left(\frac{\varphi(N)}{N}\right)^{\kappa} \prod_{p \nmid N} \left(1 - \frac{1}{p}\right)^{\kappa} \left(1 + \frac{\kappa}{p}\right) \gg (\log\log N)^{-\kappa}. \tag{3.1}$$

This is [16, Lemma 4.1] (see [15, p. 398] for a more precise result).

Lemma 3.3. *Let* $U \in [1,\infty)$ *and* $M, N \in \mathbb{N}$. *Define* $h_y(n)$ *to be the multiplicative function supported on squarefree integers such that*

$$h_y(p) = \begin{cases} -2, & \text{if } p > y, \\ 2\cos\left(\dfrac{\pi}{m+1}\right), & \text{if } y^{\frac{1}{m+1}} \leqslant p < y^{\frac{1}{m}} \text{ for some integer } m < M, \\ 2\cos\left(\dfrac{\pi}{M+1}\right), & \text{if } p \leqslant y^{\frac{1}{M}}. \end{cases} \tag{3.2}$$

Uniformly for $\exp(C_0(\log\omega(N)+3)^{\mathrm{e}\chi_0+4}) \leqslant y$ *and* $U^{-1} \leqslant u \leqslant U$, *we have*

$$\sum_{\substack{n \leqslant y^u \\ (n,N)=1}}^{\flat} h_y(n) = (\sigma_M(u) + o_{M,U}(1)) \frac{\Pi_{N,\chi_0}}{\Gamma(\chi_0)} (\log y)^{\chi_0 - 1} y^u,$$

where $C_0 = C_0(M,U)$ *is a suitable large constant,* $\sigma_M(u)$ *is a continuous function,* $\chi_0 = 2\cos(\pi/(M+1))$, $\Gamma(\cdot)$ *is the Gamma function and* Π_{N,χ_0} *is defined as in* (3.1).

In particular for $M = 100$, *we have the following*: *the lower bounds for* $\sigma_{100}(u)$ *are* 0.0924, 0.0718, 0.0445 *and* 0.008 *for* $u = \frac{11}{9}$, $u = \frac{5}{4}$, $u = \frac{9}{7}$ *and* $u = \frac{4}{3}$, *respectively.*

This is a special case of [21, Lemmas 6 and 8 and Remark 7]. Note that $\sigma_M(u)$ is the function $\sigma(u)$ given by [21, (2.3)] and the lower bounds of $\sigma_{100}(u)$ in the above table is evaluated by a C++ program provided by Professor Matomäki.

3.2 Tools from Rankin-Selberg *L*-functions

We start with the theory of the Rankin-Selberg L-function discussed in [22] and some important results on symmetric power L-functions due to [6, 11, 13].

Let π and π' be automorphic representations of $GL_d(\mathbb{A}_{\mathbb{Q}})$ and $GL_{d'}(\mathbb{A}_{\mathbb{Q}})$ as in Section 2 (and keep using the notation therein). The Rankin-Selberg L-function $L(s,\pi\times\pi')$ is given by an Euler product

$$L(s,\pi\times\pi') = \prod_{p<\infty} L_p(s,\pi\times\pi'),$$

where $L_p(s,\pi\times\pi')^{-1}$ is a polynomial of degree less than or equal to dd' in p^{-s}. The L-function $L(s,\pi\times\pi')$ extends to either a meromorphic function on $\mathbb{C}$ with two simple poles at $s=0$ and $s=1$ or an entire function according as $\pi' \simeq \widetilde{\pi}$ or not. It satisfies the functional equation

$$q_{\pi\times\pi'}^{s/2} L_\infty(s,\pi\times\pi') L(s,\pi\times\pi') = w(\pi\times\pi') q_{\pi\times\pi'}^{(1-s)/2} L_\infty(1-s,\widetilde{\pi}\times\widetilde{\pi}') L(1-s,\widetilde{\pi}\times\widetilde{\pi}'),$$

where $q_{\pi\times\pi'} \in \mathbb{N}$, $|w(\pi\times\pi')| = 1$ and

$$L_\infty(s,\pi\times\pi') = \prod_{\substack{1\leqslant i\leqslant d \\ 1\leqslant j\leqslant d'}} \Gamma_{\mathbb{R}}(s + \mu_{\pi\times\pi',i,j}).$$

Suppose that π is unramified at p. Then

$$L_p(s,\pi\times\pi') = \prod_{\substack{1\leqslant i\leqslant d \\ 1\leqslant j\leqslant d'}} (1 - \alpha_{\pi,i}(p)\alpha_{\pi',j}(p)p^{-s})^{-1} \quad (\text{if } p<\infty) \tag{3.3}$$

and

$$\{\mu_{\pi\times\pi',i,j} : 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant d'\} = \{\mu_{\pi,i} + \mu_{\pi',j} : 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant d'\}$$

if $p = \infty$. The arithmetic conductor satisfies (see [2])

$$q_{\pi\times\pi'} \leqslant q_\pi^{d'} q_{\pi'}^{d}/(q_\pi, q_{\pi'}). \tag{3.4}$$

Lemma 3.4. *Let π be a self-dual irreducible cuspidal automorphic representation of $GL_d(\mathbb{A}_{\mathbb{Q}})$ with trivial central character, and $0 < \delta < 1$ be arbitrary but fixed. Write $Q_{\pi\times\pi}$ for the analytic conductor of $L(s, \pi\times\pi)$ at $t = 0$.*

(a) *We have*

$$Q_{\pi\times\pi} \ll_d q_\pi^{2d-1} \prod_{1\leqslant i,j\leqslant d} (1 + |\mu_{\pi,i} + \mu_{\pi,j}|)^2$$

and

$$-\frac{L'}{L}(1+\delta, \pi\times\pi) \leqslant \frac{1}{2}\log Q_{\pi\times\pi} + \delta^{-1} + O(1).$$

(b) *For any $1 \leqslant \sigma \leqslant 3$ and $t \in \mathbb{R}$, there exists an absolute constant $C > 0$ such that*

$$(\sigma - 1)L(\sigma + \mathrm{i}t, \pi\times\pi) \ll \exp\left(C\frac{\log Q_{\pi\times\pi}}{\log\log Q_{\pi\times\pi}}\right)$$

and

$$L(\sigma + \mathrm{i}t, \pi\times\pi)^{-1} \ll (\sigma - 1)^{-1} \exp\left(C\frac{\log Q_{\pi\times\pi}}{\log\log Q_{\pi\times\pi}}\right),$$

where the implied $\ll$-constants are absolute.

Proof. As π is self-dual, the L-function $L(s, \pi\times\pi)$ has a simple pole at $s = 1$ and

$$\lambda_{\pi\times\pi}(n) \geqslant |\lambda_\pi(n)|^2 = \lambda_\pi(n)^2. \tag{3.5}$$

(a) We follow the proof of [9, Proposition 5.7]. The Hadamard factorization theorem for the entire function gives

$$s(1-s)L_\infty(s, \pi\times\pi)L(s, \pi\times\pi) = \mathrm{e}^{a+bs}\prod_\rho \left(1 - \frac{s}{\rho}\right)\mathrm{e}^{\frac{s}{\rho}},$$

where ρ runs over all zeros of the left-hand side. Note that

$$-\frac{L'}{L}(s, \pi\times\pi) = \frac{1}{s-1} - \frac{1}{s} + \frac{1}{2}\log q_{\pi\times\pi} + \frac{L'_\infty}{L_\infty}(s, \pi\times\pi) - b - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

As in [9, (5.29)], we have

$$\Re e\, b = -\Re e \sum_\rho \rho^{-1}.$$

With Stirling's formula

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(|s|^{-1}), \quad |\arg s| < \pi,$$

we obtain

$$-\frac{L'}{L}(1+\delta, \pi\times\pi) = \delta^{-1} + \frac{1}{2}\log Q_{\pi\times\pi} - \Re e \sum_\rho \frac{1}{1+\delta-\rho} + O(1).$$

The result follows with the observation

$$\Re e \sum_\rho \frac{1}{1+\delta-\rho} > 0.$$

(b) The first assertion follows from [18, Theorem 2] with the facts: for $1 < \sigma \leqslant 3$ and $t \in \mathbb{R}$,

$$L(\sigma, \pi\times\pi) > 0$$

and

$$|L(\sigma + \mathrm{i}t, \pi \times \pi)| \leqslant L(\sigma, \pi \times \pi)$$

by (3.5). As the Euler product of $L(s, \pi \times \pi)$ is absolutely convergent for $\sigma > 1$, we infer from (3.3) and (2.2) that for $\sigma > 1$,

$$\log L(\sigma + \mathrm{i}t, \pi \times \pi) = \sum_{p \nmid q_\pi} \frac{|\lambda_\pi(p)|^2}{p^s} + O\bigg(\sum_p p^{-2(\sigma - 2\theta)} + \sum_{p \mid q_\pi} 1\bigg), \tag{3.6}$$

where the sum over all the primes p in the O-term is $O(1)$ by (2.2). Hence,

$$|\log L(\sigma + \mathrm{i}t, \pi \times \pi)| \leqslant \log L(\sigma, \pi \times \pi) + O(\omega(q_\pi)),$$

where $\omega(q) = \sum_{p \mid q} 1$. The proof is completed with the asserted upper bound and

$$\log |L(s, \pi \times \pi)|^{-1} = -\Re e \log L(s, \pi \times \pi) \leqslant |\log L(s, \pi \times \pi)|. \qquad \square$$

The Rankin-Selberg theory produces L-functions of higher degree from two automorphic L-functions[6)]. When $d = d' = 2$, the Rankin-Selberg L-function $L(s, \pi \times \pi')$ is known to be automorphic by Ramakrishnan [26, Theorem M]. In addition, for $1 \leqslant r \leqslant 4$, the r-th symmetric power L-function $L(s, \mathrm{sym}^r \pi)$ of a cuspidal automorphic representation π of $GL_2(\mathbb{A}_\mathbb{Q})$ is automorphic, corresponding to an automorphic representation of $GL_{r+1}(\mathbb{A}_\mathbb{Q})$, by the work of Gelbart and Jacquet [6] ($r = 2$), Kim and Shahidi [13] ($r = 3$) and Kim [11] ($r = 4$). At the unramified prime $p < \infty$, the local factor $L_p(s, \mathrm{sym}^r \pi)$ is

$$L_p(s, \mathrm{sym}^r \pi) = \prod_{0 \leqslant i \leqslant r} (1 - \alpha_{\pi,1}(p)^{r-i} \alpha_{\pi,2}(p)^i p^{-s})^{-1}$$

(see Section 2 for $\alpha_{\pi,i}(p)$). The L-functions $L(s, \mathrm{sym}^r \pi)$ ($r = 2, 3, 4$) are entire, unless π is a solvable polyhedral[7)], and satisfy a functional equation of the form (2.1). For our purpose, we need a good estimate for $q_{\mathrm{sym}^r \pi}$ for a self-dual irreducible cuspidal automorphic representation π of $GL_2(\mathbb{A}_\mathbb{Q})$ with trivial central character.

Lemma 3.5. *Let π be defined as in Theorem 2.1. Write $q_{\mathrm{sym}^r \pi}$ for the arithmetic conductor of $L(s, \mathrm{sym}^r \pi)$ and $\mathfrak{Q}_r$ for the analytic conductor of $L(s, \mathrm{sym}^r \pi \times \mathrm{sym}^r \pi)$ at $t = 0$. For $r = 2$ (resp. $r = 3$, $r = 4$), $q_{\mathrm{sym}^r \pi}$ is at most q_π^3 (resp. q_π^7, q_π^{12}). Also we have the following bounds for $\mathfrak{Q}_r$: the upper bounds for $\mathfrak{Q}_r$ are Q_π^3, Q_π^{15}, Q_π^{49} and Q_π^{108} for $r = 1$, $r = 2$, $r = 3$ and $r = 4$, respectively.*

Proof. As $\mathrm{sym}^r \pi$ is an automorphic representation of $GL_{r+1}(\mathbb{A}_\mathbb{Q})$ for $1 \leqslant r \leqslant 4$, the following Rankin-Selberg L-functions are well defined and factor with the known identities into

$$\begin{aligned} &L(s, \pi \times \pi) = \zeta(s) L(s, \mathrm{sym}^2 \pi), \\ &L(s, \pi \times \mathrm{sym}^2 \pi) = L(s, \pi) L(s, \mathrm{sym}^3 \pi), \\ &L(s, \mathrm{sym}^2 \pi \times \mathrm{sym}^2 \pi) = \zeta(s) L(s, \mathrm{sym}^2 \pi) L(s, \mathrm{sym}^4 \pi) \end{aligned}$$

(see [24] and [29, p. 194]). Let p be a ramified prime (of π) and f_r be the exponent such that $p^{f_r} \parallel q_{\mathrm{sym}^r \pi}$. It follows from (3.4) that

$$3f_1 \geqslant f_2, \quad 3f_1 + 2f_2 - \min(f_1, f_2) \geqslant f_1 + f_3, \quad 5f_2 \geqslant f_2 + f_4$$

and $q_{\mathrm{sym}^r \pi \times \mathrm{sym}^r \pi} \leqslant q_{\mathrm{sym}^r \pi}^{2r+1}$. Hence $q_{\mathrm{sym}^r \pi \times \mathrm{sym}^r \pi}$ is at most q_π^3, q_π^{15}, q_π^{49} and q_π^{108} for $r = 1$, $r = 2$, $r = 3$ and $r = 4$, respectively.

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6) An automorphic L-function is the L-function associated with an automorphic representation.

7) That means π is dihedral, tetrahedral or octahedral. The L-function $L(s, \mathrm{sym}^r \pi)$ is entire for $r = 2$, $r = 3$ and $r = 4$ if and only if π is not dihedral, tetrahedral and octahedral, respectively. The representation π is dihedral if π is monomial attached to a character of a quadratic extension $K/\mathbb{Q}$ (or π admits a self-twist by a quadratic character), and π is tetrahedral (resp. octahedral), if $\mathrm{sym}^2 \pi$ is not monomial attached to a character of a cyclic cubic (resp. non-normal cubic extension $E/\mathbb{Q}$) (or $\mathrm{sym}^2 \pi$ (resp. $\mathrm{sym}^3 \pi$) is cuspidal and admits a non-trivial self-twist by a cubic (resp. quadratic, character)) (see [27, 28]).

The gamma factors of $L(s,\mathrm{sym}^r\pi\times\mathrm{sym}^r\pi)$ can be computed by the local Langlands correspondence for $GL_d(\mathbb{R})$ (see [17, Section 2], [4, Subsection 3.1.1] and [32, p. 205]). For the discrete series representation π_∞ with $\{\mu_{\pi,1},\mu_{\pi,2}\}=\{m-1/2,m+1/2\}$, we have

$$L_\infty(s,\mathrm{sym}^r\pi\times\mathrm{sym}^r\pi)=\Gamma_\mathbb{R}(s+1)^{[\frac{r}{2}]+1}\Gamma_\mathbb{R}(s)^{[\frac{r+1}{2}]}\prod_{j=0}^{r}\Gamma_\mathbb{C}(s+j(2m-1))^{r-j+1},$$

where $\Gamma_\mathbb{C}(s):=\Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1)$; for the principal series representation $\{\mu_{\pi,1},\mu_{\pi,2}\}=\{\nu+\delta,\nu-\delta\}$,

$$L_\infty(s,\mathrm{sym}^r\pi\times\mathrm{sym}^r\pi)=\prod_{0\leqslant a,b\leqslant r}\Gamma_\mathbb{R}(s+2(a+b-r)\nu).$$

The result follows by the direct check. □

Lemma 3.6. *Let π be as in Theorem 2.1 and $1\leqslant u\leqslant\frac{3}{2}$. Set $y=Q^{\frac{1}{2u}}$, where $Q:=Q_\pi$. As $Q\to\infty$, we have*

$$\sum_{\substack{y<p\leqslant y^u\\ p\nmid q_\pi}}(\max\{0,|\lambda_\pi(p)|-2\})^8p^{-1}\leqslant uA_8+B_8\log u+o(1),$$

where the constants A_8 and B_8 are less than 131 and 138, respectively.

Proof. Observing $(x-2)^8\leqslant x^8-6x^6$ for $x\geqslant 5/2$, we infer that if $x_+:=\max(0,x)$, then

$$(x-2)_+^8\leqslant x^8-6x^6+\frac{3^7}{16}+\frac{1}{2^8} \tag{3.7}$$

for all $x\geqslant 0$. [10, Lemma 11] gives that for $R\in\mathbb{N}$,

$$\lambda_\pi(p)^{2R}=\sum_{j=0}^{R}a_{R,j}\lambda_{\mathrm{sym}^{2(R-j)}\pi}(p),$$

where $a_{R,0}=1$ and

$$a_{R,j}=\binom{2R}{j}-\binom{2R}{j-1}$$

for $1\leqslant j\leqslant R$. Using the automorphicity of $\mathrm{sym}^r\pi$, $1\leqslant r\leqslant 4$ and

$$\mathrm{sym}^r\pi_p\otimes\mathrm{sym}^r\pi_p=\bigoplus_{j=0}^{r}\mathrm{sym}^{2j}\pi_p,$$

we may replace $\lambda_{\mathrm{sym}^{2r}\pi}(p)$ by

$$\lambda_{\mathrm{sym}^{2r}\pi}(p)=\lambda_{\mathrm{sym}^r\pi\times\mathrm{sym}^r\pi}(p)-\lambda_{\mathrm{sym}^{r-1}\pi\times\mathrm{sym}^{r-1}\pi}(p),$$

where $\lambda_{\mathrm{sym}^0\pi\times\mathrm{sym}^0\pi}(p):=1$. After a little computation, we obtain

$$\lambda_\pi(p)^8-6\lambda_\pi(p)^6=\sum_{j=0}^{4}b_{4,j}\lambda_{\mathrm{sym}^{4-j}\pi\times\mathrm{sym}^{4-j}\pi}(p),$$

where $b_{4,0}=1$, $b_{4,1}=0$, $b_{4,2}=-11$, $b_{4,3}=-16$ and $b_{4,4}=10$. Set $B_8=10+3^7/16+1/2^8=137.69\cdots$. By (3.7) and the positivity of $\lambda_{\mathrm{sym}^j\pi\times\mathrm{sym}^j\pi}(p)$, it follows that

$$\begin{aligned}\sum_{\substack{y<p\leqslant y^u\\ p\nmid q_\pi}}(|\lambda_\pi(p)|-2)_+^8\,p^{-1}&\leqslant\sum_{\substack{y<p\leqslant y^u\\ p\nmid q_\pi}}\lambda_{\mathrm{sym}^4\pi\times\mathrm{sym}^4\pi}(p)p^{-1}+B_8\sum_{y<p\leqslant y^u}p^{-1}\\&=\Sigma(y,u)+B_8\log u+o(1),\end{aligned}$$

where

$$\Sigma(y,u) := \sum_{\substack{y<p\leqslant y^u \\ p\nmid q_\pi}} \lambda_{\mathrm{sym}^4\pi\times\mathrm{sym}^4\pi}(p)p^{-1} \leqslant \frac{y^{\delta u}}{\log y}\sum_p \lambda_{\mathrm{sym}^4\pi\times\mathrm{sym}^4\pi}(p)\frac{\log p}{p^{1+\delta}}$$

for any $\delta > 0$, by Rankin's trick. Using the known bound for the Ramanujan conjecture, we may replace the sum over p by $-\frac{L'}{L}(1+\delta, \mathrm{sym}^4\pi\times\mathrm{sym}^4\pi)$ up to a term of $O(1)$ whose implied O-constant is independent of its (analytic) conductor $\mathcal{Q}_4$. Suppose $\log \mathcal{Q}_4 \leqslant f\log Q$. Let

$$\lambda = \frac{4}{\sqrt{1+4f}+1} \quad \text{and} \quad \delta = \frac{\lambda}{\log Q}.$$

By Lemma 3.4, we get

$$\Sigma(y,u) \leqslant \frac{y^{\delta u}}{\log y}\left(\delta^{-1} + \frac{f}{2}\log Q + O(1)\right) \leqslant u\mathrm{e}^{\lambda/2}\left(\frac{2}{\lambda}+f\right) + o(1)$$

as $Q\to\infty$. By Lemma 3.5, we may take $f = 108$. Our result hence follows with

$$A_8 := \mathrm{e}^{\lambda/2}\left(\frac{2}{\lambda}+f\right) < 131.$$

The proof is completed. □

4 Proof of Theorem 2.1

Define

$$S_\pi(x) = \sideset{}{^\flat}\sum_{\substack{n\leqslant x \\ (n,q_\pi)=1}} \lambda_\pi(n),$$

recalling the sum $\sum^\flat$ ranges over squarefree integers n.

4.1 The upper bound

We derive an upper bound for $S_\pi(x)$. Via the Euler products of $L(s,\pi)$ and $L(s,\pi\times\widetilde{\pi})$, we have

$$\sideset{}{^\flat}\sum_{\substack{n\geqslant 1 \\ (n,q_\pi)=1}} \frac{\lambda_\pi(n)}{n^s} = \prod_{p\nmid q_\pi}\left(1+\frac{\lambda_\pi(p)}{p^s}\right) = \frac{L(s,\pi)\zeta(2s)}{L(2s,\pi\times\widetilde{\pi})}D_\pi(s)$$

for $\Re e\, s > 1$, where $D_\pi(s)$ is the Dirichlet series given by

$$D_\pi(s) := \prod_{p\mid q_\pi}\frac{L_p(2s,\pi\times\widetilde{\pi})}{(1-p^{-2s})^{-1}L_p(s,\pi)}\prod_{p\nmid q_\pi}(1+\lambda_\pi(p)p^{-3s}+O(p^{4(\theta-\sigma)})).$$

Let $\varepsilon > 0$ be any sufficiently small number. The Euler product defining $D_\pi(s)$ converges absolutely in the half plane $\Re e\, s \geqslant \frac{1}{2}+\varepsilon$, and $D_\pi(s) \ll_\varepsilon Q_\pi^\varepsilon$ uniformly for $\Re e\, s\geqslant \frac{1}{2}+\varepsilon$. With (3.5), we infer from Lemma 3.4 that for $\sigma = \Re e\, s \geqslant 1+\varepsilon$,

$$L(s,\pi) \ll_\varepsilon L(\sigma,\pi\times\widetilde{\pi}) \ll_\varepsilon Q_\pi^\varepsilon.$$

The Perron-type formula in Lemma 3.1 gives

$$\sideset{}{^\flat}\sum_{\substack{n\leqslant x \\ (n,q_\pi)=1}} \lambda_f(n) = \frac{1}{2\pi \mathrm{i}}\int_{1+\varepsilon-\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T}\frac{L(s,\pi)\zeta(2s)}{L(2s,\pi\times\widetilde{\pi})}D_\pi(s)\frac{x^s}{s}ds$$

$$+ O\bigg(\sum_{x-\frac{x}{H}<n<x+\frac{x}{H}} |\lambda_\pi(n)| \bigg) + O\bigg(\frac{x^{1+\varepsilon} Q_\pi^\varepsilon H}{T} \bigg).$$

The first O-term is $\ll x^{1+\varepsilon} Q_\pi^\varepsilon / \sqrt{H}$. To see it, we apply Cauchy's inequality to get

$$\sum_{x-\frac{x}{H}<n<x+\frac{x}{H}} |\lambda_\pi(n)| \ll \frac{x}{\sqrt{H}} \bigg(\sum_{x-\frac{x}{H}<n<x+\frac{x}{H}} \frac{|\lambda_\pi(n)|^2}{n^{1+\alpha}} \bigg)^{\frac{1}{2}},$$

where $\alpha = (\log x)^{-1}$, and observe the bound (using (3.5) and Lemma 3.4(b))

$$\sum_{x-\frac{x}{H}<n<x+\frac{x}{H}} \frac{|\lambda_\pi(n)|^2}{n^{1+\alpha}} \leqslant \sum_{x-\frac{x}{H}<n<x+\frac{x}{H}} \frac{\lambda_{\pi\times\widetilde{\pi}}(n)}{n^{1+\alpha}} \leqslant L(1+\alpha, \pi\times\widetilde{\pi}) \ll (xQ_\pi)^\varepsilon.$$

For the integral, we shift the line of integration $\Re e\, s = \frac{1}{2} + \varepsilon$ and then

$$\begin{aligned}
&\int_{1+\varepsilon-\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T} \frac{L(s,\pi)\zeta(2s)}{L(2s,\pi\times\widetilde{\pi})} D_\pi(s) \frac{x^s}{s} ds \\
&\quad = \int_{\frac{1}{2}+\varepsilon-\mathrm{i}T}^{\frac{1}{2}+\varepsilon+\mathrm{i}T} \frac{L(s,\pi)\zeta(2s)}{L(2s,\pi\times\widetilde{\pi})} D_\pi(s) \frac{x^s}{s} ds + O\bigg(\frac{x^{1+\varepsilon}Q_\pi^\varepsilon}{T} + \frac{x^{\frac{1}{2}+\varepsilon} Q_\pi^{\frac{1}{4}+\varepsilon}}{T^{\frac{1}{2}}} \bigg) \\
&\quad \ll x^{\frac{1}{2}+\varepsilon} (Q_\pi T^2)^{\frac{1}{4}+\varepsilon} + \frac{x^{1+\varepsilon}Q_\pi^\varepsilon}{T}.
\end{aligned}$$

Here, we have used $D_\pi(s) \ll Q_\pi^\varepsilon$, Lemma 3.4(b) and the convexity bound

$$L\bigg(\frac{1}{2} + \varepsilon + \mathrm{i}t, \pi \bigg) \ll Q_\pi^{\frac{1}{4}+\varepsilon} (|t|+2)^{\frac{1}{2}+\varepsilon}.$$

Altogether we get

$$S_\pi(x) \ll (xT)^{\frac{1}{2}+\varepsilon} Q_\pi^{\frac{1}{4}+\varepsilon} + \frac{x^{1+\varepsilon}Q_\pi^\varepsilon}{\sqrt{H}} + \frac{x^{1+\varepsilon} H Q_\pi^\varepsilon}{T}.$$

Taking

$$H = T^{\frac{2}{3}} \quad \text{and} \quad T = x^{12\varepsilon},$$

we obtain

$$S_\pi(x) \ll x^{\frac{1}{2}+7\varepsilon} Q_\pi^{\frac{1}{4}+\varepsilon} + x^{1-3\varepsilon} Q_\pi^\varepsilon. \tag{4.1}$$

4.2 The conditional lower bound

We evaluate a lower bound for $S_\pi(y^u)$ under the condition of positive $\lambda_\pi(n)$'s. Suppose

$$\lambda_\pi(n) \geqslant 0 \quad \text{for all } n \leqslant y \quad \text{and} \quad (n, q_\pi) = 1. \tag{4.2}$$

By the argument in [8, Section 3], we know $y \leqslant Q_\pi^{\frac{1}{2}+\varepsilon}$ when Q_π is sufficiently large. For our purpose, we consider $y \geqslant Q_\pi^{\frac{1}{3}}$, and Theorem 2.1 is obviously true for the other case. Hence we assume

$$y = Q_\pi^{\frac{1}{2u}}, \quad \text{where } 1 \leqslant u \leqslant \frac{3}{2}.$$

To get the lower bound of $S_\pi(x)$, a multiplicative function is constructed to approximate $\lambda_\pi(n)$. For the case of holomorphic cusp forms, Matomäki [21] introduced the function $h_y(n)$ defined as in Lemma 3.3 and showed that

$$S_\pi(y^u) \gg \sideset{}{^\flat}\sum_{\substack{n\leqslant y^u \\ (n,N)=1}} h_y(n),$$

for which the Deligne bound (1.4) is utilized.

The GRC is still open for Maass cusp forms, so we need to get around this condition in Kowalski-Lau-Soundararajan-Wu-Matomäki's approach. We refine their construction and consider the multiplicative function $w_y(n)$ supported on squarefree integers with

$$w_y(p) = \begin{cases} -|\lambda_\pi(p)|, & \text{if } p > y, \\ 2\cos\left(\dfrac{\pi}{m+1}\right), & \text{if } y^{\frac{1}{m+1}} \leqslant p < y^{\frac{1}{m}} \text{ for some integer } 1 \leqslant m < M, \\ 2\cos\left(\dfrac{\pi}{M+1}\right), & \text{if } p \leqslant y^{\frac{1}{M}}. \end{cases} \tag{4.3}$$

Lemma 4.1. *Let $M \geqslant 4$ be an integer, $y = Q_\pi^{\frac{1}{2u}}$ $(1 \leqslant u \leqslant \frac{3}{2})$ and define w_y as in* (4.3). *Uniformly for $1 \leqslant u \leqslant \frac{3}{2}$, we have*

$$\sideset{}{^\flat}\sum_{\substack{n \leqslant y^u \\ (n,q_\pi)=1}} w_y(n) \geqslant \{\sigma_M(u) - (A_8 u + B_8 \log u)^{\frac{1}{8}} I(u)^{\frac{7}{8}} + o(1)\} \frac{\Pi_{q_\pi,\chi_0}}{\Gamma(\chi_0)} (\log y)^{\chi_0 - 1} y^u,$$

where the notations $\sigma_M(u)$, Π_{q_π,χ_0}, $\Gamma(\chi_0)$, A_8 and B_8 are defined as in Lemmas 3.3 *and* 3.6, *and*

$$I(u) := \int_1^u (u-t)^{\frac{8}{7}(\chi_0 - 1)} \frac{dt}{t}.$$

Besides, the upper bounds for $I(u)$ are 0.017439, 0.02227, 0.02936 *and* 0.040324 *for $u = \frac{11}{9}$, $u = \frac{5}{4}$, $u = \frac{9}{7}$ and $u = \frac{4}{3}$, respectively.*

Proof. In view of (4.3) and (3.2) (with $N = q_\pi$), we see that $w_y(n) = h_y(n)$ when the largest prime factor $P(n)$ of n is less than or equal to y. Thus for $1 \leqslant u \leqslant 3/2$,

$$\begin{aligned} \sideset{}{^\flat}\sum_{\substack{n \leqslant y^u \\ (n,q_\pi)=1}} w_y(n) &= \sideset{}{^\flat}\sum_{\substack{n \leqslant y^u \\ P(n) \leqslant y}} w_y(n) - \sum_{\substack{y < p \leqslant y^u \\ (p,q_\pi)=1}} |\lambda_\pi(p)| \sideset{}{^\flat}\sum_{\substack{n \leqslant \frac{y^u}{p} \\ (n,q_\pi)=1}} w_y(n) \\ &= \sideset{}{^\flat}\sum_{\substack{n \leqslant y^u \\ (n,q_\pi)=1}} h_y(n) + 2 \sum_{\substack{y < p \leqslant y^u \\ (p,q_\pi)=1}} \sideset{}{^\flat}\sum_{\substack{n \leqslant \frac{y^u}{p} \\ (n,q_\pi)=1}} h_y(n) - \sum_{\substack{y < p \leqslant y^u \\ (p,q_\pi)=1}} |\lambda_\pi(p)| \sideset{}{^\flat}\sum_{\substack{n \leqslant \frac{y^u}{p} \\ (n,q_\pi)=1}} h_y(n) \\ &=: S_1 - S_2, \end{aligned} \tag{4.4}$$

where S_1 denotes the first summation and

$$S_2 = \sum_{\substack{y < p \leqslant y^u \\ (p,q_\pi)=1}} (|\lambda_\pi(p)| - 2) \sideset{}{^\flat}\sum_{\substack{n \leqslant \frac{y^u}{p} \\ (n,q_\pi)=1}} h_y(n).$$

Since, in S_2, $0 \leqslant h_y(n) \leqslant \chi_0^{\omega(n)}$, we may bound S_2 as follows:

$$S_2 \leqslant \sum_{\substack{y < p \leqslant y^u,\, (p,q_\pi)=1 \\ |\lambda_\pi(p)| > 2}} (|\lambda_\pi(p)| - 2) \sideset{}{^\flat}\sum_{\substack{n \leqslant \frac{y^u}{p} \\ (n,q_\pi)=1}} h_y(n) \leqslant S_{2,1} + S_{2,2},$$

where for any $Z \geqslant 1$,

$$S_{2,1} = \sum_{\substack{y < p \leqslant y^u/Z \\ (p,q_\pi)=1,\, |\lambda_\pi(p)| > 2}} (|\lambda_\pi(p)| - 2) \sideset{}{^\flat}\sum_{\substack{n \leqslant \frac{y^u}{p} \\ (n,q_\pi)=1}} \chi_0^{\omega(n)}$$

and

$$S_{2,2} = \sum_{\substack{y^u/Z<p\leqslant y^u \\ (p,q_\pi)=1}} |\lambda_\pi(p)| \sum_{\substack{n\leqslant \frac{y^u}{p} \\ (n,q_\pi)=1}}^{\flat} \chi_0^{\omega(n)}.$$

Take $Z = Z(y)$ so that

$$\log Z = o(1)(\log y)^{(\chi_0-1/8)/(\chi_0+7/8)}$$

with the term $o(1) \to 0$ as $y \to \infty$. By Rankin's trick, we see that

$$\begin{aligned} S_{2,2} &\leqslant y^u \sum_{\substack{y^u/Z<p\leqslant y^u \\ (p,q_\pi)=1}} \frac{|\lambda_\pi(p)|}{p} \sum_{n\leqslant Z}^{\flat} \frac{\chi_0^{\omega(n)}}{n} \\ &\ll y^u \frac{(\log Z)^{\chi_0+7/8}}{(\log y^u)^{7/8}} \Bigg(\sum_{\substack{y^u/Z<p\leqslant y^u \\ (p,q_\pi)=1}} \frac{\lambda_\pi(p)^8}{p} \Bigg)^{1/8} \\ &= o(1) y^u (\log y)^{\chi_0-1}, \end{aligned}$$

where we have bounded the sum over n by $\prod_{p\leqslant Z}(1+\chi_0/p)$ and applied Hölder's inequality to the sum over p with Lemma 3.6. As $y < p \leqslant y^u/Z$, Lemma 3.2 can be applied to $S_{2,1}$:

$$S_{2,1} = (1+o(1)) \frac{\Pi_{N,\chi_0}}{\Gamma(\chi_0)} y^u \sum_{\substack{y<p\leqslant y^u/Z \\ (p,q_\pi)=1}} \frac{(|\lambda_\pi(p)|-2)_+}{p} \bigg(\log \frac{y^u}{p} \bigg)^{\chi_0-1},$$

where $x_+ := \max(x, 0)$. We apply Hölder's inequality to the sum over p and deduce that

$$\sum_{\substack{y<p\leqslant y^u/Z \\ (p,q_\pi)=1}} \cdots \leqslant \Bigg(\sum_{y<p\leqslant y^u} \frac{1}{p} \bigg(\log \frac{y^u}{p} \bigg)^{\frac{8}{7}(\chi_0-1)} \Bigg)^{7/8} \Bigg(\sum_{\substack{y<p\leqslant y^u \\ (p,q_\pi)=1}} \frac{(|\lambda_\pi(p)|-2)_+^8}{p} \Bigg)^{1/8}.$$

By the prime number theorem,

$$\sum_{y<p\leqslant y^u} \frac{1}{p} \bigg(\log \frac{y^u}{p} \bigg)^{\frac{8}{7}(\chi_0-1)} = (1+o(1))(\log y)^{\frac{8}{7}(\chi_0-1)} \int_1^u (u-t)^{\frac{8}{7}(\chi_0-1)} \frac{dt}{t}.$$

Together with Lemma 3.6, we infer that

$$S_2 \leqslant (1+o(1))(uA_8 + B_8 \log u)^{\frac{1}{8}} \bigg(\int_1^u (u-t)^{\frac{8}{7}(\chi_0-1)} \frac{dt}{t} \bigg)^{\frac{7}{8}} \frac{\Pi_{N,\chi_0}}{\Gamma(\chi_0)} y^u (\log y)^{\chi_0-1}.$$

Inserting it into (4.4), we obtain the required inequality with Lemma 3.3. □

Remark 4.2. One may keep $h_y(n)$, in lieu of $\chi_0^{\omega(n)}$, in the inner sum over n in $S_{2,1}$. This will lead to the integral

$$\int_1^u \sigma_M(u-t)^{8/7} dt/t$$

in place of $I(u)$, which is manageable though being more complicated. The saving is however not enough for us to take $u = 9/7$ in the next section.

4.3 Completion of the proof

Let y be the largest positive integer for which (4.2) holds. Under the positivity condition (4.2), we have

$$\lambda_\pi(p^j) \geqslant 0$$

for any $1 \leqslant j \leqslant m$ and any prime $p \nmid q_\pi$ with $y^{\frac{1}{m+1}} \leqslant p < y^{\frac{1}{m}}$. If, moreover, $\lambda_\pi(p) \leqslant 2$, the arguments in [15, Section 2] (based on the fact that $\lambda_\pi(p^j)$ is given by a Chebyshev polynomial of the second kind) will give

$$\lambda_\pi(p) \geqslant 2\cos\frac{\pi}{m+1}. \tag{4.5}$$

If $\lambda_\pi(p) > 2$, clearly (4.5) will also hold. In view of (4.3), we have the following:

$$S_\pi(y^u) \geqslant \sideset{}{^\flat}\sum_{\substack{n\leqslant y^u\\(n,q_\pi)=1}} w_y(n) \quad \text{subject to (4.2)}.$$

Next, we invoke Lemma 4.1 with $M = 100$. Since σ_M is continuous, we take $u = \frac{5}{4} + \varepsilon'$ for some sufficiently small $\varepsilon' > 0$. Then

$$\sigma_{100}(u) - (uA_8 + B_8\log u)^{1/8} I(u)^{7/8} > 2.5 \times 10^{-3}$$

and

$$y^u \ll S_\pi(y^u) \ll y^{\frac{u}{2}+10\varepsilon} Q_\pi^{\frac{1}{4}+\varepsilon} + y^{u-4\varepsilon} Q_\pi^\varepsilon$$

by (4.1). This implies

$$y \ll Q_\pi^{\frac{1}{2u}+10\varepsilon} \ll Q_\pi^{\frac{2}{5}}$$

by taking $\varepsilon = \varepsilon'/100$. The proof is now completed.

Acknowledgements The first author was supported by General Research Fund of the Research Grants Council of Hong Kong (Grant Nos. 17313616 and 17305617). The third author was supported by National Natural Science Foundation of China (Grant No. 11871193) and the Program for Young Scholar of Henan Province (Grant No. 2019GGJS026). The fourth author was supported by National Natural Science Foundation of China (Grant No. 11871344). The authors are grateful to the referees for careful reading and thank Professor Matomäki for her C++ code.

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