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Superconnections and an intrinsic Gauss-Bonnet-Chern formula for Finsler manifolds

Dedicated to Professor Shiing-Shen Chern and Professor Yibing Shen

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Abstract In this paper, we establish an intrinsic Gauss-Bonnet-Chern formula for Finsler manifolds by using the Mathai-Quillen's superconnection formalism, in which no extra vector field is involved. Furthermore, we prove a more general Lichnerowicz formula in this direction through a geometric localization procedure.

Keywords superconnection, Gauss-Bonnet-Chern formula, Finsler manifold, transgression

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1 Introduction

In the celebrated paper [6], Chern presented a simple and intrinsic proof of the following famous Gauss-Bonnet-Chern formula (also GBC-formula in short) for a closed and oriented Riemannian manifold (M, g^{TM}) of dimension $2n$:

$$\chi(M) = \left(\frac{-1}{2\pi}\right)^n \int_M \mathrm{Pf}(R^{TM}), \tag{1.1}$$

where the Pfaffian $\mathrm{Pf}(R^{TM})$ is a well-defined $2n$-form on M constructed from the curvature R^{TM} of the Levi-Civita connection $\nabla^{g^{TM}}$ associated with the Riemannian metric g^{TM}. With respect to any oriented orthonormal (local) frame $\{e_1, \dots, e_{2n}\}$ for TM,

$$\mathrm{Pf}(R^{TM}) = \frac{1}{2^n n!} \sum_{a_1,\dots,a_{2n}=1}^{2n} \epsilon_{a_1a_2\cdots a_{2n}} \Omega^{a_2}_{a_1} \wedge \cdots \wedge \Omega^{a_{2n}}_{a_{2n-1}}, \tag{1.2}$$

where $\epsilon_{a_1a_2\cdots a_{2n}}$ is the usual Kronecker symbol and

$$\Omega^b_a := g^{TM}(R^{TM}e_a, e_b). \tag{1.3}$$

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Chern's formula (1.1) expresses the Euler characteristic $\chi(M)$ by the integration of the purely geometric differential form $\mathrm{Pf}(R^{TM})$ on M and initiates the geometric theory of characteristic classes, i.e., the Chern-Weil theory, which plays a very important role in the study of modern geometry and topology. The key point in Chern's proof is his significant transgression formula

$$\left(\frac{-1}{2\pi}\right)^n \pi^*\mathrm{Pf}(R^{TM}) = -d^{SM}\Pi$$

on the unit sphere bundle $\pi : SM \to M$, where the transgression form Π lives on SM. For any vector field X on M with the isolated zero set $Z(X)$, let $[X]$ denote the normalizing of X on $M \setminus Z(X)$. Also for any $\epsilon > 0$, let $Z_\epsilon(X)$ denote the ϵ-neighbourhood of the zero set $Z(X)$ in M, and set $M_\epsilon = M \setminus Z_\epsilon(X)$. By using the above transgression formula, Chern got the following equality over M_ϵ:

$$\left(\frac{-1}{2\pi}\right)^n \mathrm{Pf}(R^{TM}) = \left(\frac{-1}{2\pi}\right)^n [X]^*\pi^*\mathrm{Pf}(R^{TM}) = -[X]^*d^{SM}\Pi = -d^M[X]^*\Pi.$$

With the help of the Poincaré-Hopf theorem and noticing that the tangent unit spheres of M have the constant volume, Chern obtained his formula (1.1) by computing the following integral:

$$\left(\frac{-1}{2\pi}\right)^n \int_M \mathrm{Pf}(R^{TM}) = \lim_{\epsilon\to 0}\int_{\partial Z_\epsilon(X)} [X]^*\Pi.$$

After Chern's work, many people tried to generalize Chern's formula (1.1) to the Finsler setting (see, e.g., [1, 12, 13, 17, 18, 22]). Inspired by Chern's work, Lichnerowicz [13] first established a GBC-formula for some special Finsler manifolds by using the Cartan connection ∇^{Car} on π^*TM. Realizing that almost all Finsler geometric quantities live actually on the unit sphere bundle SM, Lichnerowicz constructed an analogous differential $2n$-form $\mathrm{Pf}(R^{\mathrm{Car}})$ on SM and proved the following transgression formula:

$$\left(\frac{-1}{2\pi}\right)^n \mathrm{Pf}(R^{\mathrm{Car}}) = -d^{SM}\Pi^{\mathrm{Car}}$$

for some $(2n-1)$-form Π^{Car} on SM, where R^{Car} denotes the curvature of the Cartan connection on π^*TM. Following Chern's strategy, Lichnerowicz also proceeded with the computations

$$\left(\frac{-1}{2\pi}\right)^n \int_M [X]^*\mathrm{Pf}(R^{\mathrm{Car}}) := \left(\frac{-1}{2\pi}\right)^n \lim_{\epsilon\to 0}\int_{M_\epsilon} [X]^*\mathrm{Pf}(R^{\mathrm{Car}}) = \lim_{\epsilon\to 0}\int_{\partial Z_\epsilon(X)} [X]^*\Pi^{\mathrm{Car}}.$$

To get the desired Euler number $\chi(M)$ from the above computations, Lichnerowicz assumed that the space (M, F) should be a Cartan-Berwald space and all the Finsler unit spheres $S_xM = \{Y \in T_xM \mid F(Y) = 1\}$ should have the same volume as a Euclidean unit sphere, and under these assumptions, he got the following formula:

$$\chi(M) = \left(\frac{-1}{2\pi}\right)^n \int_M [X]^*\mathrm{Pf}(R^{\mathrm{Car}}). \tag{1.4}$$

Note that the above volume assumption holds automatically for all the Cartan-Berwald spaces of dimension greater than 2. Moreover, as mentioned by Bao and Shen [3] (see also [17]), when the Finsler metric is reversible, then by a theorem of Brickell, any Cartan-Berwald space of dimension greater than 2 must be Riemannian.

Around fifty years later, Bao and Chern [1] dropped the assumption of the Cartan-Berwald condition of Lichnerowicz by using the Chern connection ∇^{Ch} proposed in [8] and established the following GBC-formula for all the $2n$-dimensional oriented and closed Finsler manifolds with the constant volume of Finsler unit spheres:

$$\left(\frac{-1}{2\pi}\right)^n \int_M [X]^*[\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}}) + \mathcal{F}] = \chi(M)\frac{\mathrm{Vol}(\mathrm{Finsler}S^{2n-1})}{\mathrm{Vol}(S^{2n-1})} \tag{1.5}$$

by proving the following transgression formula

$$\left(\frac{-1}{2\pi}\right)^n [\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}}) + \mathcal{F}] = -d^{SM}\Pi^{\mathrm{Ch}},$$

where $\widehat{R}^{\mathrm{Ch}}$ is the skew-symmetrization of the curvature R^{Ch} of the Chern connection ∇^{Ch} with respect to the fundamental tensor g_F of F, $\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}})$ is the Pfaffian of $\widehat{R}^{\mathrm{Ch}}$, Π^{Ch} is the associated transgression form and

$$\mathcal{F} = (-1)^n \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(2n-2k-1)!!k!2^k} \mathcal{F}_k, \tag{1.6}$$

in which $\mathcal{F}_0 = 0$ and

$$\begin{aligned}\mathcal{F}_k = {} & k\epsilon_{\alpha_1\cdots\alpha_{2n-1}}\{\Omega^{\alpha_2}_{\alpha_1} \wedge (\omega^{\alpha_1}_{\alpha_1} - \omega^{\alpha_2}_{\alpha_2}) + \Omega^{\alpha_2}_{\alpha_2} \wedge (\omega^{\alpha_2}_{\alpha_1} + \omega^{\alpha_1}_{\alpha_2}) + (k-1)\Omega^{\alpha_3}_{\alpha_2} \wedge (\omega^{\alpha_3}_{\alpha_1} + \omega^{\alpha_1}_{\alpha_3}) \\ & + (2n-2k-1)[\Omega^{\alpha_{2k+1}}_{\alpha_2} \wedge (\omega^{\alpha_{2k+1}}_{\alpha_1} + \omega^{\alpha_1}_{\alpha_{2k+1}}) + 1/k\Omega^{\alpha_2}_{\alpha_1} \wedge \omega^{\alpha_{2k+1}}_{\alpha_{2k+1}}]\} \\ & \wedge \Omega^{\alpha_4}_{\alpha_3} \wedge \cdots \wedge \Omega^{\alpha_{2k}}_{\alpha_{2k-1}} \wedge \omega^{2n+\alpha_{2k+1}} \wedge \cdots \wedge \omega^{2n+\alpha_{2n-1}}.\end{aligned}$$

To avoid the constant volume assumption in Bao-Chern's formula (1.5), following Bao-Chern's approach, Lackey [12] and Shen [18] modified the GBC-integrand terms independently by using the unit sphere volume function $V(x) = \mathrm{Vol}(S_xM)$ and obtained some new types of GBC-formulae via the Chern and Cartan connections, respectively, for all the oriented and closed Finsler manifolds.

However, a notable difference from Chern's formula (1.1) for Riemannian manifolds, the above-mentioned generalizations in the Finsler setting had to make use of an extra vector field X on M in their GBC-integrands. As a result, all these GBC-formulae look not so intrinsic in the spirit of Chern's original formula (1.1). In [16], Shen asked explicitly whether there is a Gauss-Bonnet-Chern formula for general Finsler manifolds without using any vector fields.

In this paper, by using Mathai-Quillen's superconnection formalism, we obtain the following Gauss-Bonnet-Chern formula for Finsler manifolds.

Theorem 1.1. *Let (M,F) be a closed and oriented Finsler manifold of dimension $2n$. Let $R^{\mathrm{Ch}} = R+P$ be the curvature of the Chern connection ∇^{Ch} on the pull-back bundle π^*TM over SM. Then in the induced homogeneous coordinate charts (x^i, y^i) on SM, one has*

$$\chi(M) = \int_M e(TM, \nabla^{\mathrm{Ch}}),$$

where

$$\begin{aligned} e(TM, \nabla^{\mathrm{Ch}}) = {} & \frac{1}{(2\pi)^{2n}(2n)!}\Bigg\{\sum_{k=1}^{n}(-1)^k C^{2k}_{2n} C^{k-1}_{2k-2} \int_{SM/M} \delta^{i_1\cdots i_{2n}}_{j_1\cdots j_{2n}} R^{j_1}_{i_1} \cdots R^{j_k}_{i_k} P^{j_{k+1}}_{i_{k+1}} \cdots P^{j_{2n-k}}_{i_{2n-k}} \\ & \cdot \Upsilon^{j_{2n-k+1}}_{i_{2n-k+1}} \cdots \Upsilon^{j_{2n-1}}_{i_{2n-1}} \Xi^{j_{2n}}_{i_{2n}} + \int_{SM/M} \delta^{i_1\cdots i_{2n}}_{j_1\cdots j_{2n}} P^{j_1}_{i_1} \cdots P^{j_{2n-1}}_{i_{2n-1}} \varpi^{j_{2n}}_{i_{2n}}\Bigg\}, \end{aligned} \tag{1.7}$$

and ϖ^j_i, R^j_i, P^j_i, Υ^j_i and Ξ^j_i defined by (3.3), (3.9) *and* (3.30)*, respectively, are purely geometric data derived from the Finsler metric F on M.*

On the other hand, combining a slight generalization of a lemma of Feng and Zhang [11] and a geometric localization procedure, we obtain a precise form of Lichnerowicz's orginal GBC-formula (1.4) associated with a vector field with the isolated zero set.

Theorem 1.2. *Let (M,F) be a closed and oriented Finsler manifold of dimension $2n$ with the constant volume of Finsler unit spheres. One has*

$$\left(\frac{-1}{2\pi}\right)^n \int_M [X]^*[\mathrm{Pf}(R^{\mathrm{Car}}) + d\mathcal{H}] = \chi(M)\frac{\mathrm{Vol}(\mathrm{Finsler}S^{2n-1})}{\mathrm{Vol}(S^{2n-1})}, \tag{1.8}$$

where X is a vector field on M with the isolated zero set, and

$$\mathcal{H} := \sum_{k=1}^{n-1} \frac{(-1)^{n+k}}{(2n-2k-1)!!2^k k!} \sum \epsilon_{a_1\cdots a_{2n-1}} Q_{a_1}^{a_2} \wedge \cdots \wedge Q_{a_{2k-1}}^{a_{2k}} \wedge \omega_{a_{2k+1}}^{2n} \wedge \cdots \wedge \omega_{a_{2n-1}}^{2n} \tag{1.9}$$

with $Q = -\frac{1}{4}\Theta \wedge \Theta$ and $\Theta = g_F^{-1}(\nabla^{\mathrm{Ch}} g_F)$ being the Bismut-Zhang form (see [5]) in the Finsler setting or the Cartan endomorphism (see [10]).

The rest of this paper is organized as follows. In Section 2, we first briefly introduce the Mathai-Quillen's superconnection formalism (see [14, 15]) for the reader's convenience, and then recall the Mathai-Quillen type formula of Feng and Zhang on the Euler characteristic (see [11]) and give it a slight generalization for our purpose. In Section 3, we work out the main result Theorem 1.1 in this paper. In Section 4, we investigate some special Finsler manifolds, such as Finsler surfaces and Berwald spaces. By the special curvature property of Berwald spaces, our Finslerian GBC-formula reduces to a simple and elegant form, from which Chern's GBC-formula is deduced easily. In Section 5, we prove Theorem 1.2 by proceeding with a geometric localization procedure. In Appendix A, we give a proof of the claim (5.40).

2 Superconnections and the Euler characteristic

We first review some basic definitions and notations on superspaces and superconnections [14, 15] (see also [4, 20, 21] for more details). Then we recall the Mathai-Quillen type formula of Feng and Zhang on the Euler characteristic in [11] and prove a slight generalization of it.

2.1 Superspaces and superconnections

A super vector space E is a vector space with a $\mathbf{Z}_2$-grading $E = E_+ \oplus E_-$. Let $\tau_E \in \mathrm{End}(E)$ such that $\tau_E|_{E_\pm} = \pm 1$. Then for any $B \in \mathrm{End}(E)$, the supertrace $\mathrm{tr}_s[B]$ is defined by

$$\mathrm{tr}_s[B] = \mathrm{tr}[\tau_E B]. \tag{2.1}$$

An element B in $\mathrm{End}(E)$ is even (resp. odd) if $B(E_\pm) \subset E_\pm$ (resp. $B(E_\pm) \subset E_\mp$) and the degree $|B|$ of B is defined to be 0/1 if B is even/odd. The bracket operation in $\mathrm{End}(E)$ for a superspace E always refers to the superbracket

$$[B_1, B_2] = B_1B_2 - (-1)^{|B_1||B_2|}B_2B_1$$

for any $B_1, B_2 \in \mathrm{End}(E)$. One has

$$\mathrm{tr}_s[B_1, B_2] = 0. \tag{2.2}$$

As an example, for any vector space V of dimension m, the exterior algebra $\Lambda^*(V^*)$ generated by V is a superspace with the natural even/odd $\mathbf{Z}_2$-grading, i.e.,

$$\Lambda^*(V^*) = \Lambda^{\mathrm{even}}(V^*) \oplus \Lambda^{\mathrm{odd}}(V^*). \tag{2.3}$$

For any $B \in \mathrm{End}(V)$, the lifting $B^\natural$ of B is a derivative acting on $\Lambda^*(V^*)$, i.e., $B^\natural$ is linear, and for any k and $v^{*,1}, \ldots, v^{*,k} \in V^*$,

$$B^\natural(v^{*,1} \wedge \cdots \wedge v^{*,k}) := \sum_l v^{*,1} \wedge \cdots \wedge (B^* v^{*,l}) \wedge \cdots \wedge v^{*,k}, \tag{2.4}$$

where $(B^*v^*)(v) := -v^*(Bv)$ for any $v \in V$ and $v^* \in V^*$.

Let $\{v_1, \ldots, v_m\}$ be any basis of V and $\{v^{*,1}, \ldots, v^{*,m}\}$ be its dual basis for V^*. Set $Bv_i = B_i^j v_j$. Then

$$B^\natural = -\sum_{i,j} B_i^j v^{*,i} \wedge i_{v_j}, \tag{2.5}$$

where i_v is the interior multiplication on $\Lambda^*(V^*)$ induced by $v \in V$. One easily verifies that

$$\begin{aligned} &\operatorname{tr}_s[(v^{*,1} \wedge i_{v_1}) \cdots (v^{*,m} \wedge i_{v_m})] = (-1)^m, \\ &\operatorname{tr}_s[v^{*,i_1} \wedge \cdots \wedge v^{*,i_k} \wedge i_{v_{j_1}} \cdots i_{v_{j_l}}] = 0 \end{aligned} \tag{2.6}$$

for any $1 \leqslant i_1 < \cdots < i_k \leqslant m$ and $1 \leqslant j_1 < \cdots < j_l \leqslant m$ with $0 \leqslant k + l < 2m$.

Given a Euclidean metric g^V on V, for any $v \in V$, let v^* be the metric dual of v, and set

$$c(v) = v^* \wedge -i_v, \quad \hat{c}(v) = v^* \wedge +i_v. \tag{2.7}$$

Then for any $u, v \in V$, one has

$$\begin{aligned} &c(u)c(v) + c(v)c(u) = -2g^V(u, v), \\ &\hat{c}(u)\hat{c}(v) + \hat{c}(v)\hat{c}(u) = 2g^V(u, v), \\ &c(u)\hat{c}(v) = -\hat{c}(v)c(u). \end{aligned} \tag{2.8}$$

Also from (2.6) and (2.7), one gets that for any orthonormal basis $\{v_1, \ldots, v_m\}$ of V,

$$\begin{aligned} &\operatorname{tr}_s[\hat{c}(v_1)c(v_1) \cdots \hat{c}(v_m)c(v_m)] = 2^m, \\ &\operatorname{tr}_s[\hat{c}(v_{i_1}) \cdots \hat{c}(v_{i_k})c(v_{j_1}) \cdots c(v_{j_l})] = 0 \end{aligned} \tag{2.9}$$

for any $1 \leqslant i_1 < \cdots < i_k \leqslant m$ and $1 \leqslant j_1 < \cdots < j_l \leqslant m$ with $0 \leqslant k + l < 2m$.

A super vector bundle $E = E_+ \oplus E_-$ over a smooth manifold M is a vector bundle with fibres of super vector spaces. Let $\Omega^*(M, E) = \Gamma(\Lambda^*(T^*M)\hat{\otimes}E)$, which is, in general, an infinite-dimensional super vector space with the natural total $\mathbf{Z}_2$-grading. A superconnection $\mathbf{A}$ on E is an odd-parity first-order differential operator, i.e.,

$$\mathbf{A} : \Omega^{\pm}(M, E) \to \Omega^{\mp}(M, E),$$

which satisfies the following Leibniz rule: for any $\omega \in \Omega^k(M)$ and $s \in \Omega^*(M, E)$,

$$\mathbf{A}(\omega \wedge s) = d\omega \wedge s + (-1)^k \omega \wedge \mathbf{A}s. \tag{2.10}$$

The following two simple identities are crucial in the Chern-Weil theory related to the Mathai-Quillen's superconnection formalism:

$$[\mathbf{A}, \mathbf{A}^2] = 0, \quad \operatorname{tr}_s[\mathbf{A}, B] = d\operatorname{tr}_s[B] \tag{2.11}$$

for any superconnection $\mathbf{A}$ on E and any $B \in \Omega^*(M, \operatorname{End}(E))$.

2.2 A Mathai-Quillen type formula on the Euler characteristic

Recall that in Chern's GBC-formula (1.1), Chern [6] actually obtained a Chern-Weil geometric expression Pfaffian $\operatorname{Pf}(R^{TM})$ of the Euler class $e(M)$ of an oriented and closed manifold M of dimension $2n$ by using a metric-preserving connection on the tangent bundle $\pi : TM \to M$. For any connection ∇^{a} on TM, by applying the Mathai-Quillen's geometric construction of the Thom class (see [14]) to the exterior algebra bundle $\pi^*\Lambda^*(T^*M)$, Feng and Zhang [11] constructed an integrable top-form on TM from the connection ∇^{a} and proved the integral of this form over TM to be the Euler number $\chi(M)$ of M.

Let ∇^{a} be any connection on TM. Then it induces a connection $\nabla^{\Lambda^*(T^*M)}$ on the exterior algebra bundle $\Lambda^*(T^*M)$, which preserves the even/odd $\mathbf{Z}_2$-grading in $\Lambda^*(T^*M)$. Let $\hat{Y}$ denote the tautological section of the pull-back bundle π^*TM:

$$\hat{Y}(x, Y) := Y \in (\pi^*TM)\,|_{(x,Y)}, \tag{2.12}$$

where $(x, Y) \in TM$ with $x \in M$ and $Y \in T_xM$. For any given Euclidean metric g^{TM} on TM, let $\hat{Y}^*$ denote the dual of $\hat{Y}$ with respect to the pull-back metric π^*g^{TM} on π^*TM. Then the Clifford action $c(\hat{Y}) = \hat{Y}^* \wedge -i_{\hat{Y}}$ acts on $\pi^*\Lambda^*(T^*M)$ and exchanges the even/odd grading in $\pi^*\Lambda^*(T^*M)$. Moreover,

$$c(\hat{Y})^2 = -|\hat{Y}|^2_{\pi^*g^{TM}} = -|Y|^2. \tag{2.13}$$

For any $T > 0$, Feng and Zhang [11] used the superconnection

$$A_T = \pi^* \nabla^{\Lambda^*(T^*M)} + Tc(\hat{Y}) \tag{2.14}$$

on the bundle $\pi^*\Lambda^*(T^*M)$ and proved the following Mathai-Quillen type formula on the Euler number $\chi(M)$:

$$\chi(M) = \left(\frac{1}{2\pi}\right)^{2n} \int_{TM} \mathrm{tr}_s[\exp(A_T^2)]. \tag{2.15}$$

A key point in the formula (2.15) is that the connection ∇^{a} on TM need not preserve the metric g^{TM} used to define the Clifford action $c(\hat{Y})$.

For the purpose of this paper, we need to generalize the formula (2.15) slightly. Actually, one can choose any connection ∇ and any Euclidean metric g on the pull-back bundle π^*TM to define a superconnection on $\pi^*\Lambda^*(T^*M) \equiv \Lambda^*(\pi^*T^*M)$: let $\nabla^{\Lambda^*(\pi^*T^*M)}$ denote the lifting of the connection ∇ on $\Lambda^*(\pi^*T^*M)$; let $\hat{Y}_g^*$ denote the dual of $\hat{Y}$ with respect to the metric g and set $c_g(\hat{Y}) = \hat{Y}_g^* \wedge -i_{\hat{Y}}$; then for any $T > 0$,

$$\tilde{A}_T = \nabla^{\Lambda^*(\pi^*T^*M)} + Tc_g(\hat{Y}) \tag{2.16}$$

is also a superconnection on $\Lambda^*(\pi^*T^*M)$. Moreover, by using (2.15) and a transgression argument, one can prove the following slight generalization of (2.15) easily.

Lemma 2.1. *Let M be a closed and oriented manifold of dimension $2n$. Then for any connection ∇ and any Euclidean metric g on π^*TM, if the curvature $R = \nabla^2$ and the metric g have polynomial growth along fibres of TM, then the following formula holds for any $T > 0$:*

$$\chi(M) = \left(\frac{1}{2\pi}\right)^{2n} \int_{TM} \mathrm{tr}_s[\exp(\tilde{A}_T^2)]. \tag{2.17}$$

Proof. Here, we give a direct proof of (2.17).

We first check the formula (2.17) for $g = \pi^*g^{TM}$ and $\nabla = \pi^*\nabla^{g^{TM}}$, where $\nabla^{g^{TM}}$ is the Levi-Civita connection on TM associated with the Riemannian metric g^{TM} on M. Let $\nabla^{\Lambda^*(T^*M)}$ denote the lifting of $\nabla^{g^{TM}}$ on the exterior algebra bundle $\Lambda^*(T^*M)$ and $R^{\Lambda^*(T^*M)}$ be its curvature. So the superconnection defined by (2.16) becomes

$$A_T = \pi^* \nabla^{\Lambda^*(T^*M)} + Tc(\hat{Y}) \tag{2.18}$$

and

$$A_T^2 = (\pi^*\nabla^{\Lambda^*(T^*M)} + Tc(\hat{Y}))^2 = \pi^* R^{\Lambda^*(T^*M)} + Tc(\pi^*\nabla^{g^{TM}}\hat{Y}) - T^2|Y|^2. \tag{2.19}$$

Let $\{e_1, \dots, e_{2n}\}$ be a local oriented orthonormal frame for TM and let $\{e^{*,1}, \dots, e^{*,2n}\}$ denote the dual frame for T^*M. Then by (1.3) and (2.7), one has

$$\pi^* R^{\Lambda^*(T^*M)} = -\sum_{a,b} (\pi^*\Omega_b^a) e^{*,b} \wedge i_{e_a} = -\frac{1}{4}\sum_{a,b} (\pi^*\Omega_b^a)(\hat{c}(e_b) + c(e_b))(\hat{c}(e_a) - c(e_a)). \tag{2.20}$$

We now compute $\mathrm{tr}_s\left[\exp(A_T^2)\right]$ fiberwisely. For the simplicity of computations, we choose a local oriented orthonormal frame field $\{e_1, \dots, e_{2n}\}$ around each $x \in M$ such that $(\nabla^{g^{TM}} e_a)(x) = 0$, $a = 1, \dots, 2n$. Moreover, we write $Y = \sum_a y^a e_a$ around x. Then from (2.19), (2.20), (2.8), (2.9), (1.2) and the degree counting of differential forms, we have

$$\begin{aligned}
&\int_{T_xM} \mathrm{tr}_s[\exp(A_T^2)] \\
&\quad = \int_{T_xM} \mathrm{e}^{-T^2|Y|^2} \mathrm{tr}_s[\exp(\pi^* R^{\Lambda^*(T^*M)} + Tc(\pi^*\nabla^{g^{TM}}\hat{Y}))]
\end{aligned}$$

$$
\begin{aligned}
&= \int_{T_xM} \mathrm{e}^{-T^2|Y|^2} \mathrm{tr}_s\bigg[\exp\bigg(-\frac{1}{4}\pi^*\Omega_b^a(x)(\hat{c}(e_b)+c(e_b))(\hat{c}(e_a)-c(e_a))+Tdy^a c(e_a)\bigg)\bigg] \\
&= \int_{T_xM} \mathrm{e}^{-T^2|Y|^2} \mathrm{tr}_s\bigg[\exp\bigg(-\frac{1}{4}\pi^*\Omega_b^a(x)\hat{c}(e_b)\hat{c}(e_a)\bigg)\exp(Tdy^a c(e_a))\bigg] \\
&= \int_{T_xM} \frac{(-1)^n}{2^n}\mathrm{e}^{-T^2|Y|^2} \mathrm{tr}_s\bigg[\frac{1}{2^n n!}\bigg(\sum_{a,b}\pi^*\Omega_b^a(x)\hat{c}(e_b)\hat{c}(e_a)\bigg)^n \prod_{a=1}^{2n}(1+Tdy^a c(e_a))\bigg] \\
&= \int_{T_xM} \frac{(-1)^n T^{2n}}{2^n}\mathrm{e}^{-T^2|Y|^2} \mathrm{tr}_s\bigg[\pi^*\mathrm{Pf}(R^{TM})(x)\hat{c}(e_1)\cdots\hat{c}(e_{2n})\prod_{a=1}^{2n} dy^a c(e_a)\bigg] \\
&= \bigg(\frac{-1}{2}\bigg)^n \int_{T_xM} T^{2n}\mathrm{e}^{-T^2|Y|^2}\pi^*\mathrm{Pf}(R^{TM})(x)\wedge dy^1\wedge\cdots\wedge dy^{2n}\mathrm{tr}_s[\hat{c}(e_1)c(e_1)\cdots\hat{c}(e_{2n})c(e_{2n})] \\
&= (-2)^n\mathrm{Pf}(R^{TM})(x)\int_{\mathbb{R}^{2n}} T^{2n}\mathrm{e}^{-T^2\sum_a (y^a)^2} dy^1\wedge\cdots\wedge dy^{2n} = (-2\pi)^n\mathrm{Pf}(R^{TM})(x).
\end{aligned}
$$

Therefore, by Chern's formula (1.1), the formula (2.17) holds in the current case.

Now we prove the formula (2.17) for any metric g and any connection ∇ on the pull-back bundle π^*TM under the assumption in Lemma 2.1. For any $T>0$ and $t\in[0,1]$, set

$$
\omega_T = \tilde{A}_T - A_T = \nabla^{\Lambda^*(\pi^*T^*M)} - \pi^*\nabla^{\Lambda^*(T^*M)} + T(\hat{Y}_g^* - \hat{Y}^*)\wedge, \tag{2.21}
$$

$$
A_{T,t} = t\tilde{A}_T + (1-t)A_T = \pi^*\nabla^{\Lambda^*(T^*M)} + t\omega_T + Tc(\hat{Y}), \tag{2.22}
$$

where the superconnections $\tilde{A}_T$ and A_T are defined by (2.16) and (2.18), respectively. Moreover,

$$
A_{T,t}^2 = \pi^*R^{\Lambda^*(T^*M)} + ([\pi^*\nabla^{\Lambda^*(T^*M)}, t\omega_T + Tc(\hat{Y})] + tT[\omega_T, c(\hat{Y})] + t^2\omega_T^2) - T^2|Y|^2. \tag{2.23}
$$

By (2.11), we have

$$
\begin{aligned}
\frac{d}{dt}\mathrm{tr}_s[\exp(A_{T,t}^2)] &= \mathrm{tr}_s\bigg[\bigg(\frac{d}{dt}A_{T,t}^2\bigg)\exp(A_{T,t}^2)\bigg] = \mathrm{tr}_s\bigg[\bigg[A_{T,t}, \frac{d}{dt}A_{T,t}\bigg]\exp(A_{T,t}^2)\bigg] \\
&= \mathrm{tr}_s[[A_{T,t}, \omega_T\exp(A_{T,t}^2)]] = d^{TM}\mathrm{tr}_s[\omega_T\exp(A_{T,t}^2)].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{tr}_s[\exp(\tilde{A}_T^2)] - \mathrm{tr}_s[\exp(A_T^2)] &= \int_0^1 \frac{d}{dt}\mathrm{tr}_s[\exp(A_{T,t}^2)]dt \\
&= \int_0^1 d^{TM}\mathrm{tr}_s[\omega_T\exp(A_{T,t}^2)]dt = d^{TM}\int_0^1 \mathrm{tr}_s[\omega_T\exp(A_{T,t}^2)]dt.
\end{aligned}
$$

From (2.21) and (2.22) and the assumption that the metric g and the curvature $R=\nabla^2$ have polynomial growth along fibres of TM, one verifies easily from (2.23) that $\mathrm{tr}_s[\omega_T\exp(A_{T,t}^2)]$ decays exponentially along fibres of $\pi: TM\to M$, so $\int_{TM/M}\int_0^1 \mathrm{tr}_s[\omega_T\exp(A_{T,t}^2)]dt$ is a well-defined differential form on M. Therefore, we have

$$
\begin{aligned}
&\int_{TM}\mathrm{tr}_s[\exp(\tilde{A}_T^2)] - \int_{TM}\mathrm{tr}_s[\exp(A_T^2)] \\
&\quad= \int_{TM} d^{TM}\int_0^1 \mathrm{tr}_s[\omega_T\exp(A_{T,t}^2)]dt \\
&\quad= \int_M\int_{TM/M} d^{TM}\int_0^1 \mathrm{tr}_s[\omega_T\exp(A_{T,t}^2)]dt \\
&\quad= \int_M d^M\int_{TM/M}\int_0^1 \mathrm{tr}_s[\omega_T\exp(A_{T,t}^2)]dt = 0,
\end{aligned}
$$

from which the lemma follows. □

Set

$$e(TM,\tilde{A}_T)=\left(\frac{1}{2\pi}\right)^{2n}\int_{TM/M}\mathrm{tr}_s[\exp\tilde{A}_T^2]\in\Omega^{2n}(M).$$

From Lemma 2.1, $e(TM,\tilde{A}_T)$ gives rise to a geometric representation of the Euler class $e(TM)$ associated with the superconnection $\tilde{A}_T$.

3 An intrinsic Gauss-Bonnet-Chern formula on Finsler manifolds

In this section, we first give a brief review of some basic definitions and notations in Finsler geometry used in this paper (see [2, 10] for more details). Then starting from Lemma 2.1, we work out a Finslerian Gauss-Bonnet-Chern formula in which no extra vector field is involved.

3.1 A brief review of Finsler geometry

Let M be a smooth manifold of dimension m. For any local coordinate chart $(U;(x^1,\dots,x^m))$ on M, $(\pi^{-1}(U);(x^1,\dots,x^m,y^1,\dots,y^m))$ is an induced local coordinate chart on the total space TM of the tangent bundle $\pi:TM\to M$. Let O denote the zero section of TM and set $TM_o=TM\setminus O$.

To distinguish elements in $\pi^*\Lambda^*(T^*M)$ and π^*TM from $\Lambda^*(T^*M)$ and TM, we decorate the elements in $\pi^*\Lambda^*(T^*M)$ and π^*TM with a $\hat{}$ notation for clarity. We also use the summation convention of Einstein in computations to simplify the notations.

A Finsler metric F on M is a positive smooth function on TM_o satisfying the positive homogeneity $F(x,\lambda Y)=\lambda F(x,Y)$ for any $\lambda>0$ and that the induced fundamental tensor

$$g_F=\frac{1}{2}[F^2]_{y^iy^j}d\hat{x}^i\otimes d\hat{x}^j$$

defines a Euclidean structure on the pull-back bundle $\pi^*TM\to TM_o$. Set

$$G^i=\frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\},\quad N^i_j=\frac{\partial G^i}{\partial y^j},$$

and define

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N^j_i\frac{\partial}{\partial y^j},\quad \delta y^i=dy^i+N^i_jdx^j.$$

Let ∇^{Ch} denote the Chern connection on the pull-back bundle $\pi^*TM\to TM_o$. With respect to the pull-back frame $\{\frac{\partial}{\partial\hat{x}^i}\}$ for π^*TM, set

$$\nabla^{\mathrm{Ch}}\frac{\partial}{\partial\hat{x}^j}=\varpi^i_j\otimes\frac{\partial}{\partial\hat{x}^i}.\tag{3.1}$$

It is well known that $\varpi:=(\varpi^i_j)$ is determined uniquely by the following structure equations:

$$\begin{aligned}&0=dx^j\wedge\varpi^i_j,\\&dg_{ij}=g_{ik}\varpi^k_j+g_{jk}\varpi^k_i+2A_{ijk}\frac{\delta y^k}{F},\end{aligned}\tag{3.2}$$

where $A_{ijk}=\frac{F}{4}[F^2]_{y^iy^jy^k}$ is the Cartan tensor. The first and the second equations of (3.2) are often referred to as the torsion-free and the almost metric-preserving conditions of the Chern connection, respectively. A direct consequence of the torsion-free condition is that ϖ^i_j's are horizontal one-forms on TM_o, which can be written as

$$\varpi^i_j=\Gamma^i_{jk}dx^k\quad\text{and}\quad\Gamma^i_{jk}=\Gamma^i_{kj},\tag{3.3}$$

where

$$\Gamma^i_{jk}=\gamma^i_{jk}-g^{il}\left(A_{ljs}\frac{N^s_k}{F}+A_{lks}\frac{N^s_j}{F}-A_{jks}\frac{N^s_l}{F}\right)\tag{3.4}$$

and

$$\gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{3.5}$$

Furthermore, one has

$$N^i_j = y^k \Gamma^i_{jk} = \gamma^i_{jk} y^k - F^{-1} g^{il} A_{ljk} \gamma^k_{rs} y^r y^s, \tag{3.6}$$

and then

$$\delta y^i = dy^i + y^k \varpi^i_k. \tag{3.7}$$

One the other hand, the almost metric-preserving condition implies that

$$\nabla^{\mathrm{Ch},*}(\hat{Y}^*_{g_F}) = (\nabla^{\mathrm{Ch}} \hat{Y})^*_{g_F}, \tag{3.8}$$

where $\nabla^{\mathrm{Ch},*}$ is the dual connection of the Chern connection on $\pi^* T^* M$.

Let $R^{\mathrm{Ch}} = (\nabla^{\mathrm{Ch}})^2$ be the curvature of the Chern connection ∇^{Ch}, which is an $\mathrm{End}(\pi^* TM)$-valued two-form on TM_o. By the torsion-freeness of the Chern connection, the Chern curvature R^{Ch} is divided into two parts

$$R^{\mathrm{Ch}} = R + P,$$

where R is called the (h-h)-Chern curvature of ∇^{Ch}, which is an $\mathrm{End}(\pi^* TM)$-valued horizontal two-form on TM_o, and P is called the (h-v)-Chern curvature of ∇^{Ch}, which is an $\mathrm{End}(\pi^* TM)$-valued horizontal-vertical two-form on TM_o. Set

$$R \frac{\partial}{\partial \hat{x}^j} = R^i_j \otimes \frac{\partial}{\partial \hat{x}^i}, \quad P \frac{\partial}{\partial \hat{x}^j} = P^i_j \otimes \frac{\partial}{\partial \hat{x}^i}. \tag{3.9}$$

By (3.3), a direct computation shows that (see [2, (3.2.2), (3.3.2) and (3.3.3)])

$$\begin{aligned} R^i_j &= \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l = \frac{1}{2}\left(\frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{hk}\Gamma^h_{jl} - \Gamma^i_{hl}\Gamma^h_{jk}\right) dx^k \wedge dx^l, \\ P^i_j &= P_j{}^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F} = -dx^k \wedge \left(\frac{\partial \Gamma^i_{jk}}{\partial y^l} \delta y^l\right). \end{aligned} \tag{3.10}$$

Sometimes, it is more convenient to use the special g_F-orthonormal local frame $\{e_1, \ldots, e_{2n}\}$ of $\pi^* TM$, which is orthonormal with respect to g_F and satisfies $e_{2n} = \widehat{Y}/F$.

Set

$$\nabla^{\mathrm{Ch}} e_a = \omega^b_a \otimes e_b, \quad R^{\mathrm{Ch}} e_a = \Omega^b_a \otimes e_b = (R^b_a + P^b_a) \otimes e_b. \tag{3.11}$$

Let $\widehat{R}^{\mathrm{Ch}}$ be the skew-symmetrization of R^{Ch} with respect to g_F. Then

$$\widehat{R}^{\mathrm{Ch}} e_a = \sum_b \frac{1}{2}(\Omega^b_a - \Omega^a_b) \otimes e_b =: \widehat{\Omega}^b_a \otimes e_b, \tag{3.12}$$

and the Pfaffian $\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}})$ of $\widehat{R}^{\mathrm{Ch}}$ satisfies

$$\begin{aligned} \mathrm{Pf}(\widehat{R}^{\mathrm{Ch}}) &= \frac{1}{2^n n!} \sum_{a_1,\ldots,a_{2n}=1}^{2n} \epsilon_{a_1 \cdots a_{2n}} \widehat{\Omega}^{a_2}_{a_1} \wedge \cdots \wedge \widehat{\Omega}^{a_{2n}}_{a_{2n-1}} \\ &= \frac{1}{2^n n!} \sum_{a_1,\ldots,a_{2n}=1}^{2n} \epsilon_{a_1 \cdots a_{2n}} \Omega^{a_2}_{a_1} \wedge \cdots \wedge \Omega^{a_{2n}}_{a_{2n-1}}. \end{aligned}$$

The Cartan connection ∇^{Car} is proved to be the symmetrization of the Chern connection $\widehat{\nabla}^{\mathrm{Ch}}$ with respect to the Euclidean structure g_F on $\pi^* TM$. By (3.2), the difference between the Cartan connection and the Chern connection is given by

$$\frac{1}{2}\Theta = \widehat{\nabla}^{\mathrm{Ch}} - \nabla^{\mathrm{Ch}},$$

where $\Theta = g_F^{-1}(\nabla^{\mathrm{Ch}} g_F)$, i.e., $\Theta_i^j = (\nabla^{\mathrm{Ch}} g_F)_{ik} g^{kj} = 2A_{ikl} g^{kj} \frac{\delta y^l}{F}$, is symmetric and called the Bismut-Zhang form or the Cartan endomorphism (see [10]). Set $\Theta e_a = \Theta_a^b e_b$. By the Euler lemma for homogeneous functions, one has $\Theta_a^{2n} = 0$ for $a = 1, \ldots, 2n$.

Similar to [5, Proposition 4.5], we have the following lemma.

Lemma 3.1. *The curvatures of the Cartan connection and the Chern connection satisfy*

$$R^{\mathrm{Car}} = \widehat{R}^{\mathrm{Ch}} + Q, \tag{3.13}$$

where $Q := -\frac{1}{4}\Theta \wedge \Theta$.

Set $Qe_a := Q_a^b \otimes e_b$. By (3.12) and (3.13), the Pfaffian $\mathrm{Pf}(R^{\mathrm{Car}})$ is given by

$$\begin{aligned} \mathrm{Pf}(R^{\mathrm{Car}}) &= \frac{1}{2^n n!} \sum_{a_1,\ldots,a_{2n}=1}^{2n} \epsilon_{a_1\cdots a_{2n}} (\widehat{\Omega}_{a_1}^{a_2} + Q_{a_1}^{a_2}) \wedge \cdots \wedge (\widehat{\Omega}_{a_{2n-1}}^{a_{2n}} + Q_{a_{2n-1}}^{a_{2n}}) \\ &= \frac{1}{2^n n!} \sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} \sum_{a_1,\ldots,a_{2n}=1}^{2n} \epsilon_{a_1\cdots a_{2n}} Q_{a_1}^{a_2} \wedge \cdots \wedge Q_{a_{2k-1}}^{a_{2k}} \wedge \Omega_{a_{2k+1}}^{a_{2k+2}} \wedge \cdots \wedge \Omega_{a_{2n-1}}^{a_{2n}}. \end{aligned} \tag{3.14}$$

Notice that the Finsler metric F on TM_o is homogeneous of degree one, and all the geometric data, such as ∇^{Ch}, R^{Ch}, R and P, can be reduced naturally onto SM. In this paper, we use the same notations to denote their reductions on SM.

3.2 A Finslerian Gauss-Bonnet-Chern formula

For any $r > 0$, set

$$DM(r) := \{Y \in TM \mid F(Y) \leqslant r\}.$$

Let ρ be a non-negative smooth function on TM with $0 \leqslant \rho \leqslant 1$ and $\rho(Y) \equiv 1$ for $F(Y) \leqslant 1/4$ and $\rho(Y) \equiv 0$ for $F(Y) \geqslant 1/2$.

For any connection ∇^{a} on $\pi : TM \to M$, we get an extension of the Chern connection, i.e.,

$$\widetilde{\nabla}_\rho = (1-\rho)\nabla^{\mathrm{Ch}} + \rho\pi^*\nabla^{\mathrm{a}} \tag{3.15}$$

on π^*TM over the total space TM. Clearly, the curvature $(\widetilde{\nabla}_\rho)^2$ is bounded along the fibres of TM. Let $\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)}$ denote the induced connection on $\Lambda^*(\pi^*T^*M)$ of $\widetilde{\nabla}_\rho$.

Let $\tilde{g}_F$ be any Euclidean metric on $\pi^*TM \to TM$ with $\tilde{g}_F = g_F$ over $TM \setminus DM(1/2)$. We first prove the following lemma by Lemma 2.1 for the metric $\tilde{g}_F$ and the connection $\widetilde{\nabla}_\rho$ over TM.

Lemma 3.2. *Let* (M, F) *be a closed and oriented Finsler manifold of dimension* $2n$. *Then for any connection* ∇^{a} *on* $\pi : TM \to M$, *one has*

$$\chi(M) = \int_M e(TM, \nabla^{\mathrm{Ch}}),$$

where

$$\begin{aligned} e(TM, \nabla^{\mathrm{Ch}}) = \frac{1}{(2\pi)^{2n}(2n)!} \bigg\{ &\sum_{k=1}^{n} C_{2n}^{2k} \int_{SM/M} \mathrm{tr}_s[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2k-1}(R^\natural)^k (P^\natural)^{2n-2k}] \\ &+ \int_{SM/M} \mathrm{tr}_s[\theta^\natural (P^\natural)^{2n-1}] \bigg\}, \end{aligned} \tag{3.16}$$

$\mathbf{e} = \hat{Y}|_{SM}$ *and* $R^\natural$, $P^\natural$ *and* $\theta^\natural$ *are the natural liftings of* R, P *and* θ *respectively on* $\Lambda^*(\pi^*T^*M)$, *and* θ *is defined by* $\theta := \nabla^{\mathrm{Ch}} - \pi^*\nabla^{\mathrm{a}}$.

Proof. For any $T>0$, similar to (2.16), we define the following superconnection:

$$\tilde{A}_{\rho,T}=\widetilde{\nabla}_{\rho}^{\Lambda^*(\pi^*T^*M)}+Tc_{\tilde{g}_F}(\hat{Y}).\tag{3.17}$$

Noticing that the curvature $(\widetilde{\nabla}_{\rho})^2$ is bounded along the fibres of TM, we obtain that by (2.17),

$$\chi(M)=\left(\frac{1}{2\pi}\right)^{2n}\int_{TM}\mathrm{tr}_s[\exp(\tilde{A}^2_{\rho,T})].\tag{3.18}$$

Since $\exp(\tilde{A}^2_{\rho,T})$ decays exponentially along fibres of TM and $\chi(M)$ does not depend on $T>0$, we get

$$\chi(M)=\lim_{T\to+\infty}\left(\frac{1}{2\pi}\right)^{2n}\int_{TM}\mathrm{tr}_s[\exp(\tilde{A}^2_{\rho,T})]=\lim_{T\to+\infty}\left(\frac{1}{2\pi}\right)^{2n}\int_{DM(1)}\mathrm{tr}_s[\exp(\tilde{A}^2_{\rho,T})].\tag{3.19}$$

Noting that for any connection ∇^{a} on TM, the curvature $(\pi^*\nabla^{\Lambda^*(T^*M)})^2$ of the connection $\pi^*\nabla^{\Lambda^*(T^*M)}$ involves no vertical differential forms on TM, where $\nabla^{\Lambda^*(T^*M)}$ is the lifting of ∇^{a} on $\Lambda^*(T^*M)$, we have

$$\int_{DM(1)}\mathrm{tr}_s[\exp((\pi^*\nabla^{\Lambda^*(T^*M)})^2)]=0.$$

Therefore, we obtain

$$\begin{aligned}\int_{DM(1)}\mathrm{tr}_s[\exp(\tilde{A}^2_{\rho,T})]&=\int_{DM(1)}(\mathrm{tr}_s[\exp(\tilde{A}^2_{\rho,T})]-\mathrm{tr}_s[\exp((\widetilde{\nabla}_{\rho}^{\Lambda^*(\pi^*T^*M)})^2)])\\&\quad+\int_{DM(1)}(\mathrm{tr}_s[\exp((\widetilde{\nabla}_{\rho}^{\Lambda^*(\pi^*T^*M)})^2)]-\mathrm{tr}_s[\exp((\pi^*\nabla^{\Lambda^*(T^*M)})^2)]).\end{aligned}\tag{3.20}$$

For the first term on the right-hand side of (3.20), we have

$$\begin{aligned}&\int_{DM(1)}(\mathrm{tr}_s[\exp(\tilde{A}^2_{\rho,T})]-\mathrm{tr}_s[\exp((\widetilde{\nabla}_{\rho}^{\Lambda^*(\pi^*T^*M)})^2)])\\&=\int_{DM(1)}\int_0^1\frac{\partial}{\partial t}\mathrm{tr}_s[\exp((\widetilde{\nabla}_{\rho}^{\Lambda^*(\pi^*T^*M)}+tTc_{\tilde{g}_F}(\hat{Y}))^2)]dt\\&=\int_{DM(1)}\int_0^1 d^{TM}\mathrm{tr}_s[Tc_{\tilde{g}_F}(\hat{Y})\exp((\widetilde{\nabla}_{\rho}^{\Lambda^*(\pi^*T^*M)}+tTc_{\tilde{g}_F}(\hat{Y}))^2)]dt\\&=\int_{DM(1)}d^{TM}\int_0^1\mathrm{tr}_s[Tc_{\tilde{g}_F}(\hat{Y})\exp((\widetilde{\nabla}_{\rho}^{\Lambda^*(\pi^*T^*M)}+tTc_{\tilde{g}_F}(\hat{Y}))^2)]dt\\&=\int_{SM}i^*\int_0^1\mathrm{tr}_s[Tc_{\tilde{g}_F}(\hat{Y})\exp((\widetilde{\nabla}_{\rho}^{\Lambda^*(\pi^*T^*M)}+tTc_{\tilde{g}_F}(\hat{Y}))^2)]dt\\&=\int_{SM}\int_0^1\mathrm{tr}_s[Tc(\mathbf{e})\exp((\nabla^{\mathrm{Ch},\natural}+tTc(\mathbf{e}))^2)]dt\\&=\int_{SM}\int_0^1\mathrm{e}^{-t^2T^2}\mathrm{tr}_s[Tc(\mathbf{e})\exp(R^{\mathrm{Ch},\natural}+tT[\nabla^{\mathrm{Ch},\natural},c(\mathbf{e})])]dt\\&=\int_{SM}\int_0^1\mathrm{e}^{-t^2T^2}\mathrm{tr}_s[Tc(\mathbf{e})\exp(R^{\mathrm{Ch},\natural}+tTc(\nabla^{\mathrm{Ch}}\mathbf{e}))]dt,\end{aligned}\tag{3.21}$$

where $i:SM\hookrightarrow TM$ denotes the natural embedding of the unit sphere bundle SM into TM, and $\nabla^{\mathrm{Ch},\natural}$ is the lifting of ∇^{Ch} on $\Lambda^*(\pi^*T^*M)$ and $R^{\mathrm{Ch},\natural}=(\nabla^{\mathrm{Ch},\natural})^2=R^\natural+P^\natural$, and the last equation in (3.21) comes from (3.8).

Now noticing that the term $R^{\mathrm{Ch},\natural}$ is a two-form with two Clifford elements and the term $c(\nabla^{\mathrm{Ch}}\mathbf{e})$ is a one-form with one Clifford element, hence by the property (2.6) or (2.9) of the supertrace and the degree counting, we get from (3.21) that

$$\lim_{T\to+\infty}\int_{DM(1)}(\mathrm{tr}_s[\exp(\tilde{A}^2_{\rho,T})]-\mathrm{tr}_s[\exp((\widetilde{\nabla}_{\rho}^{\Lambda^*(\pi^*T^*M)})^2)])$$

$$
\begin{aligned}
&= \lim_{T\to+\infty}\int_{SM}\int_0^1 \mathrm{e}^{-t^2T^2}\mathrm{tr}_s[Tc(\mathbf{e})\exp(R^{\mathrm{Ch},\natural}+tTc(\nabla^{\mathrm{Ch}}\mathbf{e}))]dt,\\
&= \lim_{T\to+\infty}\int_{SM}\int_0^1 \mathrm{e}^{-t^2T^2}\mathrm{tr}_s[Tc(\mathbf{e})\exp(tTc(\nabla^{\mathrm{Ch}}\mathbf{e}))\exp(R^{\mathrm{Ch},\natural})]dt,\\
&= \sum_{k=1}^n \frac{1}{(2k-1)!}\int_{SM}\mathrm{tr}_s[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2k-1}\exp(R^{\mathrm{Ch},\natural})]\cdot\lim_{T\to+\infty}\int_0^1 \mathrm{e}^{-t^2T^2}T^{2k}t^{2k-1}dt,\\
&= \sum_{k=1}^n \frac{(k-1)!}{2(2k-1)!k!(2n-2k)!}\int_{SM}\mathrm{tr}_s[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2k-1}(R^\natural)^k(P^\natural)^{2n-2k}]\\
&= \sum_{k=1}^n \frac{C_{2n}^{2k}}{(2n)!}\int_{SM}\mathrm{tr}_s[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2k-1}(R^\natural)^k(P^\natural)^{2n-2k}]. \qquad (3.22)
\end{aligned}
$$

For the second term on the right-hand side of (3.20), by setting

$$\theta_\rho = \widetilde{\nabla}_\rho - \pi^*\nabla^{\mathrm{a}},$$

we have

$$
\begin{aligned}
&\int_{DM(1)}(\mathrm{tr}_s[\exp((\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)})^2)]-\mathrm{tr}_s[\exp((\pi^*\nabla^{\Lambda^*(T^*M)})^2)])\\
&\quad= \int_{DM(1)}\int_0^1 \frac{\partial}{\partial t}\mathrm{tr}_s[\exp((\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)}-(1-t)\theta_\rho^\natural)^2)]dt\\
&\quad= \int_{DM(1)} d^{TM}\int_0^1 \mathrm{tr}_s[\theta_\rho^\natural\exp((\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)}-(1-t)\theta_\rho^\natural)^2)]dt\\
&\quad= \int_{SM} i^*\int_0^1 \mathrm{tr}_s[\theta_\rho^\natural\exp((\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)}-(1-t)\theta_\rho^\natural)^2)]dt\\
&\quad= \int_{SM}\int_0^1 \mathrm{tr}_s[\theta^\natural\exp(R^{\mathrm{Ch},\natural}-(1-t)[\nabla^{\mathrm{Ch},\natural},\theta^\natural]+(1-t)^2\theta^\natural\wedge\theta^\natural)]dt. \qquad (3.23)
\end{aligned}
$$

Note that

$$
\begin{aligned}
[\nabla^{\mathrm{Ch},\natural},\theta^\natural] &= [\nabla^{\mathrm{Ch},\natural}-\pi^*\nabla^{\Lambda^*(T^*M)},\theta^\natural]+[\pi^*\nabla^{\Lambda^*(T^*M)},\nabla^{\mathrm{Ch},\natural}-\pi^*\nabla^{\Lambda^*(T^*M)}]\\
&= [\theta^\natural,\theta^\natural]-[\pi^*\nabla^{\Lambda^*(T^*M)},\pi^*\nabla^{\Lambda^*(T^*M)}]+[\pi^*\nabla^{\Lambda^*(T^*M)},\nabla^{\mathrm{Ch},\natural}]\\
&= 2\theta^\natural\wedge\theta^\natural-2(\pi^*\nabla^{\Lambda^*(T^*M)})^2+[\pi^*\nabla^{\Lambda^*(T^*M)}-\nabla^{\mathrm{Ch},\natural},\nabla^{\mathrm{Ch},\natural}]+[\nabla^{\mathrm{Ch},\natural},\nabla^{\mathrm{Ch},\natural}]\\
&= 2\theta^\natural\wedge\theta^\natural-2(\pi^*\nabla^{\Lambda^*(T^*M)})^2-[\nabla^{\mathrm{Ch},\natural},\theta^\natural]+2R^{\mathrm{Ch},\natural},
\end{aligned}
$$

so

$$[\nabla^{\mathrm{Ch},\natural},\theta^\natural]=\theta^\natural\wedge\theta^\natural-(\pi^*\nabla^{\Lambda^*(T^*M)})^2+R^{\mathrm{Ch},\natural}. \qquad (3.24)$$

Combining (3.23) and (3.24), we get

$$
\begin{aligned}
&\int_{DM(1)}(\mathrm{tr}_s[\exp((\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)})^2)]-\mathrm{tr}_s[\exp((\pi^*\nabla^{\Lambda^*(T^*M)})^2)])\\
&\quad= \int_{SM}\int_0^1 \mathrm{tr}_s[\theta^\natural\exp(R^{\mathrm{Ch},\natural}+(1-t)((\pi^*\nabla^{\Lambda^*(T^*M)})^2-R^{\mathrm{Ch},\natural}-\theta^\natural\wedge\theta^\natural)+(1-t)^2\theta^\natural\wedge\theta^\natural)]dt\\
&\quad= \int_{SM}\int_0^1 \mathrm{tr}_s[\theta^\natural\exp(tR^{\mathrm{Ch},\natural}+(1-t)(\pi^*\nabla^{\Lambda^*(T^*M)})^2-t(1-t)\theta^\natural\wedge\theta^\natural)]dt\\
&\quad= \int_{SM}\int_0^1 \mathrm{tr}_s[\theta^\natural\exp(tP^\natural+tR^\natural+(1-t)(\pi^*\nabla^{\Lambda^*(T^*M)})^2-t(1-t)\theta^\natural\wedge\theta^\natural)]dt. \qquad (3.25)
\end{aligned}
$$

By (3.3), the term $\theta^\natural$ is an $\mathrm{End}(\Lambda^*(\pi^*T^*M))$-valued horizontal one-form, so

$$tR^\natural+(1-t)(\pi^*\nabla^{\Lambda^*(T^*M)})^2-t(1-t)\theta^\natural\wedge\theta^\natural$$

is an $\mathrm{End}(\Lambda^*(\pi^*T^*M))$-valued horizontal two-form. Hence from (3.25), we get

$$\begin{aligned}
&\int_{DM(1)} (\mathrm{tr}_s[\exp((\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)})^2)] - \mathrm{tr}_s[\exp((\pi^*\nabla^{\Lambda^*(T^*M)})^2)]) \\
&\quad = \int_{SM}\int_0^1 \mathrm{tr}_s[\theta^\natural \exp(tP^\natural)]dt \\
&\quad = \int_{SM} \frac{1}{(2n-1)!}\int_0^1 t^{2n-1}dt\ \mathrm{tr}_s[\theta^\natural (P^\natural)^{2n-1}] \\
&\quad = \frac{1}{(2n)!}\int_{SM} \mathrm{tr}_s[\theta^\natural (P^\natural)^{2n-1}]. \qquad (3.26)
\end{aligned}$$

By (3.19), (3.20), (3.22) and (3.26), we complete the proof of Lemma 3.2. □

Remark 3.3. The Chern connection is essential to get the formula (3.16), in which the first term follows from the almost metric-preserving property, while the second term follows from the torsion-freeness.

In the following, by using the induced homogeneous coordinate charts (x^i, y^i) on SM, we prove Theorem 1.1 by working out a local version of the formula (3.16). More precisely, we give an explicit GBC-integrand on M through the integration along the fibres, in which no information of the pull-back connection $\pi^*\nabla^{\mathrm{a}}$ is involved.

Proof of Theorem 1.1. We first compute the term $\mathrm{tr}_s[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2k-1}(R^\natural)^k(P^\natural)^{2n-2k}]$ for $k = 1, \ldots, n$.

From (3.1), (3.9) and (2.5), with respect to the pull-back frame $\{\frac{\partial}{\partial \hat{x}^j}\}$ of π^*TM, we have

$$\varpi^\natural = -\varpi_i^j d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}}, \quad R^\natural = -R_i^j d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}}, \quad P^\natural = -P_i^j d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}}. \qquad (3.27)$$

We also have

$$c(\mathbf{e}) = \omega \wedge -i_{\mathbf{e}} = F_{y^j} d\hat{x}^j \wedge -\frac{y^i}{F} i_{\frac{\partial}{\partial \hat{x}^i}}, \qquad (3.28)$$

where $\omega = \mathbf{e}^* = F_{y^i}dx^i$ is the Hilbert form on SM. Define

$$(\nabla^{\mathrm{Ch}}\mathbf{e})^i := \frac{\delta y^i}{F} - \frac{y^i}{F} d\log F, \quad (\nabla^{\mathrm{Ch},*}\omega)_i := g_{ik}\left(\frac{\delta y^k}{F} - \frac{y^k}{F} d\log F\right), \qquad (3.29)$$

$$\Upsilon_i^j := (\nabla^{\mathrm{Ch},*}\omega)_i (\nabla^{\mathrm{Ch}}\mathbf{e})^j, \quad \Xi_i^j := \left(F_{y^i}(\nabla^{\mathrm{Ch}}\mathbf{e})^j - \frac{y^j}{F}(\nabla^{\mathrm{Ch},*}\omega)_i\right). \qquad (3.30)$$

By (3.8), we have

$$\begin{aligned}
c(\nabla^{\mathrm{Ch}}\mathbf{e}) &= (\nabla^{\mathrm{Ch}}\mathbf{e})^* \wedge -i_{\nabla^{\mathrm{Ch}}\mathbf{e}} = (\nabla^{\mathrm{Ch},*}\omega) \wedge -i_{\nabla^{\mathrm{Ch}}\mathbf{e}} \\
&= (\nabla^{\mathrm{Ch},*}\omega)_j d\hat{x}^j \wedge -(\nabla^{\mathrm{Ch}}\mathbf{e})^i i_{\frac{\partial}{\partial \hat{x}^i}}. \qquad (3.31)
\end{aligned}$$

From (3.27)–(3.31), we get

$$\begin{aligned}
&\mathrm{tr}_s[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2k-1}(R^\natural)^k(P^\natural)^{2n-2k}] \\
&\quad = \mathrm{tr}_s[(R^\natural)^k(P^\natural)^{2n-2k}c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2k-1}c(\mathbf{e})] \\
&\quad = \mathrm{tr}_s\bigg[(-R_i^j d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}})^k(-P_i^j d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}})^{2n-2k}((\nabla^{\mathrm{Ch},*}\omega)_j d\hat{x}^j \wedge -(\nabla^{\mathrm{Ch}}\mathbf{e})^i i_{\frac{\partial}{\partial \hat{x}^i}})^{2k-2} \\
&\qquad \cdot ((\nabla^{\mathrm{Ch},*}\omega)_p d\hat{x}^p \wedge -(\nabla^{\mathrm{Ch}}\mathbf{e})^q i_{\frac{\partial}{\partial \hat{x}^q}})\left(F_{y^l}d\hat{x}^l \wedge -\frac{y^r}{F} i_{\frac{\partial}{\partial \hat{x}^r}}\right)\bigg] \\
&\quad = (-1)^k \mathrm{tr}_s\bigg[(R_{i_1}^{j_1}\cdots R_{i_k}^{j_k} d\hat{x}^{i_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_1}}} \cdots d\hat{x}^{i_k} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_k}}})(P_{s_1}^{t_1}\cdots P_{s_{2n-2k}}^{t_{2n-2k}} d\hat{x}^{s_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{t_1}}} \\
&\qquad \cdots d\hat{x}^{s_{2n-2k}} \wedge i_{\frac{\partial}{\partial \hat{x}^{t_{2n-2k}}}}) \cdot C_{2k-2}^{k-1}((\nabla^{\mathrm{Ch},*}\omega)_{p_1}(\nabla^{\mathrm{Ch}}\mathbf{e})^{q_1}\cdots(\nabla^{\mathrm{Ch},*}\omega)_{p_{k-1}}(\nabla^{\mathrm{Ch}}\mathbf{e})^{q_{k-1}} \\
&\qquad \cdot d\hat{x}^{p_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{q_1}}} \cdots d\hat{x}^{p_{k-1}} \wedge i_{\frac{\partial}{\partial \hat{x}^{q_{k-1}}}})\left(F_{y^l}(\nabla^{\mathrm{Ch}}\mathbf{e})^q d\hat{x}^l \wedge i_{\frac{\partial}{\partial \hat{x}^q}} - \frac{y^r}{F}(\nabla^{\mathrm{Ch},*}\omega)_p d\hat{x}^p \wedge i_{\frac{\partial}{\partial \hat{x}^r}}\right)\bigg]
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^k C_{2k-2}^{k-1} \mathrm{tr}_s[(R_{i_1}^{j_1} \cdots R_{i_k}^{j_k} d\hat{x}^{i_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_1}}} \cdots d\hat{x}^{i_k} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_k}}})(P_{s_1}^{t_1} \cdots P_{s_{2n-2k}}^{t_{2n-2k}} d\hat{x}^{s_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{t_1}}} \\
&\quad \cdots d\hat{x}^{s_{2n-2k}} \wedge i_{\frac{\partial}{\partial \hat{x}^{t_{2n-2k}}}})(\Upsilon_{p_1}^{q_1} \cdots \Upsilon_{p_{k-1}}^{q_{k-1}} d\hat{x}^{p_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{q_1}}} \cdots d\hat{x}^{p_{k-1}} \wedge i_{\frac{\partial}{\partial \hat{x}^{q_{k-1}}}})(\Xi_l^r d\hat{x}^l \wedge i_{\frac{\partial}{\partial \hat{x}^r}})] \\
&= (-1)^k C_{2k-2}^{k-1} \mathrm{tr}_s[R_{i_1}^{j_1} \cdots R_{i_k}^{j_k} P_{s_1}^{t_1} \cdots P_{s_{2n-2k}}^{t_{2n-2k}} \Upsilon_{p_1}^{q_1} \cdots \Upsilon_{p_{k-1}}^{q_{k-1}} \Xi_l^r d\hat{x}^{i_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_1}}} \cdots d\hat{x}^{i_k} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_k}}} \\
&\quad \cdot d\hat{x}^{s_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{t_1}}} \cdots d\hat{x}^{s_{2n-2k}} \wedge i_{\frac{\partial}{\partial \hat{x}^{t_{2n-2k}}}} d\hat{x}^{p_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{q_1}}} \cdots d\hat{x}^{p_{k-1}} \wedge i_{\frac{\partial}{\partial \hat{x}^{q_{k-1}}}} d\hat{x}^l \wedge i_{\frac{\partial}{\partial \hat{x}^r}}] \\
&= (-1)^k C_{2k-2}^{k-1} R_{i_1}^{j_1} \cdots R_{i_k}^{j_k} P_{i_{k+1}}^{j_{k+1}} \cdots P_{i_{2n-k}}^{j_{2n-k}} \Upsilon_{i_{2n-k+1}}^{j_{2n-k+1}} \cdots \Upsilon_{i_{2n-1}}^{j_{2n-1}} \Xi_{i_{2n}}^{j_{2n}} \\
&\quad \cdot \mathrm{tr}_s[d\hat{x}^{i_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_1}}} \cdots d\hat{x}^{i_{2n}} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_{2n}}}}] \\
&= (-1)^k C_{2k-2}^{k-1} \delta_{j_1 \cdots j_{2n}}^{i_1 \cdots i_{2n}} R_{i_1}^{j_1} \cdots R_{i_k}^{j_k} P_{i_{k+1}}^{j_{k+1}} \cdots P_{i_{2n-k}}^{j_{2n-k}} \Upsilon_{i_{2n-k+1}}^{j_{2n-k+1}} \cdots \Upsilon_{i_{2n-1}}^{j_{2n-1}} \Xi_{i_{2n}}^{j_{2n}}. \qquad (3.32)
\end{aligned}
$$

Now we compute the second term $\mathrm{tr}_s[\theta^\natural (P^\natural)^{2n-1}]$ in (3.16). Recall that θ is defined by $\theta = \nabla^{\mathrm{Ch}} - \pi^* \nabla^{\mathrm{a}}$ in Lemma 3.2. Let $\vartheta = (\vartheta_i^j)$ be the connection matrix of ∇^{a} with respect to the frame $\{\frac{\partial}{\partial \hat{x}^i}\}$. Then we get

$$
\theta^\natural = -(\varpi_i^j - \pi^* \vartheta_i^j) d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}}.
$$

Using (2.6), we get

$$
\begin{aligned}
\mathrm{tr}_s[\theta^\natural (P^\natural)^{2n-1}] &= \mathrm{tr}_s[-(\varpi_i^j - \pi^* \vartheta_i^j) d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}} (-P_k^l d\hat{x}^k \wedge i_{\frac{\partial}{\partial \hat{x}^l}})^{2n-1}] \\
&= P_{i_1}^{j_1} \cdots P_{i_{2n-1}}^{j_{2n-1}} (\varpi_{i_{2n}}^{j_{2n}} - \pi^* \vartheta_{i_{2n}}^{j_{2n}}) \mathrm{tr}_s[d\hat{x}^{i_1} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_1}}} \cdots d\hat{x}^{i_{2n}} \wedge i_{\frac{\partial}{\partial \hat{x}^{j_{2n}}}}] \\
&= \delta_{j_1 \cdots j_{2n}}^{i_1 \cdots i_{2n}} P_{i_1}^{j_1} \cdots P_{i_{2n-1}}^{j_{2n-1}} (\varpi_{i_{2n}}^{j_{2n}} - \pi^* \vartheta_{i_{2n}}^{j_{2n}}). \qquad (3.33)
\end{aligned}
$$

Furthermore, by using (3.10), we get

$$
\int_{SM/M} P_{i_1}^{j_1} \cdots P_{i_{2n-1}}^{j_{2n-1}} (\pi^* \vartheta_{i_{2n}}^{j_{2n}}) = \vartheta_{i_{2n}}^{j_{2n}} \int_{SM/M} P_{i_1}^{j_1} \cdots P_{i_{2n-1}}^{j_{2n-1}} = 0. \qquad (3.34)
$$

So combining (3.33) and (3.34), we have

$$
\int_{SM/M} \mathrm{tr}_s[\theta^\natural (P^\natural)^{2n-1}] = \int_{SM/M} \delta_{j_1 \cdots j_{2n}}^{i_1 \cdots i_{2n}} P_{i_1}^{j_1} \cdots P_{i_{2n-1}}^{j_{2n-1}} \varpi_{i_{2n}}^{j_{2n}}. \qquad (3.35)
$$

Finally, from Lemma 3.2, (3.32) and (3.35), we finish the proof. □

Remark 3.4. Note that the terms

$$
\delta_{j_1 \cdots j_{2n}}^{i_1 \cdots i_{2n}} R_{i_1}^{j_1} \cdots R_{i_k}^{j_k} P_{i_{k+1}}^{j_{k+1}} \cdots P_{i_{2n-k}}^{j_{2n-k}} \Upsilon_{i_{2n-k+1}}^{j_{2n-k+1}} \cdots \Upsilon_{i_{2n-1}}^{j_{2n-1}} \Xi_{i_{2n}}^{j_{2n}}
$$

are globally defined differential forms on SM, while the term $\delta_{j_1 \cdots j_{2n}}^{i_1 \cdots i_{2n}} P_{i_1}^{j_1} \cdots P_{i_{2n-1}}^{j_{2n-1}} \varpi_{i_{2n}}^{j_{2n}}$ is not. However, the vertical exactness property (3.10) of the (h-v)-Chern curvature P guarantees that the following integral along fibres:

$$
\int_{SM/M} \delta_{j_1 \cdots j_{2n}}^{i_1 \cdots i_{2n}} P_{i_1}^{j_1} \cdots P_{i_{2n-1}}^{j_{2n-1}} \varpi_{i_{2n}}^{j_{2n}}
$$

is a well-defined global differential form on M.

4 Some special Finsler spaces

In this section, we investigate the GBC-formulae for some special Finsler spaces.

4.1 Finsler surfaces

As an example of Theorem 1.1, we give an explicit GBC-formula for a closed and oriented Finsler surface (M,F). Note that in the induced homogeneous coordinate charts (x^i,y^i) on SM, we obtain that from (1.7),

$$\chi(M)=\frac{1}{8\pi^2}\bigg\{-\int_{SM}\delta^{i_1i_2}_{j_1j_2}R^{j_1}_{i_1}\Xi^{j_2}_{i_2}+\int_M\int_{SM/M}\delta^{i_1i_2}_{j_1j_2}P^{j_1}_{i_1}\varpi^{j_2}_{i_2}\bigg\}. \tag{4.1}$$

For further investigation, it is more convenient to rewrite (4.1) with respect to the following special g_F-orthonormal oriented frame $\{e_1,e_2\}$, where

$$e_1:=\frac{F_{y^2}}{\sqrt{g}}\frac{\partial}{\partial\hat{x}^1}-\frac{F_{y^1}}{\sqrt{g}}\frac{\partial}{\partial\hat{x}^2},\quad e_2:=\frac{y^1}{F}\frac{\partial}{\partial\hat{x}^1}+\frac{y^2}{F}\frac{\partial}{\partial\hat{x}^2}.$$

In this case, the dual frame $\{\omega^1,\omega^2\}$ is given by

$$\omega^1=\frac{\sqrt{g}}{F}y^2d\hat{x}^1-\frac{\sqrt{g}}{F}y^1d\hat{x}^2,\quad \omega^2=F_{y^1}d\hat{x}^1+F_{y^2}d\hat{x}^2.$$

Set

$$\omega^3:=\omega^1_2=-\omega^2_1=\frac{\sqrt{g}}{F}\left(y^2\frac{\delta y^1}{F}-y^1\frac{\delta y^2}{F}\right).$$

Then under the special g_F-orthonormal frame above, the Chern curvature forms are

$$(R^{\mathrm{Ch}})^b_a=R^b_a+P^b_a={R_a}^b{}_{12}\omega^1\omega^2+{P_a}^b{}_{11}\omega^1\omega^3+{P_a}^b{}_{21}\omega^2\omega^3.$$

By (4.1), one easily verifies the following corollary .

Corollary 4.1. *For any closed and oriented Finsler surfaces (M,F), we have*

$$\begin{aligned}\chi(M)=\frac{1}{(2\pi)^2}\bigg\{&\int_{SM}{R_1}^2{}_{12}\omega^1\omega^2\omega^3-\int_M\int_{SM/M}\frac{1}{F^3}(G_1y^1+G_2y^2){P_1}^1{}_{11}\omega^1\omega^2\omega^3\\&+\int_M\int_{SM/M}\left[\frac{F_{y^2}}{\sqrt{g}}\left((\log F)_{x^1}-\frac{1}{F^2}G_1\right)-\frac{F_{y^1}}{\sqrt{g}}\left((\log F)_{x^2}-\frac{1}{F^2}G_2\right)\right]{P_2}^1{}_{11}\omega^1\omega^2\omega^3\bigg\},\end{aligned} \tag{4.2}$$

where $G_i:=\frac{1}{4}(y^j[F^2]_{y^ix^j}-[F^2]_{x^i})$.

Now we assume that (M,F) is a Landsberg surface, i.e., ${P_2}^1{}_{11}=0$. Due to an observation of Chern (see [9]), ${R_1}^2{}_{12}\omega^1\omega^2$ in fact lives on M. For closed and oriented Landsberg surfaces, Bao-Chern's formula (1.5) is the same as (1.8), which has the following form:

$$\int_M-{R_1}^2{}_{12}\omega^1\omega^2=\chi(M)\mathrm{Vol}(\mathrm{Finsler}S^1). \tag{4.3}$$

The following integral formula for Landsberg surfaces is interesting.

Corollary 4.2. *Let (M,F) be a closed and oriented Landsberg surface. A Gauss-Bonnet type formula for the Chern curvature holds*:

$$[(\mathrm{Vol}(\mathrm{Finsler}S^1))^2-(2\pi)^2]\chi(M)=\int_M\int_{SM/M}\frac{1}{F^3}(G_1y^1+G_2y^2){P_1}^1{}_{11}\omega^1\omega^2\omega^3. \tag{4.4}$$

Proof. When the surface (M,F) in Corollary 4.1 is a Landsberg space, we get

$$\chi(M)=\frac{1}{(2\pi)^2}\bigg\{\int_{SM}{R_1}^2{}_{12}\omega^1\omega^2\omega^3-\int_M\int_{SM/M}\frac{1}{F^3}(G_1y^1+G_2y^2){P_1}^1{}_{11}\omega^1\omega^2\omega^3\bigg\}. \tag{4.5}$$

Note that for a Landsberg surface, the volume function

$$\int_{SM/M}-\omega^3=\mathrm{Vol}(\mathrm{Finsler}S^1) \tag{4.6}$$

is constant. As a consequence of (4.3), (4.5) and (4.6), we get (4.4). □

By Corollary 4.2, closed Berwald surfaces either are tori or have 2π as the length of their Finsler circles. This fact can be derived from Szabó's rigidity theorem (see [2, p. 278]), which shows that any of the Berwald surfaces must be locally Minkowskian or Riemannian. By (4.1) or (4.3), a closed locally Minkowskian surface has zero Euler number.

4.2 A Finslerian GBC-formula for Berwald spaces

Recall that a Finsler manifold (M,F) is a Berwald manifold if and only if the (h-v)-Chern curvature P vanishes. Moreover, for a Berwald space (M,F), the Chern connection is the pull-back of the Levi-Civita connection on $TM \to M$ for a certain Riemannian metric g^{TM} on M (see [2, Chapter 10] for more details). Note that when P vanishes, the formula (3.16) becomes very simple, from which we deduce the following Finslerian GBC-formula for Berwald spaces easily.

Theorem 4.3. *Let (M,F) be a closed and oriented Berwald space of dimension $2n$. Then one has*

$$\chi(M)=\left(\frac{-1}{2\pi}\right)^n\frac{1}{\mathrm{Vol}(S^{2n-1})}\int_M\int_{SM/M}\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}})\wedge\omega_1^{2n}\wedge\cdots\wedge\omega_{2n-1}^{2n}, \tag{4.7}$$

where $\omega_1^{2n}\wedge\cdots\wedge\omega_{2n-1}^{2n}$ gives the volume form of the fibre when restricted to a fibre of SM.

Proof. Since the (h-v)-Chern curvature P vanishes for Berwald spaces, from Lemma 3.2 we get

$$\chi(M)=\frac{1}{(2\pi)^{2n}(2n)!}\int_{SM}\mathrm{tr}_s[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2n-1}(R^\natural)^n].$$

Hence, under the special g_F-orthonormal frame $\{e_1,\dots,e_{2n}\}$ of π^*TM with $e_{2n}=\mathbf{e}$, we have

$$\begin{aligned}
\chi(M)&=\frac{1}{(2\pi)^{2n}(2n)!}\int_{SM}\mathrm{tr}_s[c(\mathbf{e})c(\nabla^{\mathrm{Ch}}\mathbf{e})^{2n-1}(R^\natural)^n]\\
&=\frac{1}{(2\pi)^{2n}(2n)!}\int_{SM}\mathrm{tr}_s\left[c(e_{2n})\left(\sum_{\gamma=1}^{2n-1}\omega_{2n}^{\gamma}c(e_\gamma)\right)^{2n-1}(R^\natural)^n\right]\\
&=-\frac{1}{(2\pi)^{2n}(2n)2^{2n}}\int_{SM}\mathrm{tr}_s[\omega_{2n}^1\wedge\cdots\wedge\omega_{2n}^{2n-1}c(e_1)\cdots c(e_{2n})(\widehat{R}_a^b\hat{c}(e_a)\hat{c}(e_b))^n]\\
&=\frac{1}{(2\pi)^{2n}(2n)2^{2n}}\int_{SM}\epsilon_{a_1\cdots a_{2n}}\widehat{R}_{a_1}^{a_2}\wedge\cdots\wedge\widehat{R}_{a_{2n-1}}^{a_{2n}}\wedge\omega_1^{2n}\wedge\cdots\wedge\omega_{2n-1}^{2n}\\
&\quad\cdot\mathrm{tr}_s[c(e_1)\cdots c(e_{2n})\hat{c}(e_1)\cdots\hat{c}(e_{2n})]\\
&=\frac{(-1)^n}{(2\pi)^{2n}(2n)}\int_{SM}\epsilon_{a_1\cdots a_{2n}}\widehat{R}_{a_1}^{a_2}\wedge\cdots\wedge\widehat{R}_{a_{2n-1}}^{a_{2n}}\wedge\omega_1^{2n}\wedge\cdots\wedge\omega_{2n-1}^{2n}\\
&=\frac{(-1)^n(n-1)!}{2^{n+1}\pi^{2n}}\int_{SM}\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}})\wedge\omega_1^{2n}\wedge\cdots\wedge\omega_{2n-1}^{2n}\\
&=\left(\frac{-1}{2\pi}\right)^n\frac{1}{\mathrm{Vol}(S^{2n-1})}\int_M\int_{SM/M}\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}})\wedge\omega_1^{2n}\wedge\cdots\wedge\omega_{2n-1}^{2n}.
\end{aligned}$$

This completes the proof. □

Remark 4.4. One should notice that in (4.7), the datum $\widehat{R}^{\mathrm{Ch}}$, defined by (3.12), is the skew-symmetrization of R^{Ch} with respect to g_F. Hence the differential form $\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}})$ is dependent in general on the vertical coordinates y^i. However, if the Finsler metric F is induced by a Riemannian metric g^{TM} on M, then M itself is a Riemannian manifold. In this case, $\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}})$ is exactly the Pfaffian $\mathrm{Pf}(R^{TM})$ defined by (1.2), which is constant along fibres of SM, and therefore, we recover Chern's formula (1.1) from the formula (4.7) easily.

5 A general Lichnerowicz formula for Finsler manifolds

In this section, by using Lemma 2.1 and a geometric localization procedure, we prove Theorem 1.2, which gives a precise form of Lichnerowicz's original GBC-formula (1.4). The proof depends on a series of lemmas below.

Assume that X is a vector field on M with isolated zeros. Then the zero set $Z(X)$ is finite. The normalized vector field $[X]$ on $M \setminus Z(X)$ is defined by $[X](x) = X(x)/F(x, X)$ for any $x \in M \setminus Z(X)$. We choose a background Riemannian metric g^{TM} such that g^{TM} is Euclidean near each $p \in Z(X)$. For a sufficiently small $\delta > 0$, let $Z_\delta(X)$ be the open δ-neighbourhood of $Z(X)$ in M with respect to the Riemannian metric g^{TM}, and set $M_\delta = M \setminus Z_\delta(X)$. Then $[X]$ determines a pull-back section $\widehat{[X]}$ of $\pi^*TM \to TM_\delta$.

Using $\widehat{[X]}$, we introduce the following family of superconnections on $\Lambda^*(\pi^*T^*M) \to TM_\delta$ for $t \in [0, 1]$:

$$\tilde{A}_{\rho,T,t} = \widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)} + Tc_{\tilde{g}_F}(\hat{Y}) - tTc_{\tilde{g}_F}(\widehat{[X]}) = \widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)} + Tc_{\tilde{g}_F}(\hat{Y} - t\widehat{[X]}). \tag{5.1}$$

Clearly, $\tilde{A}_{\rho,T,0} = \tilde{A}_{\rho,T}$ is just the superconnection defined by the extended Chern connection (3.17) used in Section 2, and the curvature of $\tilde{A}_{\rho,T,t}$ is given by

$$\tilde{A}^2_{\rho,T,t} = \widetilde{R}^\natural_\rho + T[\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)}, c_{\tilde{g}_F}(\hat{Y} - t\widehat{[X]})] - T^2|\hat{Y} - t\widehat{[X]}|^2_{\tilde{g}_F}.$$

For the family of superconnections defined by (5.1), we have the following transgression formula:

$$\begin{aligned}
&\lim_{T\to\infty}\int_{TM_\delta} \mathrm{tr}_s[\exp \tilde{A}^2_{\rho,T,1}] - \lim_{T\to\infty}\int_{TM_\delta} \mathrm{tr}_s[\exp \tilde{A}^2_{\rho,T,0}] \\
&\quad = -\lim_{T\to\infty}\int_{TM_\delta} d^{TM}\int_0^1 \mathrm{tr}_s[Tc_{\tilde{g}_F}(\widehat{[X]})\exp(\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)} + Tc_{\tilde{g}_F}(\hat{Y} - t\widehat{[X]}))^2]dt \\
&\quad = \lim_{T\to\infty}\int_{TM\,|_{\partial Z_\delta(X)}}\int_0^1 \mathrm{tr}_s[Tc_{\tilde{g}_F}(\widehat{[X]})\exp(\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)} + Tc_{\tilde{g}_F}(\hat{Y} - t\widehat{[X]}))^2]dt.
\end{aligned} \tag{5.2}$$

First, we have the following lemma.

Lemma 5.1. *One has*

$$[\nabla^{\mathrm{Ch},\natural}, c_{g_F}(\hat{Y} - \widehat{[X]})] = (\delta y^i - (\nabla^{\mathrm{Ch}}\widehat{[X]})^i)c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right) + \Lambda, \tag{5.3}$$

where

$$\Lambda := \frac{1}{2}(y^i - [X]^i)\Theta_i^j\left(\hat{c}_{g_F}\left(\frac{\partial}{\partial \hat{x}^j}\right) + c_{g_F}\left(\frac{\partial}{\partial \hat{x}^j}\right)\right),$$

and $\nabla^{\mathrm{Ch}}\widehat{[X]} = (\nabla^{\mathrm{Ch}}\widehat{[X]})^i\frac{\partial}{\partial \hat{x}^i}$ *is the covariant differential of the section* $\widehat{[X]}$.

Proof. Note that near each $x \in M_\delta$, we have that from (3.2) and (3.27),

$$\begin{aligned}
\left[\nabla^{\mathrm{Ch},\natural}, c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right] &= \left[d^{TM} + \varpi^\natural, c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right] \\
&= (d^{TM}g_{ij})d\hat{x}^j \wedge + [-\varpi_k^l d\hat{x}^k \wedge i_{\frac{\partial}{\partial \hat{x}^l}}, g_{ik}d\hat{x}^k \wedge - i_{\frac{\partial}{\partial \hat{x}^i}}] \\
&= (g_{ik}\varpi_j^k + g_{jk}\varpi_i^k + 2F^{-1}A_{ijk}\delta y^k)d\hat{x}^j \wedge - (g_{il}\varpi_k^l d\hat{x}^k \wedge + \varpi_i^k i_{\frac{\partial}{\partial \hat{x}^k}}) \\
&= \varpi_i^j c_{g_F}\left(\frac{\partial}{\partial \hat{x}^j}\right) + 2F^{-1}A_{ijk}\delta y^k d\hat{x}^j \wedge.
\end{aligned} \tag{5.4}$$

Write $[X] = [X]^i\frac{\partial}{\partial x^i}$ near x, so $\hat{Y} - \widehat{[X]} = (y^i - [X]^i)\frac{\partial}{\partial \hat{x}^i}$ near $[X](x)$. Then from (2.7), (3.2), (3.7) and (5.4), we get

$$\begin{aligned}
&[\nabla^{\mathrm{Ch},\natural}, c_{g_F}(\hat{Y} - \widehat{[X]})] \\
&\quad = \left[\nabla^{\mathrm{Ch},\natural}, (y^i - [X]^i)c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right] \\
&\quad = d^{TM}(y^i - [X]^i)c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right) + (y^i - [X]^i)\left[\nabla^{\mathrm{Ch},\natural}, c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right]
\end{aligned}$$

$$= d^{TM}(y^i - [X]^i)c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right) + (y^i - [X]^i)\varpi_i^j c_{g_F}\left(\frac{\partial}{\partial \hat{x}^j}\right) + 2(y^i - [X]^i)F^{-1}A_{ijk}\delta y^k d\hat{x}^j \wedge$$
$$= (\delta y^i - (\nabla^{\mathrm{Ch}}\widehat{[X]})^i)c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right) + \frac{1}{2}(y^i - [X]^i)\Theta_i^j\left(\hat{c}_{g_F}\left(\frac{\partial}{\partial \hat{x}^j}\right) + c_{g_F}\left(\frac{\partial}{\partial \hat{x}^j}\right)\right).$$

This completes the proof. □

Lemma 5.2. *For $\tilde{A}_{\rho,T,1}$, we have the following localization formula*:

$$\lim_{T\to\infty}\int_{TM_\delta}\mathrm{tr}_s[\exp\tilde{A}^2_{\rho,T,1}] = (-2\pi)^n\int_{M_\delta}[X]^*\mathrm{Pf}(R^{\mathrm{Car}}). \tag{5.5}$$

Proof. It is clear that

$$\int_{TM_\delta}\mathrm{tr}_s[\exp\tilde{A}^2_{\rho,T,1}] = \int_{M_\delta}\int_{TM_\delta/M_\delta}\mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2_{\tilde{g}_F}}\mathrm{tr}_s[\exp(\widetilde{R}^\natural_\rho + T[\widetilde{\nabla}^{\Lambda^*(\pi^*T^*M)}_\rho, c_{\tilde{g}_F}(\hat{Y} - \widehat{[X]})])]. \tag{5.6}$$

Note that the zero set of the section $\hat{Y} - \widehat{[X]}$ in TM_δ is exactly $[X](M_\delta)$. For a fixed $\tau > 0$, let $B_\tau([X](M_\delta))$ be the open τ-tube neighbourhood of $[X](M_\delta)$ in TM_δ. So when τ is small enough, one has $\widetilde{\nabla}^{\Lambda^*(\pi^*T^*M)}_\rho = \nabla^{\mathrm{Ch},\natural}$ and $\tilde{g}_F = g_F$ on $B_\tau([X](M_\delta))$. Now by the exponential decay property of the integrand in (5.6) along fibres as $T \to +\infty$, we have

$$\lim_{T\to\infty}\int_{M_\delta}\int_{TM_\delta/M_\delta}\mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2_{\tilde{g}_F}}\mathrm{tr}_s[\exp(\widetilde{R}^\natural_\rho + T[\widetilde{\nabla}^{\Lambda^*(\pi^*T^*M)}_\rho, c_{\tilde{g}_F}(\hat{Y} - \widehat{[X]})])]$$
$$= \int_{[X](M_\delta)}\lim_{T\to\infty}\int_{B_\tau([X](M_\delta))/[X](M_\delta)}\mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\{\mathrm{tr}_s[\exp(R^{\mathrm{Ch},\natural} + T[\nabla^{\mathrm{Ch},\natural}, c_{g_F}(\hat{Y} - \widehat{[X]})])]\}^{(4n)}. \tag{5.7}$$

During the proof of this lemma, we use $|\hat{Y} - \widehat{[X]}|$ instead of $|\hat{Y} - \widehat{[X]}|_{g_F}$ for simplicity. By (5.3), for any $x \in M_\delta$, we obtain

$$\lim_{T\to\infty}\int_{B_\tau([X](x))}\mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\{\mathrm{tr}_s[\exp(R^{\mathrm{Ch},\natural} + T[\nabla^{\mathrm{Ch},\natural}, c_{g_F}(\hat{Y} - \widehat{[X]})])]\}^{(4n)}$$
$$= \lim_{T\to\infty}\int_{B_\tau([X](x))}\mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\left\{\mathrm{tr}_s\left[\exp\left(R^\natural + P^\natural + T\Lambda + T\delta y^i c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right.\right.\right.$$
$$\left.\left.\left.- T(\nabla^{\mathrm{Ch}}\widehat{[X]})^i c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right)\right]\right\}^{(4n)}. \tag{5.8}$$

Note that

$$\widetilde{P} := \left(\sum_{k,l}F^{-1}P_i{}^j{}_{kl}dx^k \wedge (\nabla^{\mathrm{Ch}}\widehat{[X]})^l\right) \tag{5.9}$$

gives a well-defined endomorphism $\widetilde{P}$ on $\pi^*TM \to M \setminus Z(X)$, and by (2.4), its lifting $\widetilde{P}^\natural$ on $\Lambda^*(\pi^*T^*M) \to M \setminus Z(X)$ is given by

$$\widetilde{P}^\natural = -\sum_{i,j}F^{-1}P_i{}^j{}_{kl}dx^k \wedge (\nabla^{\mathrm{Ch}}\widehat{[X]})^l d\hat{x}^i \wedge i_{\frac{\partial}{\partial \hat{x}^j}} =: P_l^\natural(\nabla^{\mathrm{Ch}}\widehat{[X]})^l. \tag{5.10}$$

Similarly, we define

$$\widetilde{\Lambda} := \frac{1}{2}\widetilde{\Theta}^i\left(\hat{c}_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right) + c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right) := \frac{1}{2}(y^j - [X]^j)\widetilde{\Theta}_j^i\left(\hat{c}_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right) + c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right)$$
$$:= (y^j - [X]^j)g^{ik}A_{kjl}\frac{(\nabla^{\mathrm{Ch}}\widehat{[X]})^l}{F}\left(\hat{c}_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right) + c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right) =: \Lambda_l(\nabla^{\mathrm{Ch}}\widehat{[X]})^l. \tag{5.11}$$

Moreover, from (3.3), one sees that $(\nabla^{\mathrm{Ch}}\widehat{[X]})^i = \pi^* d^M[X]^i + [X]^j \varpi_j^i$ are purely horizontal one-forms. Therefore, the right-hand side of (5.8) becomes

$$
\begin{aligned}
&\lim_{T\to\infty}\int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\bigg\{\mathrm{tr}_s\bigg[\exp(R^\natural)\exp(P^\natural+T\Lambda)\exp\bigg(-T(\nabla^{\mathrm{Ch}}\widehat{[X]})^i c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^i}\bigg)\bigg)\\
&\quad\cdot\exp\bigg(T\delta y^i c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^i}\bigg)\bigg)\bigg]\bigg\}^{(4n)}\\
&=\lim_{T\to\infty}\int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\bigg\{\mathrm{tr}_s\bigg[\exp(R^\natural)\exp(P^\natural+T\Lambda)\prod_{i=1}^{2n}\bigg(1-T(\nabla^{\mathrm{Ch}}\widehat{[X]})^i c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^i}\bigg)\bigg)\\
&\quad\cdot\prod_{i=1}^{2n}\bigg(1+T\delta y^i c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^i}\bigg)\bigg)\bigg]\bigg\}^{(4n)}\\
&=\lim_{T\to\infty}\int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\bigg\{\mathrm{tr}_s\bigg[\exp(R^\natural)\sum_{k=0}^{2n}\exp(P^\natural+T\Lambda)\sum_{1\leqslant i_1<\cdots<i_k\leqslant 2n}(-1)^k T^{2n}(\nabla^{\mathrm{Ch}}\widehat{[X]})^{i_1}\\
&\quad\cdot c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^{i_1}}\bigg)\cdots(\nabla^{\mathrm{Ch}}\widehat{[X]})^{i_k}c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^{i_k}}\bigg)\delta y^{i_{k+1}}c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^{i_{k+1}}}\bigg)\cdots\delta y^{i_{2n}}c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^{i_{2n}}}\bigg)\bigg]\bigg\}^{(4n)}\\
&=\lim_{T\to\infty}\int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\bigg\{\mathrm{tr}_s\bigg[\exp(R^\natural)\sum_{k=0}^{2n}\bigg(\frac{1}{k!}(P^\natural_{s_1}+T\Lambda_{s_1})\delta y^{s_1}\cdots(P^\natural_{s_k}+T\Lambda_{s_k})\delta y^{s_k}\bigg)\\
&\quad\cdot\sum_{1\leqslant i_1<\cdots<i_k\leqslant 2n}(-1)^k T^{2n}(\nabla^{\mathrm{Ch}}\widehat{[X]})^{i_1}c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^{i_1}}\bigg)\cdots(\nabla^{\mathrm{Ch}}\widehat{[X]})^{i_k}c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^{i_k}}\bigg)\delta y^{i_{k+1}}c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^{i_{k+1}}}\bigg)\\
&\quad\cdots\delta y^{i_{2n}}c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^{i_{2n}}}\bigg)\bigg]\bigg\}^{(4n)}\\
&=\lim_{T\to\infty}\int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\bigg\{\mathrm{tr}_s\bigg[\exp(R^\natural)\bigg(\sum_{k=0}^{2n}\frac{1}{k!}(P^\natural_{s_1}+T\Lambda_{s_1})(\nabla^{\mathrm{Ch}}\widehat{[X]})^{s_1}\cdots(P^\natural_{s_k}+T\Lambda_{s_k})\\
&\quad\cdot(\nabla^{\mathrm{Ch}}\widehat{[X]})^{s_k}\bigg)T^{2n}\delta y^1 c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^1}\bigg)\cdots\delta y^{2n}c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^{2n}}\bigg)\bigg]\bigg\}^{(4n)}\\
&=\lim_{T\to\infty}\int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\bigg\{\mathrm{tr}_s\bigg[\exp(R^\natural)\exp(\widetilde{P}^\natural+T\widetilde{\Lambda})\exp\bigg(Tdy^i c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^i}\bigg)\bigg)\bigg]\bigg\}^{(4n)}\\
&=\lim_{T\to\infty}\int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2}\bigg\{\mathrm{tr}_s\bigg[\exp(R^\natural+\widetilde{P}^\natural)\exp(T\widetilde{\Lambda})\exp\bigg(Tdy^i c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^i}\bigg)\bigg)\bigg]\bigg\}^{(4n)}, \quad (5.12)
\end{aligned}
$$

where $\{i_1,\dots,i_k,i_{k+1},\dots,i_{2n}\}$ denotes any of the rearrangements of $\{1,\dots,2n\}$.

Now we use the special g_F-orthonormal frame field $\{e_1,\dots,e_{2n}\}$ with $e_{2n}=\hat{Y}/F$. Let $\{\omega^1,\dots,\omega^{2n}\}$ be its dual frame field. Set

$$e_a = u_a^j\frac{\partial}{\partial\hat{x}^j},\quad \frac{\partial}{\partial\hat{x}^i}=v_i^a e_a. \tag{5.13}$$

Then we have

$$g_{ij}=\sum_{a=1}^{2n}v_i^a v_j^a,\quad \sqrt{\det(g_{ij})}=\det(v_i^a),\quad c_{g_F}\bigg(\frac{\partial}{\partial\hat{x}^i}\bigg)=v_i^a c(e_a). \tag{5.14}$$

Set

$$(R^\natural+\widetilde{P}^\natural)\omega^a =: -\widetilde{\Omega}_b^a\omega^b,\quad \widetilde{\Theta}_b^a := \widetilde{\Theta}_i^j v_j^a u_b^i,\quad \widetilde{\Theta}^a := \widetilde{\Theta}^j v_j^a. \tag{5.15}$$

By the Euler lemma for homogeneous functions, one has $\widetilde{\Theta}_a^{2n}=0$ for $a=1,\dots,2n$.

Letting β_k denote any $2k$ indices $1\leqslant b_1,b_2,\dots,b_{2k}\leqslant 2n$ with repetition, for $k=0,1,2,\dots,n$, one has the following integral formula:

$$\int_{\mathbb{R}^{2n}}\mathrm{e}^{-T^2\sum_{i=1}^{2n}(y^i)^2}y^{b_1}\cdots y^{b_{2k}}T^{2n+2k}dy^1\wedge\cdots\wedge dy^{2n}$$

$$= \left[\prod_{i=1}^{2n} \frac{1+(-1)^{\beta_k(i)}}{2} \Gamma\left(\frac{\beta_k(i)+1}{2}\right)\right] = \frac{\pi^n}{2^k}\left[\prod_{i=1}^{2n} \frac{1+(-1)^{\beta_k(i)}}{2}(\beta_k(i)-1)!!\right], \tag{5.16}$$

where $\beta_k(i)$ denotes the number of times that the value $i \in \{1, 2, \ldots, 2n\}$ has occurred in β_k, and clearly,

$$\sum_{i=1}^{2n} \beta_k(i) = 2k.$$

For any bounded smooth function f on T_xM, similar to (5.16), one has the following localization formula:

$$\lim_{T\to\infty} \int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2} (y^{p_1} - [X]^{p_1}) v_{p_1}^{b_1} \cdots (y^{p_{2k}} - [X]^{p_{2k}}) v_{p_{2k}}^{b_{2k}} f(Y) T^{2n+2k}$$
$$\cdot \sqrt{\det(g_{ij})} dy^1 \wedge \cdots \wedge dy^{2n} = \frac{\pi^n}{2^k}\left[\prod_{i=1}^{2n} \frac{1+(-1)^{\beta_k(i)}}{2}(\beta_k(i)-1)!!\right] f([X](x)). \tag{5.17}$$

Let $\widetilde{Q}_i^j = \frac{1}{4}\widetilde{\Theta}_i^p \wedge \widetilde{\Theta}_p^j$ be the coefficients of $\widetilde{Q} := -\frac{1}{4}\widetilde{\Theta} \wedge \widetilde{\Theta}$. For a fixed $k = 0, 1, 2, \ldots, n$, and any fixed $2k$ numbers $1 \leqslant j_1 < \cdots < j_{2k} \leqslant 2n$, the following combinatorial fact holds:

$$\frac{1}{2^k k!} \sum_{i_1\cdots i_{2k}} \delta_{j_1 j_2 \cdots j_{2k}}^{i_1 i_2 \cdots i_{2k}} \widetilde{Q}_{i_1}^{i_2} \wedge \cdots \wedge \widetilde{Q}_{i_{2k-1}}^{i_{2k}}$$
$$= \frac{1}{2^k k!} \frac{1}{4^k} \sum_{i_1\cdots i_{2k}} \delta_{j_1 j_2 \cdots j_{2k}}^{i_1 i_2 \cdots i_{2k}} (\widetilde{\Theta}_{i_1}^{p_1} \wedge \widetilde{\Theta}_{p_1}^{i_2}) \wedge \cdots \wedge (\widetilde{\Theta}_{i_{2k-1}}^{p_k} \wedge \widetilde{\Theta}_{p_k}^{i_{2k}})$$
$$= \frac{1}{4^k} \sum_{\beta_k} \left[\prod_{i=1}^{2n} \frac{1+(-1)^{\beta_k(i)}}{2}(\beta_k(i)-1)!!\right] \widetilde{\Theta}_{j_1}^{p_1} \wedge \widetilde{\Theta}_{p_2}^{j_2} \wedge \cdots \wedge \widetilde{\Theta}_{j_{2k-1}}^{p_{2k-1}} \wedge \widetilde{\Theta}_{p_{2k}}^{j_{2k}}, \tag{5.18}$$

where β_k runs over all the $2k$ indices $1 \leqslant p_1, p_2, \ldots, p_{2k} \leqslant 2n$ with repetition.

By (5.13)–(5.18), we have

$$\lim_{T\to\infty} \int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2} \left\{\mathrm{tr}_s\left[\exp(R^\natural + \widetilde{P}^\natural)\exp(T\widetilde{\Lambda})\exp\left(T dy^i c_{g_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right)\right]\right\}^{(4n)}$$
$$= \lim_{T\to\infty} \int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2} \{\mathrm{tr}_s[\exp(R^\natural + \widetilde{P}^\natural)\exp(T\widetilde{\Lambda})\exp(T dy^i v_i^a c_{g_F}(e_a))]\}^{(4n)}$$
$$= \sum_{k=0}^{n-1} \frac{(-1)^n}{(2k)!(n-k)!} \lim_{T\to\infty} \int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2} \sum \epsilon_{a_1\cdots a_{2n}} \widetilde{\Theta}^{a_1} \wedge \cdots \wedge \widetilde{\Theta}^{a_{2k}} \wedge \widetilde{\Omega}_{a_{2k+1}}^{a_{2k+2}} \wedge \cdots \wedge \widetilde{\Omega}_{a_{2n-1}}^{a_{2n}}$$
$$\cdot T^{2n+2k} \det(v_i^a) dy^1 \wedge \cdots \wedge dy^{2n}$$
$$= \sum_{k=0}^{n-1} \frac{(-1)^n}{(2k)!(n-k)!} \lim_{T\to\infty} \int_{B_\tau([X](x))} \mathrm{e}^{-T^2|\hat{Y}-\widehat{[X]}|^2} \sum_{\beta_{2k}} (y^{p_1} - [X]^{p_1}) v_{p_1}^{b_1} \cdots (y^{p_{2k}} - [X]^{p_{2k}}) v_{p_{2k}}^{b_{2k}}$$
$$\cdot \sum \epsilon_{a_1\cdots a_{2n}} \widetilde{\Theta}_{b_1}^{a_1} \wedge \cdots \wedge \widetilde{\Theta}_{b_{2k}}^{a_{2k}} \wedge \widetilde{\Omega}_{a_{2k+1}}^{a_{2k+2}} \wedge \cdots \wedge \widetilde{\Omega}_{a_{2n-1}}^{a_{2n}} T^{2n+2k} \det(v_i^a) dy^1 \wedge \cdots \wedge dy^{2n}$$
$$= \sum_{k=0}^{n-1} \frac{(-1)^n \pi^n}{(2k)!(n-k)!2^k} \sum_{\beta_k} \left[\prod_{i=1}^{2n} \frac{1+(-1)^{\beta_k(i)}}{2}(\beta_k(i)-1)!!\right]\left[\sum \epsilon_{a_1\cdots a_{2n}} \widetilde{\Theta}_{b_1}^{a_1} \wedge \cdots \wedge \widetilde{\Theta}_{b_{2k}}^{a_{2k}}\right.$$
$$\left.\wedge \widetilde{\Omega}_{a_{2k+1}}^{a_{2k+2}} \wedge \cdots \wedge \widetilde{\Omega}_{a_{2n-1}}^{a_{2n}}\right]([X](x))$$
$$= \sum_{k=0}^{n-1} \frac{(-\pi)^n}{k!(n-k)!} \sum [\epsilon_{a_1\cdots a_{2n}} \widetilde{Q}_{a_1}^{a_2} \wedge \cdots \wedge \widetilde{Q}_{a_{2k-1}}^{a_{2k}} \wedge \widetilde{\Omega}_{a_{2k+1}}^{a_{2k+2}} \wedge \cdots \wedge \widetilde{\Omega}_{a_{2n-1}}^{a_{2n}}]([X](x)). \tag{5.19}$$

Because the map $[X] : M \setminus Z(X) \to TM$ is given by $[X](x) = (x, [X])$ for any $x \in M \setminus Z(X)$, we have

$$[X]_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{\partial [X]^j}{\partial x^i} \frac{\partial}{\partial y^j},$$

and then

$$\begin{aligned}[X]^*\delta y^i &= [X]^*(dy^i + y^j\Gamma^i_{jk}dx^k) = \frac{\partial [X]^i}{\partial x^j}dx^j + [X]^j\Gamma^i_{jk}([X])dx^k \\ &= d[X]^i + [X]^j([X]^*\varpi^i_j) = [X]^*(\nabla^{\mathrm{Ch}}\widehat{[X]})^i. \end{aligned} \tag{5.20}$$

By (3.11), (5.9), (5.10), (5.11), (5.15) and (5.20), we have

$$[X]^*\Omega^a_b = [X]^*\widetilde{\Omega}^a_b, \quad [X]^*\Theta^a_b = [X]^*\widetilde{\Theta}^a_b, \quad [X]^*Q^a_b = [X]^*\widetilde{Q}^a_b. \tag{5.21}$$

From (3.12), (3.14), (5.7), (5.8), (5.12), (5.19) and (5.21), we obtain

$$\begin{aligned} &\lim_{T\to\infty}\int_{TM_\delta} \mathrm{tr}_s[\exp(\widetilde{\nabla}^{\Lambda^*(\pi^*TM)}_\rho + Tc_{\tilde{g}_F}(\hat{Y} - \widehat{[X]}))^2] \\ &= \sum_{k=0}^{n-1}\frac{(-\pi)^n}{k!(n-k)!}\int_{[X](M_\delta)}\left[\sum \epsilon_{a_1\cdots a_{2n}}\widetilde{Q}^{a_2}_{a_1}\wedge\cdots\wedge\widetilde{Q}^{a_{2k}}_{a_{2k-1}}\wedge\widetilde{\Omega}^{a_{2k+2}}_{a_{2k+1}}\wedge\cdots\wedge\widetilde{\Omega}^{a_{2n}}_{a_{2n-1}}\right] \\ &= \sum_{k=0}^{n-1}\frac{(-\pi)^n}{k!(n-k)!}\int_{M_\delta}[X]^*\left[\sum \epsilon_{a_1\cdots a_{2n}}Q^{a_2}_{a_1}\wedge\cdots\wedge Q^{a_{2k}}_{a_{2k-1}}\wedge\Omega^{a_{2k+2}}_{a_{2k+1}}\wedge\cdots\wedge\Omega^{a_{2n}}_{a_{2n-1}}\right] \\ &= (-2\pi)^n\frac{1}{2^n n!}\sum_{k=0}^{n-1}\frac{n!}{k!(n-k)!}\int_{M_\delta}[X]^*\left[\sum \epsilon_{a_1\cdots a_{2n}}Q^{a_2}_{a_1}\wedge\cdots\wedge Q^{a_{2k}}_{a_{2k-1}}\wedge\widehat{\Omega}^{a_{2k+2}}_{a_{2k+1}}\wedge\cdots\wedge\widehat{\Omega}^{a_{2n}}_{a_{2n-1}}\right] \\ &= (-2\pi)^n\int_{M_\delta}[X]^*\mathrm{Pf}(R^{\mathrm{Car}}). \end{aligned}$$

Thus (5.5) holds. □

Define

$$\int_M [X]^*\mathrm{Pf}(R^{\mathrm{Car}}) := \lim_{\delta\to 0}\int_{M_\delta}[X]^*\mathrm{Pf}(R^{\mathrm{Car}}). \tag{5.22}$$

By using Lemmas 2.1 and 5.2, we prove the general Lichnerowicz GBC-formulae under the assumption that the Finsler metrics are locally Minkowskian near the isolated zeros of the vector field X.

Lemma 5.3. *Let (M,F) be a closed and oriented Finsler manifold of dimension $2n$. Let X be a vector field on M with isolated zeros. Assume that (M,F) is locally Minkowskian near the zeros of X. Then*

$$\left(\frac{-1}{2\pi}\right)^n\int_M [X]^*[\mathrm{Pf}(R^{\mathrm{Car}}) + d\mathcal{H}] = \sum_{p\in Z(X)}\mathrm{ind}_p\frac{\mathrm{Vol}(S_pM)}{\mathrm{Vol}(S^{2n-1})},$$

where

$$\mathcal{H} := \sum_{k=1}^{n-1}\frac{(-1)^{n+k}}{(2n-2k-1)!!2^k k!}\sum \epsilon_{a_1\cdots a_{2n-1}}Q^{a_2}_{a_1}\wedge\cdots\wedge Q^{a_{2k}}_{a_{2k-1}}\wedge\omega^{2n}_{a_{2k+1}}\wedge\cdots\wedge\omega^{2n}_{a_{2n-1}}.$$

Proof. Assume that there is a sufficiently small $\epsilon > 0$ such that the background Riemannian metric g^{TM} is Euclidean on $Z_\epsilon(X)$ and (M,F) is locally Minkowskian on $Z_{\epsilon/2}(X)$. From now on, we always assume that $0 < \delta < \epsilon/2$. Note that for locally Minkowski spaces, the Chern connection $\nabla^{\mathrm{Ch}} = d$. By our choice of the background Riemannian metric g^{TM}, one has

$$\nabla^{\mathrm{Ch}}g^{TM} = dg^{TM} = 0 \tag{5.23}$$

on $Z_{\epsilon/2}(X)$. In this case, we define

$$\widetilde{\nabla}_\rho = \nabla^{\mathrm{Ch}}, \quad \tilde{g}_F = (1-\rho)g_F + \rho g^{TM}, \tag{5.24}$$

where ρ is the cutoff function used in Section 2.

Because $\partial Z_\delta(X) \subset Z_{\epsilon/2}(X)$, we calculate the last term of (5.2) on $Z_{\epsilon/2}(X)$.

From (5.23) and (5.24), a similar computation to that in Lemma 5.1 shows that on $Z_{\epsilon/2}(X)$,

$$[\nabla^{\mathrm{Ch},\natural}, c_{\tilde{g}_F}(\hat{Y}-t\widehat{[X]})] = \left[d, (y^i - t[X]^i)c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right] = (dy^i - t\pi^* d^M[X]^i)c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^i}\right) + \Lambda(t), \tag{5.25}$$

where

$$\begin{aligned}\Lambda(t) &:= (y^i - t[X]^i)(g^{TM}_{ij} - g_{ij})d^{TM}\rho d\hat{x}^j \wedge +2(1-\rho)(y^i - t[X]^i)A_{ijk}F^{-1}dy^k d\hat{x}^j \wedge \\ &= \frac{1}{2}(y^i - t[X]^i)(\tilde{g}_F)^{jk}d(\tilde{g}_F)_{ki}\left(\hat{c}_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^j}\right) + c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^j}\right)\right) \\ &=: \frac{1}{2}(y^i - t[X]^i)\Theta^j_{ik}dy^k\left(\hat{c}_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^j}\right) + c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^j}\right)\right) \\ &=: \frac{1}{2}(y^i - t[X]^i)\Theta^j_{i}\left(\hat{c}_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^j}\right) + c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^j}\right)\right) \\ &=: \frac{1}{2}\Theta^j\left(\hat{c}_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^j}\right) + c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^j}\right)\right) =: \Lambda_l(t)dy^l.\end{aligned}$$

During the proof of this lemma, we use $|\hat{Y} - \widehat{[X]}|$ instead of $|\hat{Y} - \widehat{[X]}|_{\tilde{g}_F}$ for simplicity.

Since $R^{\mathrm{Ch}} = 0$ for locally Minkowski spaces and $\Lambda(t)$ contains only vertical forms, by (5.24) and (5.25), $\forall\, x \in \partial Z_{\epsilon/2}(X)$, we obtain

$$\begin{aligned}&\lim_{T\to\infty}\int_{T_xM}\int_0^1 \{\mathrm{tr}_s[Tc_{\tilde{g}_F}(\widehat{[X]})\exp(\widetilde{\nabla}^{\Lambda^*(\pi^*T^*M)}_\rho + Tc_{\tilde{g}_F}(\hat{Y}-t\widehat{[X]}))^2]\}^{(4n-1)}dt \\ &= \int_0^1 dt \lim_{T\to\infty}\int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2}\{\mathrm{tr}_s[Tc_{\tilde{g}_F}(\widehat{[X]})\exp(T[\nabla^{\mathrm{Ch},\natural}, c_{\tilde{g}_F}(\hat{Y}-t\widehat{[X]})])]\}^{(4n-1)} \\ &= \int_0^1 dt \lim_{T\to\infty}\int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2}\left\{\mathrm{tr}_s\left[T[X]^i c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\exp\left(-tT\pi^*d^M[X]^i c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right.\right.\right. \\ &\quad \left.\left.\left.+ Tdy^i c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^i}\right) + T\Lambda(t)\right)\right]\right\}^{(4n-1)} \\ &= \int_0^1 dt \lim_{T\to\infty}\int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2}\left\{\mathrm{tr}_s\left[T[X]^i c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\prod_{i=1}^{2n}\left(1 - tT\pi^*d^M[X]^i c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^i}\right)\right)\right.\right. \\ &\quad \left.\left.\cdot \prod_{l=1}^{2n}(1 + T\Lambda_l(t)dy^l)\right]\right\}^{(4n-1)} \\ &= \int_0^1 dt \lim_{T\to\infty}\int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2}t^{2n-1}T^{2n} \\ &\quad \cdot \left\{\mathrm{tr}_s\left[\sum_{i=1}^{2n}(-1)^{n+i-1}[X]^i\pi^*(d[X]^1\wedge\cdots\wedge\widehat{d[X]^i}\wedge\cdots\wedge d[X]^{2n})\right.\right. \\ &\quad \left.\left.\cdot \prod_{l=1}^{2n}\left(1 + \frac{T}{2}\Theta^j\hat{c}_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^j}\right)\right)c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^1}\right)\cdots c_{\tilde{g}_F}\left(\frac{\partial}{\partial \hat{x}^{2n}}\right)\right]\right\}^{(4n-1)}.\end{aligned} \tag{5.26}$$

Use a local orthonormal frame field $\{e_a\}$ of $\tilde{g}_F$ around $t[X](x)$ and assume

$$e_a = u^j_a \frac{\partial}{\partial \hat{x}^j}, \quad \frac{\partial}{\partial \hat{x}^i} = v^a_i e_a.$$

Define

$$\Theta^b_{ac} = \Theta^j_{ik}u^i_a u^k_c v^b_j, \quad \Theta^b_a = \Theta^j_i u^i_a v^b_j, \quad \Theta^b = \Theta^j v^b_j.$$

By (5.26) and an integral formula similar to (5.17), we obtain

$$\lim_{T\to\infty}\int_{T_xM}\int_0^1 \{\mathrm{tr}_s[Tc_{\tilde{g}_F}(\widehat{[X]})\exp(\widetilde{\nabla}^{\Lambda^*(\pi^*T^*M)}_\rho + Tc_{\tilde{g}_F}(\hat{Y}-t\widehat{[X]}))^2]\}^{(4n-1)}dt$$

$$
\begin{aligned}
&= \int_0^1 t^{2n-1} dt \lim_{T\to\infty} \int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2} T^{2n} \\
&\quad \cdot \bigg\{ \mathrm{tr}_s \bigg[\sum_{i=1}^{2n} (-1)^{n+i-1} [X]^i \pi^*(d[X]^1 \wedge \cdots \wedge \widehat{d[X]^i} \wedge \cdots \wedge d[X]^{2n}) \\
&\quad \cdot \prod_{l=1}^{2n} \bigg(1 + \frac{T}{2} \Theta^b \hat{c}_{\tilde{g}_F}(e_b) \bigg) \det(v_i^a) c_{\tilde{g}_F}(e_1) \cdots c_{\tilde{g}_F}(e_{2n}) \bigg] \bigg\}^{(4n-1)} \\
&= \int_0^1 t^{2n-1} dt \lim_{T\to\infty} \int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2} \frac{T^{4n}}{2^{2n}} \sum_{i=1}^{2n} (-1)^{n+i-1} [X]^i \\
&\quad \cdot \pi^*(d[X]^1 \wedge \cdots \wedge \widehat{d[X]^i} \wedge \cdots \wedge d[X]^{2n}) (-1)^n \Theta^1 \wedge \cdots \wedge \Theta^{2n} \det(v_i^a) \\
&\quad \cdot \mathrm{tr}_s[\hat{c}_{\tilde{g}_F}(e_1) \cdots \hat{c}_{\tilde{g}_F}(e_{2n}) c_{\tilde{g}_F}(e_1) \cdots c_{\tilde{g}_F}(e_{2n})] \\
&= \int_0^1 t^{2n-1} dt \lim_{T\to\infty} \int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2} T^{4n} \sum_{i=1}^{2n} (-1)^{n+i-1} [X]^i \\
&\quad \cdot \pi^*(d[X]^1 \wedge \cdots \wedge \widehat{d[X]^i} \wedge \cdots \wedge d[X]^{2n}) \det(v_i^a) \Theta^1 \wedge \cdots \wedge \Theta^{2n} \\
&= \int_0^1 t^{2n-1} dt \sum_{i=1}^{2n} (-1)^{n+i-1} [X]^i \pi^*(d[X]^1 \wedge \cdots \wedge \widehat{d[X]^i} \wedge \cdots \wedge d[X]^{2n}) \\
&\quad \cdot \lim_{T\to\infty} \int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2} T^{4n} \det(v_i^a) \sum_{\beta_n} (y^{p_1} - [X]^{p_1}) v_{p_1}^{b_1} \cdots (y^{p_{2n}} - [X]^{p_{2n}}) v_{p_{2n}}^{b_{2n}} \Theta_{b_1}^1 \wedge \cdots \wedge \Theta_{b_{2n}}^{2n} \\
&= \int_0^1 t^{2n-1} dt \sum_{i=1}^{2n} (-1)^{n+i-1} [X]^i \pi^*(d[X]^1 \wedge \cdots \wedge \widehat{d[X]^i} \wedge \cdots \wedge d[X]^{2n}) \lim_{T\to\infty} \int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2} T^{4n} \\
&\quad \cdot \det(v_i^a) \sum_{\beta_n} (y^{p_1} - [X]^{p_1}) v_{p_1}^{b_1} \cdots (y^{p_{2n}} - [X]^{p_{2n}}) v_{p_{2n}}^{b_{2n}} \Theta_{b_1 c_1}^1 \cdots \Theta_{b_{2n} c_{2n}}^{2n} v_{q_1}^{c_1} dy^{q_1} \wedge \cdots \wedge v_{q_{2n}}^{c_{2n}} dy^{q_{2n}} \\
&= \int_0^1 t^{2n-1} dt \sum_{i=1}^{2n} (-1)^{n+i-1} [X]^i \pi^*(d[X]^1 \wedge \cdots \wedge \widehat{d[X]^i} \wedge \cdots \wedge d[X]^{2n}) \lim_{T\to\infty} \int_{T_xM} \mathrm{e}^{-T^2|\hat{Y}-t\widehat{[X]}|^2} T^{4n} \\
&\quad \cdot (\det(v_i^a))^2 \sum_{\beta_n} (y^{p_1} - [X]^{p_1}) v_{p_1}^{b_1} \cdots (y^{p_{2n}} - [X]^{p_{2n}}) v_{p_{2n}}^{b_{2n}} \\
&\quad \cdot \sum \epsilon_{c_1 \cdots c_{2n}} \Theta_{b_1 c_1}^1 \cdots \Theta_{b_{2n} c_{2n}}^{2n} dy^1 \wedge \cdots \wedge dy^{2n} \\
&= \int_0^1 t^{2n-1} dt \sum_{i=1}^{2n} (-1)^{i-1} [X]^i \pi^*(d[X]^1 \wedge \cdots \wedge \widehat{d[X]^i} \wedge \cdots \wedge d[X]^{2n}) \det(v_i^a)(t[X](x)) \\
&\quad \cdot \frac{(-\pi)^n}{2^n} \sum_{\beta_n} \bigg[\prod_{i=1}^{2n} \frac{1 + (-1)^{\beta_n(i)}}{2} (\beta_n(i) - 1)!! \bigg] \\
&\quad \cdot \sum \epsilon_{c_1 \cdots c_{2n}} \Theta_{b_1 c_1}^1 (t[X](x)) \cdots \Theta_{b_{2n} c_{2n}}^{2n} (t[X](x)). \qquad (5.27)
\end{aligned}
$$

For convenience, we introduce the following differential form of degree $2n-1$ on $TZ_{\epsilon/2}(X)$:

$$
\begin{aligned}
\varphi := \frac{(-\pi)^n}{2^n} \sum_{\beta_n} \bigg[\prod_{i=1}^{2n} \frac{1 + (-1)^{\beta_n(i)}}{2} (\beta_n(i) - 1)!! \bigg] \sum \epsilon_{c_1 \cdots c_{2n}} \Theta_{b_1 c_1}^1 \cdots \Theta_{b_{2n} c_{2n}}^{2n} \\
\cdot \sum_{i=1}^{2n} (-1)^{i-1} \frac{y^i}{F(y)} (dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^{2n}) \sqrt{\det((\tilde{g}_F)_{ij})}. \qquad (5.28)
\end{aligned}
$$

From (5.27) and (5.28), we have

$$
\lim_{T\to\infty} \int_{TM\,|_{\partial Z_\delta(X)}} \int_0^1 \mathrm{tr}_s [T c_{\tilde{g}_F}(\widehat{[X]}) \exp(\widetilde{\nabla}_\rho^{\Lambda^*(\pi^* T^* M)} + T c_{\tilde{g}_F}(\hat{Y} - t\widehat{[X]}))^2] dt
$$

$$= \int_0^1 dt \int_{t[X](\partial Z_\delta(X))} \varphi. \tag{5.29}$$

At any $p \in Z(X)$, we denote the Finsler sphere (or Finsler disc, respectively) of radius $t > 0$ by $S_pM(t)$ (or $D_pM(t)$, respectively). For the case $t = 1$, we also use S_pM instead of $S_pM(1)$ for simplicity.

Let $\delta \to 0$. By the mapping degree theory, we have

$$\lim_{\delta\to 0} \int_0^1 dt \int_{t[X](\partial Z_\delta(X))} \varphi = \sum_{p\in Z(X)} \mathrm{ind}_p \int_0^1 dt \int_{S_pM(t)} \varphi. \tag{5.30}$$

By (5.28) and (5.18), we obtain

$$\begin{aligned}
-\int_0^1 dt \int_{S_pM(t)} \varphi &= -\int_0^1 \int_{S_pM(t)} dF \wedge \sum_{i=1}^{2n} (-1)^{i-1} \frac{y^i}{F(y)} (dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^{2n}) \\
&\quad \cdot \frac{(-\pi)^n}{2^n} \sum_{\beta_n} \left[\prod_{i=1}^{2n} \frac{1 + (-1)^{\beta_n(i)}}{2} (\beta_n(i) - 1)!! \right] \epsilon_{c_1\cdots c_{2n}} \Theta^1_{b_1c_1} \cdots \Theta^{2n}_{b_{2n}c_{2n}} \sqrt{\det((\tilde{g}_F)_{ij})} \\
&= \int_{D_pM(1)} \frac{(-\pi)^n}{2^n} \sum_{\beta_n} \left[\prod_{i=1}^{2n} \frac{1 + (-1)^{\beta_n(i)}}{2} (\beta_n(i) - 1)!! \right] \epsilon_{c_1\cdots c_{2n}} \Theta^1_{b_1c_1} \cdots \Theta^{2n}_{b_{2n}c_{2n}} \\
&\quad \cdot \sqrt{\det((\tilde{g}_F)_{ij})} dy^1 \wedge \cdots \wedge dy^{2n} \\
&= \int_{D_pM(1)} (-2\pi)^n \frac{1}{2^{2n}} \sum_{\beta_n} \left[\prod_{i=1}^{2n} \frac{1 + (-1)^{\beta_n(i)}}{2} (\beta_n(i) - 1)!! \right] \Theta^1_{b_1} \wedge \cdots \wedge \Theta^{2n}_{b_{2n}} \\
&= (-2\pi)^n \int_{D_pM(1)} \mathrm{Pf}(Q),
\end{aligned} \tag{5.31}$$

where we set $Q := -\frac{1}{4}\Theta \wedge \Theta$ as usual.

For any $p \in Z(X)$, the tangent space T_pM is a flat manifold with the flat connection d and the Riemannian metric

$$\tilde{\bar{g}}_F := (\tilde{g}_F)_{ij} dy^i \otimes dy^j.$$

Let $\hat{d}$ be the symmetrization of d. According to [5, Proposition 4.3], the curvature of $\hat{d}$ is just Q. Denote the curvature of the Levi-Civita connection of $\tilde{\bar{g}}_F$ by R^{T_pM}. By [21, Proposition 3.6], one has

$$\mathrm{Pf}(Q) = \mathrm{Pf}(R^{T_pM}) + d\psi \tag{5.32}$$

for some differential form ψ. Furthermore, on the set $D_pM(1) \setminus D_pM(1/2)$, $\tilde{\bar{g}}_F$ is a Hessian metric, i.e.,

$$\tilde{\bar{g}}_F = \bar{g}_F = \frac{1}{2}[F]^2_{y^iy^j} dy^i \otimes dy^j.$$

In this case, the curvature form of the Levi-Civita connection of the Hessian metric is just Q (see [19]). Therefore, $d\psi = 0$ holds on $D_pM(1) \setminus D_pM(1/2)$.

By (5.31), (5.32) and Stokes' theorem, we have

$$-\int_0^1 dt \int_{S_pM(t)} \varphi = (-2\pi)^n \int_{D_pM(1)} [\mathrm{Pf}(R^{T_pM}) + d\psi] = (-2\pi)^n \int_{D_pM(1)} \mathrm{Pf}(R^{T_pM}). \tag{5.33}$$

Following Chern [6], we introduce some differential forms on SM:

$$\bar{\Phi}_k := \sum \epsilon_{a_1\cdots a_{2n-1}} Q^{a_2}_{a_1} \wedge \cdots \wedge Q^{a_{2k}}_{a_{2k-1}} \wedge \omega^{2n}_{a_{2k+1}} \wedge \cdots \wedge \omega^{2n}_{a_{2n-1}}, \quad k = 0, \dots, n-1,$$

and

$$\bar{\Pi} := \left(\frac{1}{2\pi}\right)^n \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k-1)!!2^k k!} \bar{\Phi}_k, \quad \mathcal{H} := \sum_{k=1}^{n-1} \frac{(-1)^{n+k}}{(2n-2k-1)!!2^k k!} \bar{\Phi}_k. \tag{5.34}$$

Using the GBC-formula for Riemannian manifolds with boundary (see [7]) and (5.34), we obtain

$$\begin{aligned}\left(\frac{-1}{2\pi}\right)^n\int_{D_pM(1)}\mathrm{Pf}(R^{T_pM}) &= 1-\int_{S_pM}\bar{\Pi}\\ &= 1-\left(\frac{1}{2\pi}\right)^n\int_{S_pM}\frac{(2n-1)!}{(2n-1)!!}\omega_1^{2n}\wedge\cdots\wedge\omega_{2n-1}^{2n}-\left(\frac{-1}{2\pi}\right)^n\int_{S_pM}\mathcal{H}\\ &= 1-\frac{\mathrm{Vol}(S_pM)}{\mathrm{Vol}(S^{2n-1})}-\left(\frac{-1}{2\pi}\right)^n\int_{S_pM}\mathcal{H}. \end{aligned} \tag{5.35}$$

Combining (5.29), (5.30), (5.33) and (5.35), when $\delta\to 0$, we see that the last term in (5.2) is

$$\begin{aligned}&\lim_{\delta\to 0}\lim_{T\to\infty}\left(\frac{1}{2\pi}\right)^{2n}\int_{TM|_{\partial Z_\delta(X)}}\int_0^1\mathrm{tr}_s[Tc_{\tilde{g}_F}(\widehat{[X]})\exp(\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)}+Tc_{\tilde{g}_F}(\hat{Y}-t\widehat{[X]}))^2]dt\\ &\quad=-\sum_{p\in Z(X)}\mathrm{ind}_p\left(1-\frac{\mathrm{Vol}(S_pM)}{\mathrm{Vol}(S^{2n-1})}-\left(\frac{-1}{2\pi}\right)^n\int_{S_pM}\mathcal{H}\right).\end{aligned} \tag{5.36}$$

By (3.18), (5.2), (5.5), (5.36) and the Poincaré-Hopf theorem, we obtain

$$\begin{aligned}&\left(\frac{-1}{2\pi}\right)^n\int_M[X]^*\mathrm{Pf}(R^{\mathrm{Car}})\\ &=\left(\frac{-1}{2\pi}\right)^n\lim_{\delta\to 0}\int_{M_\delta}[X]^*\mathrm{Pf}(R^{\mathrm{Car}})\\ &=\left(\frac{1}{2\pi}\right)^{2n}\lim_{\delta\to 0}\lim_{T\to\infty}\int_{TM_\delta}\mathrm{tr}_s[\exp\tilde{A}^2_{\rho,T,1}]\\ &=\left(\frac{1}{2\pi}\right)^{2n}\lim_{\delta\to 0}\lim_{T\to\infty}\int_{TM_\delta}\mathrm{tr}_s[\exp\tilde{A}^2_{\rho,T,0}]\\ &\quad+\left(\frac{1}{2\pi}\right)^{2n}\lim_{\delta\to 0}\lim_{T\to\infty}\int_{TM|_{\partial Z_\delta(X)}}\int_0^1\mathrm{tr}_s[Tc_{\tilde{g}_F}(\widehat{[X]})\exp(\widetilde{\nabla}_\rho^{\Lambda^*(\pi^*T^*M)}+Tc_{\tilde{g}_F}(\hat{Y}-t\widehat{[X]}))^2]dt\\ &=\left(\frac{1}{2\pi}\right)^{2n}\lim_{T\to\infty}\lim_{\delta\to 0}\int_{TM_\delta}\mathrm{tr}_s[\exp\tilde{A}^2_{\rho,T,0}]-\sum_{p\in Z(X)}\mathrm{ind}_p\left(1-\frac{\mathrm{Vol}(S_pM)}{\mathrm{Vol}(S^{2n-1})}-\left(\frac{-1}{2\pi}\right)^n\int_{S_pM}\mathcal{H}\right)\\ &=\chi(M)-\sum_{p\in Z(X)}\mathrm{ind}_p\left(1-\frac{\mathrm{Vol}(S_pM)}{\mathrm{Vol}(S^{2n-1})}-\left(\frac{-1}{2\pi}\right)^n\int_{S_pM}\mathcal{H}\right)\\ &=\sum_{p\in Z(X)}\mathrm{ind}_p\frac{\mathrm{Vol}(S_pM)}{\mathrm{Vol}(S^{2n-1})}+\left(\frac{-1}{2\pi}\right)^n\sum_{p\in Z(X)}\mathrm{ind}_p\int_{S_pM}\mathcal{H}.\end{aligned}$$

Similar to (5.22), we define

$$\int_M[X]^*d\mathcal{H}:=\lim_{\delta\to 0}\int_{M_\delta}[X]^*d\mathcal{H}. \tag{5.37}$$

Following the strategy in [1, 6, 7], by the mapping degree theorem, one has

$$-\int_M[X]^*d\mathcal{H}=-\lim_{\delta\to 0}\int_{M_\delta}[X]^*d\mathcal{H}=\sum_{p\in Z(X)}\mathrm{ind}_p\int_{S_pM}\mathcal{H}. \tag{5.38}$$

This completes the proof. □

By Lemma 5.3 and modifying the Finsler metric near the isolated zeros of a given vector field, we are able to give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let $p\in Z(X)$ be any one of the zero points of X. We can find a local coordinate system $(U_p;x^1,\dots,x^{2n})$ around p with $x^i(p)=0$. For simplicity, we change the background Riemannian metric g^{TM} such that

$$g^{TM}|_{U_p}=(dx^1)^2+\cdots+(dx^1)^{2n}.$$

Now we define a Finsler metric $\tilde{F}$ on TU_p as follows: for any $x \in U_p$ and $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$,

$$\tilde{F}\bigg(x, y^i \frac{\partial}{\partial x^i}\bigg|_x\bigg) := F\bigg(p, y^i \frac{\partial}{\partial x^i}\bigg|_p\bigg).$$

It is clear that $\tilde{F}$ is a locally Minkowski metric on TU_p. For a sufficiently small positive number $\epsilon > 0$,

$$B_p(\epsilon) = \left\{x \in U_p \mid r(x) := \sqrt{\sum_{i=1}^{2n} (x^i)^2} < \epsilon\right\}$$

denotes the g^{TM} ball of radius ϵ enclosed in U_p. Set

$$Z_\epsilon(X) = \bigcup_{p \in Z(X)} B_p(\epsilon)$$

and

$$M_\epsilon = M \setminus Z_\epsilon(X).$$

Let $\phi(t)$ be any smooth cutoff function with $0 \leqslant \phi(t) \leqslant 1$, $\phi(t) \equiv 1$ for $t < 0$ and $\phi(t) \equiv 0$ for $t > 1$. It is clear that $|\phi'|$ and $|\phi''|$ are bounded and $\mathrm{supp}(\phi^{(k)}) = [0, 1]$ for any $k = 0, 1, 2, \dots$ Set $C_0 := \max\{|\phi'|, |\phi''|\}$.

For each $p \in Z(X)$, we define the following cutoff function:

$$\phi_{p,\epsilon}(x) = \phi\bigg(\frac{r(x) - \epsilon/2}{\epsilon/2}\bigg)$$

and the following modified metric:

$$F_{p,\epsilon}(x, y) = \sqrt{(1 - \phi_{p,\epsilon}(x))F^2(x, y) + \phi_{p,\epsilon}(x)\tilde{F}^2(x, y)} = \sqrt{(1 - \phi_{p,\epsilon}(x))F^2(x, y) + \phi_{p,\epsilon}(x)F^2(p, y)}.$$

Because

$$g_{p,\epsilon,ij}(x, y) := \frac{1}{2}[F^2_{p,\epsilon}(x, y)]_{y_i y_j} = (1 - \phi_{p,\epsilon}(x))g_{ij}(x, y) + \phi_{p,\epsilon}(x)g_{ij}(p, y)$$

is positive definite, one can easily verify that $F_{p,\epsilon}$ is a well-defined Finsler metric on M.

Set

$$F_\epsilon = \sqrt{\bigg[\prod_{p \in Z(X)} (1 - \phi_{p,\epsilon}(x))\bigg]F^2(x, y) + \sum_{p \in Z(X)} [\phi_{p,\epsilon}(x)F^2(p, y)]}.$$

One verifies that $F_\epsilon = F_{p,\epsilon}$ around $p \in Z(X)$. By definition, $F_\epsilon \equiv F$ on M_ϵ, while it is locally Minkowskian on

$$Z_{\epsilon/2}(X) = \bigcup_{p \in Z(X)} B_p(\epsilon/2).$$

We use $R^{\mathrm{Car}}_\epsilon$ and $\mathcal{H}_\epsilon$ to denote the geometric invariants related to F_ϵ. Obviously, one has $R^{\mathrm{Car}}_\epsilon = R^{\mathrm{Car}}$ on TM_ϵ. On the other hand, by the construction of F_ϵ and the definition (5.34), one directly verifies that $\mathcal{H}_\epsilon = \mathcal{H}$ along $S_p M$ for any $p \in Z(X)$.

Applying Lemma 5.3 to F_ϵ, for any $0 < \delta < \epsilon/2$, we get

$$\begin{aligned}
&\sum_{p \in Z(X)} \mathrm{ind}_p \frac{\mathrm{Vol}(S_p M)}{\mathrm{Vol}(S^{2n-1})} + \bigg(\frac{-1}{2\pi}\bigg)^n \sum_{p \in Z(X)} \mathrm{ind}_p \int_{S_p M} \mathcal{H} \\
&\quad = \sum_{p \in Z(X)} \mathrm{ind}_p \frac{\mathrm{Vol}(S_p M)}{\mathrm{Vol}(S^{2n-1})} + \bigg(\frac{-1}{2\pi}\bigg)^n \sum_{p \in Z(X)} \mathrm{ind}_p \int_{S_p M} \mathcal{H}_\epsilon \\
&\quad = \bigg(\frac{-1}{2\pi}\bigg)^n \lim_{\delta \to 0} \int_{M_\delta} [X]^* \mathrm{Pf}(R^{\mathrm{Car}}_\epsilon)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{-1}{2\pi}\right)^n \int_M [X]^*\mathrm{Pf}(R_\epsilon^{\mathrm{Car}}) \\
&= \left(\frac{-1}{2\pi}\right)^n \int_{M_\epsilon} [X]^*\mathrm{Pf}(R^{\mathrm{Car}}) + \left(\frac{-1}{2\pi}\right)^n \int_{Z_\epsilon(X)} [X]^*\mathrm{Pf}(R_\epsilon^{\mathrm{Car}}). \qquad (5.39)
\end{aligned}
$$

We claim that

$$
\lim_{\epsilon\to 0}\int_{Z_\epsilon(X)} [X]^*\mathrm{Pf}(R_\epsilon^{\mathrm{Car}}) = 0. \qquad (5.40)
$$

The proof of this claim will be presented in Appendix A.

Combining (5.38)–(5.40), we have

$$
\begin{aligned}
&\sum_{p\in Z(X)} \mathrm{ind}_p \frac{\mathrm{Vol}(S_pM)}{\mathrm{Vol}(S^{2n-1})} - \left(\frac{-1}{2\pi}\right)^n \int_M [X]^* d\mathcal{H} \\
&= \sum_{p\in Z(X)} \mathrm{ind}_p \frac{\mathrm{Vol}(S_pM)}{\mathrm{Vol}(S^{2n-1})} + \left(\frac{-1}{2\pi}\right)^n \sum_{p\in Z(X)} \mathrm{ind}_p \int_{S_pM} \mathcal{H} \\
&= \left(\frac{-1}{2\pi}\right)^n \lim_{\epsilon\to 0}\int_{M_\epsilon} [X]^*\mathrm{Pf}(R^{\mathrm{Car}}) + \left(\frac{-1}{2\pi}\right)^n \lim_{\epsilon\to 0}\int_{Z_\epsilon(X)} [X]^*\mathrm{Pf}(R_\epsilon^{\mathrm{Car}}) \\
&= \left(\frac{-1}{2\pi}\right)^n \lim_{\epsilon\to 0}\int_{M_\epsilon} [X]^*\mathrm{Pf}(R^{\mathrm{Car}}) \\
&= \left(\frac{-1}{2\pi}\right)^n \int_M [X]^*\mathrm{Pf}(R^{\mathrm{Car}}).
\end{aligned}
$$

Hence the proof is completed by the assumption on the volumes of the Finsler unit spheres. □

Remark 5.4. It is well known that a Finsler manifold is locally Minkowskian if and only if the Chern curvature $R^{\mathrm{Ch}} = 0$. Thus Bao-Chern's GBC-formula (1.5) immediately implies that the Euler characteristic $\chi(M) = 0$ for locally Minkowski spaces M, whereas it is hard to get this result directly from the Lichnerowicz GBC-formula (1.8). On the other hand, our Theorem 1.1 also directly implies the same vanishing result for locally Minkowski spaces.

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Appendix A

In this appendix, we give a proof of the claim (5.40). We present the following estimations on Z_ϵ. We only need to deal with any one point $p \in Z(X)$, because $Z(X)$ is discrete and finite. First, we have

$$\frac{\partial r}{\partial x^k} = \frac{x^k}{r}, \quad \frac{\partial^2 r}{\partial x^k \partial x^l} = \frac{1}{r}\left(\delta^{kl} - \frac{x^k}{r}\frac{x^l}{r}\right).$$

Then we have the estimations for the first derivatives of $\phi_{p,\epsilon}$:

$$\left|\frac{\partial \phi_{p,\epsilon}}{\partial x^k}(x)\right| = \frac{2}{\epsilon}\left|\phi'\left(\frac{r(x)-\epsilon/2}{\epsilon/2}\right)\frac{\partial r}{\partial x^k}\right| \leqslant 2C_0\frac{1}{\epsilon}.$$

For the second derivatives of $\phi_{p,\epsilon}$, noticing that the support of the derivatives of $\phi_{p,\epsilon}$ is just $\overline{B_p(\epsilon)}\backslash B_p(\epsilon/2)$, we have

$$\left|\frac{\partial^2 \phi_{p,\epsilon}}{\partial x^k \partial x^l}(x)\right| = \frac{4}{\epsilon^2}\left|\phi''\left(\frac{r(x)-\epsilon/2}{\epsilon/2}\right)\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l} + \phi'\left(\frac{r(x)-\epsilon/2}{\epsilon/2}\right)\frac{\epsilon}{2}\frac{\partial^2 r}{\partial x^k \partial x^l}\right| \leqslant 12C_0\frac{1}{\epsilon^2}.$$

As $\epsilon \to 0$, the first and the second derivatives of $g_{p,\epsilon,ij}(x,y)$ with respect to x^i satisfy

$$\begin{aligned}\frac{\partial g_{p,\epsilon,ij}}{\partial x^k}(x,y) &= (1-\phi_{p,\epsilon}(x))\frac{\partial g_{ij}}{\partial x^k}(x,y) + \frac{\partial \phi_{p,\epsilon}}{\partial x^k}(x)(g_{ij}(p,y) - g_{ij}(x,y)) \\ &= (1-\phi_{p,\epsilon}(x))\frac{\partial g_{ij}}{\partial x^k}(x,y) + \frac{\partial \phi_{p,\epsilon}}{\partial x^k}(x)\frac{\partial g_{ij}}{\partial x^t}(p+\theta\cdot(x-p),y)x^t \\ &= O(1)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 g_{p,\epsilon,ij}}{\partial x^k \partial x^l}(x,y) = (1-\phi_{p,\epsilon}(x))&\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(x,y) - \frac{\partial \phi_{p,\epsilon}}{\partial x^k}(x)\frac{\partial g_{ij}}{\partial x^l}(x,y) - \frac{\partial \phi_{p,\epsilon}}{\partial x^l}(x)\frac{\partial g_{ij}}{\partial x^k}(x,y) \\ &+ \frac{\partial^2 \phi_{p,\epsilon}}{\partial x^k \partial x^l}(x)(g_{ij}(p,y) - g_{ij}(x,y)) \\ = (1-\phi_{p,\epsilon}(x))&\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(x,y) - \frac{\partial \phi_{p,\epsilon}}{\partial x^k}(x)\frac{\partial g_{ij}}{\partial x^l}(x,y) - \frac{\partial \phi_{p,\epsilon}}{\partial x^l}(x)\frac{\partial g_{ij}}{\partial x^k}(x,y) \\ &+ \frac{\partial^2 \phi_{p,\epsilon}}{\partial x^k \partial x^l}(x)\frac{\partial g_{ij}}{\partial x^t}(p+\theta\cdot(x-p),y)x^t \\ = O\left(\frac{1}{\epsilon}\right)&,\end{aligned}$$

where $\theta \in [0,1]$. On the other hand, as $\epsilon \to 0$, we have

$$F_{p,\epsilon}\frac{\partial g_{p,\epsilon,ij}}{\partial y^k}(x,y) = O(1), \quad F_{p,\epsilon}\frac{\partial^2 g_{p,\epsilon,ij}}{\partial x^k \partial y^l}(x,y) = O(1), \quad F_{p,\epsilon}^2\frac{\partial^2 g_{p,\epsilon,ij}}{\partial y^k \partial y^l}(x,y) = O(1).$$

By (3.4)–(3.6) and (3.10), as $\epsilon \to 0$, we have

$$(\Gamma_{p,\epsilon})^i_{jk} = O(1), \quad \frac{\partial}{\partial x^l}(\Gamma_{p,\epsilon})^i_{jk} = O\bigg(\frac{1}{\epsilon}\bigg), \quad (P_{p,\epsilon})_j{}^i{}_{kl} = F_{p,\epsilon}\frac{\partial}{\partial y^l}(\Gamma_{p,\epsilon})^i_{jk} = O(1).$$

By (3.10), as $\epsilon \to 0$, we have

$$(R_{p,\epsilon})_j{}^i{}_{kl} = \frac{\delta(\Gamma_{p,\epsilon})^i_{jk}}{\delta x^l} - \frac{\delta(\Gamma_{p,\epsilon})^i_{jl}}{\delta x^k} + O(1) = \frac{\partial(\Gamma_{p,\epsilon})^i_{jk}}{\partial x^k} - \frac{\partial(\Gamma_{p,\epsilon})^i_{jl}}{\partial x^k} + O(1) = O\bigg(\frac{1}{\epsilon}\bigg).$$

Now we estimate $\nabla^{\mathrm{Ch}}\widehat{[X]}$ as $\epsilon \to 0$. By definition,

$$\widehat{[X]}(x,y) = [X]^i \frac{\partial}{\partial \hat{x}^i}\bigg|_{(x,y)} = \frac{X^i(x)}{F_{p,\epsilon}(x,X)}\frac{\partial}{\partial \hat{x}^i}\bigg|_{(x,y)}$$

and

$$\nabla^{\mathrm{Ch}}\widehat{[X]} = (d^M[X]^i + [X]^j(\Gamma_{p,\epsilon})^i_{jk}dx^k)\frac{\partial}{\partial \hat{x}^i}.$$

Assume that the vector field X has the following Taylor expansion near $p \in Z(X)$:

$$X(x) = X^i(x)\frac{\partial}{\partial x^i}\bigg|_x = \sum_{|\alpha|=s} a^i_\alpha x^\alpha \frac{\partial}{\partial x^i}\bigg|_x + o(r^s), \quad \forall\, x \in U_p, \tag{A.1}$$

where the sum is taken over all the multi-indices $\alpha = (\alpha_1, \dots, \alpha_{2n})$ with $|\alpha| = \alpha_1 + \cdots + \alpha_{2n} = s$, and $x^\alpha := (x^1)^{\alpha_1} \cdots (x^{2n})^{\alpha_{2n}}$. When $\epsilon \to 0$, one easily has

$$X^i = O(\epsilon^s), \quad \frac{\partial X^i}{\partial x^j} = O(\epsilon^{s-1}) \quad \text{for } i,j = 1,\dots,2n.$$

Set

$$\xi_\epsilon(x) := F_{p,\epsilon}(x,X) = \sqrt{g_{p,\epsilon,ij}(x,X)X^iX^j}.$$

Because a^i_α's are constants and $g_{p,\epsilon}$ is uniformly bounded and positive definite on $B_p(\epsilon)$, we have

$$\xi_\epsilon(x) = \sqrt{g_{p,\epsilon,ij}(x,X)\sum_{|\alpha|=|\beta|=s} a^i_\alpha a^j_\beta x^\alpha x^\beta} + o(r^s) \geqslant C_1 r^s(x)$$

for some constant $C_1 > 0$. Thus as $\epsilon \to 0$, we obtain

$$[X]^i = \frac{X^i}{\xi_\epsilon} = \frac{\sum_{|\alpha|=s} a^i_\alpha x^\alpha + o(r^s)}{\xi_\epsilon} = O(1) \quad \text{for } i = 1,\dots,2n.$$

Furthermore, by the Euler lemma for homogeneous functions, as $\epsilon \to 0$ we have

$$\begin{aligned}\frac{\partial \xi_\epsilon}{\partial x^k} &= \frac{1}{2\xi_\epsilon}\frac{\partial \xi_\epsilon^2}{\partial x^k} = \frac{1}{2\xi_\epsilon}\bigg[\frac{\partial g_{p,\epsilon,ij}}{\partial x^k}(x,X)X^iX^j + \frac{\partial g_{p,\epsilon,ij}}{\partial y^s}(x,X)\frac{\partial X^s}{\partial x^k}X^iX^j + g_{p,\epsilon,ij}(x,X)\frac{\partial}{\partial x^k}(X^iX^j)\bigg]\\ &= \frac{1}{2\xi_\epsilon}\bigg[\frac{\partial g_{p,\epsilon,ij}}{\partial x^k}(x,X)X^iX^j + g_{p,\epsilon,ij}(x,X)\frac{\partial}{\partial x^k}(X^iX^j)\bigg]\\ &= g_{p,\epsilon,ij}(x,X)\frac{\partial X^i}{\partial x^k}[X]^j + O(\epsilon^s)\\ &= O(\epsilon^{s-1}).\end{aligned}$$

Hence,

$$\begin{aligned}[X]^*(\nabla^{\mathrm{Ch}}\widehat{[X]}) &= \frac{\partial [X]^i}{\partial x^k}\frac{\partial}{\partial x^i} + O(1) = \left(\frac{\partial}{\partial x^k}\frac{X^i}{\xi_\epsilon}\right)\frac{\partial}{\partial x^i} + O(1) \\ &= \frac{1}{\xi_\epsilon}\left[\frac{\partial X^i}{\partial x^k} - [X]^i\frac{\partial \xi_\epsilon}{\partial x^k}\right]\frac{\partial}{\partial x^i} + O(1) = O\left(\frac{1}{\epsilon}\right) \quad \text{as } \epsilon \to 0.\end{aligned}$$

When $\epsilon \to 0$, we get

$$[X]^*\Theta_{p,\epsilon} = \left(g^{ik}_{p,\epsilon}(x,[X])(A_{p,\epsilon})_{kjl}(x,[X])[X]^*\frac{(\nabla^{\mathrm{Ch}}\widehat{[X]})^l}{F_{p,\epsilon}}\right) = O\left(\frac{1}{\epsilon}\right)$$

and

$$[X]^*R^{\mathrm{Ch}}_{p,\epsilon} = \left(\frac{1}{2}(R_{p,\epsilon})_j{}^i{}_{kl}(x,[X])dx^k \wedge dx^l + (P_{p,\epsilon})_j{}^i{}_{kl}(x,[X])dx^k \wedge [X]^*(\nabla^{\mathrm{Ch}}\widehat{[X]})^l\right) = O\left(\frac{1}{\epsilon}\right).$$

By (3.14), as $\epsilon \to 0$, we obtain

$$[X]^*\mathrm{Pf}(R^{\mathrm{Car}}_{p,\epsilon}) = [X]^*\mathrm{Pf}(\widehat{R}^{\mathrm{Ch}}_{p,\epsilon} + Q_{p,\epsilon}) = O\left(\frac{1}{\epsilon^{2n-1}}\right).$$

But the volume of $B_p(\epsilon)$ is

$$\mathrm{Vol}(B_p(\epsilon)) = O(\epsilon^{2n}),$$

and then we get

$$\left|\int_{Z_\epsilon(X)}[X]^*\mathrm{Pf}(R^{\mathrm{Car}}_\epsilon)\right| = \left|\sum_{p\in Z(X)}\int_{B_p(\epsilon)}[X]^*\mathrm{Pf}(R^{\mathrm{Car}}_{p,\epsilon})\right| \leqslant C\epsilon \to 0 \quad \text{as } \epsilon \to 0,$$

where C is a constant. So the claim is valid.