中国科学 : 数学 2025 年 第 55 卷 第 4 期 : 791 ∼ 802
SCIENTIA SINICA Mathematica
论 文

《中国科学》杂志社
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加权 Lebesgue 空间中超齐次核有界积分算子的构造条件及范数计算公式

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收稿日期: 2023-05-25; 接受日期: 2024-02-28; 网络出版日期: 2024-03-18
广东省基础与应用基础研究基金 (批准号: 2022A1515012429) 和广州华商学院科研团队基金 (批准号: 2021HSKT03) 资助项目

摘要 本文引入超齐次函数的概念, 用超齐次核统一齐次核、广义齐次核及若干非齐次核. 首先利用权函数方法讨论具有超齐次核的 Hilbert 型积分不等式; 然后根据 Hilbert 型积分不等式与同核积分算子的联系, 讨论具有超齐次核的积分算子, 得到在加权 Lebesgue 空间中构造超齐次核有界积分算子的充分必要条件及算子范数的计算公式.

关键词 超齐次核 有界积分算子 Hilbert 型积分不等式 加权 Lebesgue 空间 算子范数 构造条件

MSC (2020) 主题分类 26D15

1 引言

设 $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, 针对 -1 阶齐次核 $\frac{1}{x+y}$, Hardy [1] 得到了著名的 Hilbert 积分不等式

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y}dxdy\leqslant\frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_p\|g\|_q,$$

其中常数因子 $\frac{\pi}{\sin(\frac{\pi}{p})}$ 是最佳值. 利用该不等式, 可以证明积分算子 $T(f)(y)=\int_0^{+\infty}\frac{1}{x+y}f(x)dx$ 是空间 $L_p(0,+\infty)$ 中的有界算子, 且 T 的算子范数 $\|T\|=\frac{\pi}{\sin(\frac{\pi}{p})}$. 文献 [10,18] 讨论了核为 $-\lambda$ 阶齐次函数 $\frac{1}{x^\lambda+y^\lambda}$ 和 $\frac{1}{(x+y)^\lambda}$ 的情形, 分别得到

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x^\lambda+y^\lambda}dxdy\leqslant\frac{\pi}{\lambda\sin(\frac{\pi}{2\lambda})}\|f\|_{p,1-\lambda}\|g\|_{q,1-\lambda},\tag{1.1}$$

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x+y)^\lambda}dxdy\leqslant B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\|f\|_{p,1-\lambda}\|g\|_{q,1-\lambda},\tag{1.2}$$

英文引用格式: Hong Y. Construction conditions for the bounded integral operator with the super-homogeneous kernel in the weighted Lebesgue space and the formula of the operator norm (in Chinese). Sci Sin Math, 2025, 55: 791–802, doi: 10.1360/SSM-2023-0156

其中 (1.1) 和 (1.2) 的常数因子均为最佳值. 利用 (1.1) 和 (1.2) 可以证明积分算子 $T_1(y)=\int_0^{+\infty}\frac{1}{x^\lambda+y^\lambda}dx$ 和 $T_2(y)=\int_0^{+\infty}\frac{1}{(x+y)^\lambda}dx$ 都是 $L_p^{1-\lambda}(0,+\infty)$ 中的有界算子, 且 T_1 和 T_2 的算子范数分别为

$$\|T_1\|=\frac{\pi}{\lambda\sin(\frac{\pi}{2\lambda})},\quad \|T_2\|=B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right),$$

其中 $L_p^r(0,+\infty)$ 表示加权 Lebesgue 空间

$$L_p^r(0,+\infty)=\left\{f(x):\|f\|_{p,r}=\left(\int_0^{+\infty}x^r|f(x)|^pdx\right)^{\frac{1}{p}}<+\infty\right\}.$$

之后, 这些结果被推广到各种各样的齐次核、广义齐次核和若干非齐次核的情形 (参见文献 [3,9,13,16,17,19,20]).

一个自然的问题是, 对于具有某些参数的核 $K(x,y)$ 的积分算子 T:

$$T(f)(y)=\int_0^{+\infty}K(x,y)f(x)dx, \tag{1.3}$$

它在什么条件下是 $L_p^\alpha(0,+\infty)$ 到 $L_p^\gamma(0,+\infty)$ 的有界算子? 其算子范数 $\|T\|$ 又如何计算? 显然这不仅与核 $K(x,y)$ 的参数有关, 还与 Lebesgue 空间的参数 p、α 和 γ 有关. 这些问题的解决对于算子理论和与算子理论相关的分析类学科有重要意义, 例如, Riesz 位势算子 $T_\alpha(f)(y)=\int_{\mathbb{R}^n}\frac{1}{|x-y|^\alpha}f(x)\,dx$ 在各类空间上的有界性质在近代调和分析中就具有广泛应用.

文献 [8] 针对抽象的齐次核 $K(x,y)$ 首次讨论了构建 Hilbert 型级数不等式及同核离散有界算子的参数条件, 得到了一个充分条件, 但未能证明条件的必要性. 文献 [4] 进一步讨论了齐次核 $K(x,y)$ 的积分算子情形, 并得到了充分必要条件. 之后, 该结果被推广, 得到更多有相关的结果 (参见文献 [2,5–7,11,12,14,15]).

为了统一齐次核、广义齐次核和若干非齐次核, 从更一般的角度讨论问题, 本文引入超齐次函数概念, 并讨论超齐次核积分算子的构造条件.

定义 1.1 设 $\sigma_1,\sigma_2,\tau_1,\tau_2\in\mathbb{R}$, 若函数 $K(x,y)$ 满足: 对于任意 $t>0$, 有

$$K(tx,y)=t^{\sigma_1}K(x,t^{\tau_1}y),\quad K(x,ty)=t^{\sigma_2}K(t^{\tau_2}x,y),$$

则称 $K(x,y)$ 是具有参数 $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$ 的超齐次函数.

若 $K_1(x,y)$ 是 λ 阶齐次函数, 则 $K_1(x,y)$ 是具有参数 $\{\lambda,\lambda,-1,-1\}$ 的超齐次函数, 广义齐次函数 $K_2(x,y)=K_1(x^{\lambda_1},y^{\lambda_2})$ 是具有参数 $\{\lambda\lambda_1,\lambda\lambda_2,-\frac{\lambda_1}{\lambda_2},-\frac{\lambda_2}{\lambda_1}\}$ 的超齐次函数. 若 $G(u)$ 是一个实函数, 则非齐次函数 $K_3(x,y)=G(x^{\lambda_1}y^{\lambda_2})$ 是具有参数 $\{0,0,\frac{\lambda_1}{\lambda_2},\frac{\lambda_2}{\lambda_1}\}$ 的超齐次函数. 由此可见, 超齐次函数统一了常见的 Hilbert 型不等式及相应算子研究中的核函数.

若 $K(x,y)$ 是具有参数 $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$ 的超齐次函数, 由于

$$K(tx,y)=t^{\sigma_1}K(x,t^{\tau_1}y)=t^{\sigma_1+\tau_1\sigma_2}K(t^{\tau_1\tau_2}x,y),$$

可见在一般情形下都有 $\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, 因此不失一般性, 本文的讨论总是在 $\tau_1\tau_2=1$ 及 $\sigma_1+\tau_1\sigma_2=0$ 的条件下进行.

2　预备引理

为了避免不必要的重复, 记

$$A(K,f,g)=\int_0^{+\infty}\int_0^{+\infty}K(x,y)f(x)g(y)dxdy,$$

$$W_1(s)=\int_0^{+\infty}K(1,t)t^sdt,\quad W_2(s)=\int_0^{+\infty}K(t,1)t^sdt.$$

引理 2.1　设 $K(x,y)$ 是具有参数 $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$ 的超齐次函数, 则

(i) 当 $\tau_2\frac{\alpha+1}{p}-\frac{\beta+1}{q}=\tau_2-\sigma_2-1$ 时, 有

$$W_1\bigg(-\frac{\beta+1}{q}\bigg)=\frac{1}{|\tau_2|}W_2\bigg(-\frac{\alpha+1}{p}\bigg);$$

(ii) 当 $\tau_1\frac{\beta+1}{q}-\frac{\alpha+1}{p}=\tau_1-\sigma_1-1$ 时, 有

$$W_2\bigg(-\frac{\alpha+1}{p}\bigg)=\frac{1}{|\tau_1|}W_1\bigg(-\frac{\beta+1}{q}\bigg).$$

证明　(i) 因为 $\tau_2\frac{\alpha+1}{p}-\frac{\beta+1}{q}=\tau_2-\sigma_2-1$, 故 $\frac{1}{\tau_2}(\sigma_2-\frac{\beta+1}{q})+\frac{1}{\tau_2}-1=-\frac{\alpha+1}{p}$, 于是

$$\begin{aligned}W_1\bigg(-\frac{\beta+1}{q}\bigg)&=\int_0^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}}dt\\&=\int_0^{+\infty}K(t^{\tau_2},1)t^{\sigma_2-\frac{\beta+1}{q}}dt\\&=\frac{1}{|\tau_2|}\int_0^{+\infty}K(u,1)u^{\frac{1}{\tau_2}(\sigma_2-\frac{\beta+1}{q})+\frac{1}{\tau_2}-1}du\\&=\frac{1}{|\tau_2|}\int_0^{+\infty}K(u,1)u^{-\frac{\alpha+1}{p}}du\\&=\frac{1}{|\tau_2|}W_2\bigg(-\frac{\alpha+1}{p}\bigg).\end{aligned}$$

(ii) 类似可证 $W_2(-\frac{\alpha+1}{p})=\frac{1}{|\tau_1|}W_1(-\frac{\beta+1}{q})$. □

引理 2.2　当且仅当 $\tau_1\tau_2=1$ 和 $\sigma_1+\tau_1\sigma_2=0$ 时, $\tau_1\frac{\beta+1}{q}-\frac{\alpha+1}{p}=\tau_1-\sigma_1-1$ 与 $\tau_2\frac{\alpha+1}{p}-\frac{\beta+1}{q}=\tau_2-\sigma_2-1$ 等价.

证明　记 $\frac{\alpha+1}{p}=x_1$ 和 $\frac{\beta+1}{q}=x_2$, 则 $\tau_1\frac{\beta+1}{q}-\frac{\alpha+1}{p}=\tau_1-\sigma_1-1$ 与 $\tau_2\frac{\alpha+1}{p}-\frac{\beta+1}{q}=\tau_2-\sigma_2-1$ 等价的充分必要条件是二元线性方程组

$$\begin{cases}-x_1+\tau_1x_2=\tau_1-\sigma_1-1,\\ \tau_2x_1-x_2=\tau_2-\sigma_2-1\end{cases}$$

的系数对应成比例, 即 $\frac{-1}{\tau_2}=\frac{\tau_1}{-1}=\frac{\tau_1-\sigma_1-1}{\tau_2-\sigma_2-1}$, 而这等价于 $\tau_1\tau_2=1$ 和 $\sigma_1+\tau_1\sigma_2=0$, 故引理 2.2 成立. □

引理 2.3　设 $K(x,y)$ 是具有参数 $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$ 的超齐次函数, 则

$$\omega_1(x,\beta)=\int_0^{+\infty}K(x,y)y^{-\frac{\beta+1}{q}}dy=x^{\sigma_1+\tau_1(\frac{\beta+1}{q}-1)}W_1\bigg(-\frac{\beta+1}{q}\bigg),$$

$$\omega_2(y,\alpha)=\int_0^{+\infty}K(x,y)y^{-\frac{\alpha+1}{p}}dx=y^{\sigma_2+\tau_2(\frac{\alpha+1}{p}-1)}W_2\Big(-\frac{\alpha+1}{p}\Big).$$

证明 根据 $K(x,y)$ 的性质, 有

$$\begin{aligned}\omega_1(x,\beta)&=x^{\sigma_1}\int_0^{+\infty}K(1,x^{\tau_1}y)y^{-\frac{\beta+1}{q}}dy\\&=x^{\sigma_1-\tau_1+\tau_1\frac{\beta+1}{q}}\int_0^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}}dt\\&=x^{\sigma_1+\tau_1(\frac{\beta+1}{q}-1)}W_1\Big(-\frac{\beta+1}{q}\Big).\end{aligned}$$

同理可证 $\omega_2(y,\alpha)=y^{\sigma_2+\tau_2(\frac{\alpha+1}{p}-1)}W_2(-\frac{\alpha+1}{p})$. □

3 超齐次核 Hilbert 型积分不等式的构建条件

定理 3.1 设 $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\alpha,\beta\in\mathbb{R}$, $K(x,y)$ 是具有参数 $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$ 的超齐次非负可测函数, $\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, $0<W_2(-\frac{\alpha+1}{p})<+\infty$.

(i) 当且仅当 $\tau_1(\frac{1}{p}-\frac{\beta}{q})-(\frac{1}{q}-\frac{\alpha}{p})=\sigma_1$ 时, 存在常数 $M>0$ 使得

$$A(K,f,g)=\int_0^{+\infty}\int_0^{+\infty}K(x,y)f(x)g(y)dxdy\leqslant M\|f\|_{p,\alpha}\|g\|_{q,\beta},\tag{3.1}$$

且 $M\geqslant W_1^{\frac{1}{p}}(-\frac{\beta+1}{q})W_2^{\frac{1}{q}}(-\frac{\alpha+1}{p})$, 其中, $f\in L_p^{\alpha}(0,+\infty)$, $g\in L_q^{\beta}(0,+\infty)$.

(ii) 当 $\tau_1(\frac{1}{p}-\frac{\beta}{q})-(\frac{1}{q}-\frac{\alpha}{p})=\sigma_1$ 时, (3.1) 的最佳常数因子为

$$M_0=\inf\{M\}=\Big(\frac{1}{|\tau_1|}\Big)^{\frac{1}{q}}W_1\Big(-\frac{\beta+1}{q}\Big)=\Big(\frac{1}{|\tau_2|}\Big)^{\frac{1}{p}}W_2\Big(-\frac{\alpha+1}{p}\Big).$$

证明 (i) 设 $\tau_1(\frac{1}{p}-\frac{\beta}{q})-(\frac{1}{q}-\frac{\alpha}{p})=\sigma_1$, 则

$$\tau_1\frac{\beta+1}{q}-\frac{\alpha+1}{p}=\tau_1-\sigma_1-1,$$

由引理 2.2, 有 $\tau_2\frac{\alpha+1}{p}-\frac{\beta+1}{q}=\tau_2-\sigma_2-1$. 因 $0<W_2(-\frac{\alpha+1}{p})<+\infty$, 故由引理 2.1, 有 $0<W_1(-\frac{\beta+1}{q})<+\infty$. 于是根据 Hölder 不等式及引理 2.3, 有

$$\begin{aligned}A(K,f,g)&=\int_0^{+\infty}\int_0^{+\infty}\bigg(\frac{x^{\frac{\alpha+1}{pq}}}{y^{\frac{\beta+1}{pq}}}f(x)\bigg)\bigg(\frac{y^{\frac{\beta+1}{pq}}}{x^{\frac{\alpha+1}{pq}}}g(y)\bigg)K(x,y)dxdy\\&\leqslant\bigg(\int_0^{+\infty}\int_0^{+\infty}x^{(\alpha+1)/q}y^{-(\beta+1)/q}|f(x)|^pK(x,y)dxdy\bigg)^{\frac{1}{p}}\\&\quad\times\bigg(\int_0^{+\infty}\int_0^{+\infty}y^{(\beta+1)/p}x^{-(\alpha+1)/p}|g(y)|^qK(x,y)dxdy\bigg)^{\frac{1}{q}}\\&=\bigg(\int_0^{+\infty}x^{(\alpha+1)/q}|f(x)|^p\omega_1(x,\beta)dx\bigg)^{\frac{1}{p}}\bigg(\int_0^{+\infty}y^{(\beta+1)/p}|g(y)|^q\omega_2(y,\alpha)dy\bigg)^{\frac{1}{q}}\\&\leqslant W_1^{\frac{1}{p}}\Big(-\frac{\beta+1}{q}\Big)W_2^{\frac{1}{q}}\Big(-\frac{\alpha+1}{p}\Big)\bigg(\int_0^{+\infty}x^{\frac{\alpha+1}{q}+\sigma_1+\tau_1(\frac{\beta+1}{q}-1)}|f(x)|^pdx\bigg)^{\frac{1}{p}}\end{aligned}$$

$$
\begin{aligned}
&\times\bigg(\int_0^{+\infty} y^{\frac{\beta+1}{p}+\sigma_2+\tau_2(\frac{\alpha+1}{p}-1)}|g(y)|^q dy\bigg)^{\frac{1}{q}}\\
&=W_1^{\frac{1}{p}}\bigg(-\frac{\beta+1}{q}\bigg)W_2^{\frac{1}{q}}\bigg(-\frac{\alpha+1}{p}\bigg)\bigg(\int_0^{+\infty} x^{\alpha}|f(x)|^p dx\bigg)^{\frac{1}{p}}\bigg(\int_0^{+\infty} y^{\beta}|g(y)|^q dy\bigg)^{\frac{1}{q}}\\
&=W_1^{\frac{1}{p}}\bigg(-\frac{\beta+1}{q}\bigg)W_2^{\frac{1}{q}}\bigg(-\frac{\alpha+1}{p}\bigg)\|f\|_{p,\alpha}\|g\|_{q,\beta},
\end{aligned}
$$

任取常数 $M \geqslant W_1^{\frac{1}{p}}(-\frac{\beta+1}{q})W_2^{\frac{1}{q}}(-\frac{\alpha+1}{p})$, (3.1) 都成立.

反之, 若存在常数 M 使 (3.1) 成立, 记 $\tau_1\frac{\beta+1}{q}-\frac{\alpha+1}{p}-(\tau_1-\sigma_1-1)=c$.

若 $c>0$, 先讨论 $\tau_1<0$ 的情形, 此时取 $0<\varepsilon<\frac{c}{-\tau_1}$, 令

$$
f(x)=\begin{cases} x^{-(\alpha+1-\tau_1\varepsilon)/p}, & x\geqslant 1,\\ 0, & 0<x<1,\end{cases}
$$

$$
g(y)=\begin{cases} y^{-(\beta+1+\varepsilon)/q}, & y\geqslant 1,\\ 0, & 0<y<1,\end{cases}
$$

则有

$$
\begin{aligned}
\|f\|_{p,\alpha}\|g\|_{q,\beta}&=\bigg(\int_1^{+\infty} x^{-1+\tau_1\varepsilon}dx\bigg)^{\frac{1}{p}}\bigg(\int_1^{+\infty} y^{-1-\varepsilon}dy\bigg)^{\frac{1}{q}}=\frac{1}{\varepsilon}\bigg(\frac{1}{|\tau_1|}\bigg)^{\frac{1}{p}},\\
A(K,f,g)&=\int_1^{+\infty} x^{-\frac{\alpha+1}{p}+\frac{\tau_1\varepsilon}{p}}\bigg(\int_1^{+\infty} K(x,y)y^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dy\bigg)dx\\
&=\int_1^{+\infty} x^{\sigma_1-\frac{\alpha+1}{p}+\frac{\tau_1\varepsilon}{p}}\bigg(\int_1^{+\infty} K(1,x^{\tau_1}y)y^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dy\bigg)dx\\
&=\int_1^{+\infty} x^{\sigma_1-\frac{\alpha+1}{p}+\frac{\tau_1\varepsilon}{p}+\tau_1(\frac{\beta+1}{q}+\frac{\varepsilon}{q})-\tau_1}\bigg(\int_{x^{\tau_1}}^{+\infty} K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\bigg)dx\\
&=\int_1^{+\infty} x^{-1+c+\tau_1\varepsilon}\bigg(\int_{x^{\tau_1}}^{+\infty} K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\bigg)dx.
\end{aligned}
$$

因为 $\tau_1<0$, 故当 $x\geqslant 1$ 时, 有 $x^{\tau_1}\leqslant 1$, 于是

$$
A(K,f,g)\geqslant\int_1^{+\infty} x^{-1+c+\tau_1\varepsilon}dx\int_1^{+\infty} K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt,
$$

从而

$$
\int_1^{+\infty} x^{-1+c+\tau_1\varepsilon}dx\int_1^{+\infty} K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\leqslant\frac{M}{\varepsilon}\bigg(\frac{1}{|\tau_1|}\bigg)^{\frac{1}{p}}<+\infty. \tag{3.2}
$$

又因为 $0<\varepsilon<\frac{c}{-\tau_1}$, 所以 $1-c-\tau_1\varepsilon<1$, 故 $\int_1^{+\infty} x^{-1+c+\tau_1\varepsilon}dx=+\infty$, 这与 (3.2) 矛盾.

再讨论 $\tau_1>0$ 的情形, 此时取 $0<\varepsilon<\frac{c}{\tau_1}$, 令

$$
f(x)=\begin{cases} x^{-(\alpha+1+\tau_1\varepsilon)/p}, & x\geqslant 1,\\ 0, & 0<x<1,\end{cases}
$$

$$g(y)=\begin{cases}y^{-(\beta+1-\varepsilon)/q}, & 0<y\leqslant 1,\\ 0, & y>1,\end{cases}$$

则有

$$\begin{aligned}\|f\|_{p,\alpha}\|g\|_{q,\beta}&=\left(\int_1^{+\infty}x^{-1-\tau_1\varepsilon}dx\right)^{\frac{1}{p}}\left(\int_0^1 y^{-1+\varepsilon}dy\right)^{\frac{1}{q}}=\frac{1}{\varepsilon}\left(\frac{1}{\tau_1}\right)^{\frac{1}{p}},\\ A(K,f,g)&=\int_1^{+\infty}x^{-\frac{\alpha+1}{p}-\frac{\tau_1\varepsilon}{p}}\left(\int_0^1 K(x,y)y^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dy\right)dx\\ &=\int_1^{+\infty}x^{\sigma_1-\frac{\alpha+1}{p}-\frac{\tau_1\varepsilon}{p}}\left(\int_0^1 K(1,x^{\tau_1}y)y^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dy\right)dx\\ &=\int_1^{+\infty}x^{-1+c-\tau_1\varepsilon}\left(\int_0^{x^{\tau_1}}K(1,t)t^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dt\right)dx\\ &\geqslant\int_1^{+\infty}x^{-1+c-\tau_1\varepsilon}dx\int_0^1 K(1,t)t^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dt,\end{aligned}$$

于是

$$\int_1^{+\infty}x^{-1+c-\tau_1\varepsilon}dx\int_0^1 K(1,t)t^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dt\leqslant\frac{M}{\varepsilon}\left(\frac{1}{\tau_1}\right)^{\frac{1}{p}}<+\infty. \tag{3.3}$$

因为 $0<\varepsilon<\frac{c}{\tau_1}$, 所以 $1-c+\tau_1\varepsilon<1$, 从而 $\int_1^{+\infty}x^{-1+c-\tau_1\varepsilon}dx=+\infty$, 这与 (3.3) 矛盾.

综上讨论知 $c>0$ 不能成立.

若 $c<0$, 先讨论 $\tau_1<0$ 的情形, 取 $0<\varepsilon<\frac{c}{\tau_1}$, 令

$$f(x)=\begin{cases}x^{-(\alpha+1+\tau_1\varepsilon)/p}, & 0<x\leqslant 1,\\ 0, & x>1,\end{cases}$$

$$g(y)=\begin{cases}y^{-(\beta+1-\varepsilon)/q}, & 0<y\leqslant 1,\\ 0, & y>1.\end{cases}$$

使用推导 (3.2) 的类似方法, 可得

$$\int_0^1 x^{-1+c-\tau_1\varepsilon}dx\int_0^1 K(1,t)t^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dt\leqslant\frac{M}{\varepsilon}\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{p}}<+\infty. \tag{3.4}$$

因为 $0<\varepsilon<\frac{c}{\tau_1}$, 所以 $1-c+\tau_1\varepsilon>1$, 从而 $\int_0^1 x^{-1+c-\tau_1\varepsilon}dx=+\infty$, 这与 (3.4) 矛盾.

再讨论 $\tau_1>0$ 的情形, 此时取 $0<\varepsilon<\frac{-c}{\tau_1}$, 令

$$f(x)=\begin{cases}x^{-(\alpha+1-\tau_1\varepsilon)/p}, & 0<x\leqslant 1,\\ 0, & x>1,\end{cases}$$

$$g(y)=\begin{cases}y^{-(\beta+1+\varepsilon)/q}, & y\geqslant 1,\\ 0, & 0<y<1,\end{cases}$$

使用推导 (3.3) 的类似方法, 可得

$$\int_0^1 x^{-1+c+\tau_1\varepsilon}dx\int_1^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\leqslant\frac{M}{\varepsilon}\left(\frac{1}{\tau_1}\right)^{\frac{1}{p}}<+\infty. \tag{3.5}$$

因为 $0<\varepsilon<\frac{-c}{\tau_1}$, 所以 $1-c-\tau_1\varepsilon>1$, 从而 $\int_0^1 x^{-1+c+\tau_1\varepsilon}dx=+\infty$, 这与 (3.5) 矛盾.

综上讨论知 $c<0$ 也不能成立, 从而 $c=0$, 故 $\tau_1\frac{\beta+1}{q}-\frac{\alpha+1}{p}=\tau_1-\sigma_1-1$, 即 $\tau_1(\frac{1}{p}-\frac{\beta}{q})-(\frac{1}{q}-\frac{\alpha}{p})=\sigma_1$.

(ii) 当 $\tau_1(\frac{1}{p}-\frac{\beta}{q})-(\frac{1}{q}-\frac{\alpha}{p})=\sigma_1$ 时, 有 $\tau_1\frac{\beta+1}{q}-\frac{\alpha+1}{p}=\tau_1-\sigma_1-1$. 由引理 2.2, 此时也有 $\tau_2\frac{\alpha+1}{p}-\frac{\beta+1}{q}=\tau_2-\sigma_2-1$. 根据引理 2.1, 有

$$W_1^{\frac{1}{p}}\left(-\frac{\beta+1}{q}\right)W_2^{\frac{1}{q}}\left(-\frac{\alpha+1}{p}\right)=\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1\left(-\frac{\beta+1}{q}\right)=\left(\frac{1}{|\tau_2|}\right)^{\frac{1}{p}}W_2\left(-\frac{\alpha+1}{p}\right),$$

于是由 (i) 可得

$$A(K,f,g)\leqslant\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1\left(-\frac{\beta+1}{q}\right)\|f\|_{p,\alpha}\|g\|_{q,\beta}.$$

若 (3.1) 的最佳常数因子不是 $(\frac{1}{|\tau_1|})^{\frac{1}{q}}W_1(-\frac{\beta+1}{q})$, 则存在常数 M_0 使得

$$0<M_0<\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1\left(-\frac{\beta+1}{q}\right),\quad A(K,f,g)\leqslant M_0\|f\|_{p,\alpha}\|g\|_{q,\beta}.$$

若 $\tau_1<0$, 取充分小的 $\varepsilon>0$ 及足够大的 $a>0$, 令

$$f(x)=\begin{cases}x^{-(\alpha+1-\tau_1\varepsilon)/p}, & x\geqslant a,\\ 0, & 0<x<a,\end{cases}$$

$$g(y)=\begin{cases}y^{-(\beta+1+\varepsilon)/q}, & y\geqslant 1,\\ 0, & 0<y<1,\end{cases}$$

则

$$\|f\|_{p,\alpha}\|g\|_{q,\beta}=\left(\int_a^{+\infty}x^{-1+\tau_1\varepsilon}dx\right)^{\frac{1}{p}}\left(\int_1^{+\infty}y^{-1-\varepsilon}dy\right)^{\frac{1}{q}}=\frac{1}{\varepsilon}\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{p}}a^{\frac{\tau_1\varepsilon}{q}}.$$

因为 $\tau_1<0$, 所以当 $x\geqslant a$ 时, 有 $x^{\tau_1}\leqslant a^{\tau_1}$. 于是

$$\begin{aligned}A(K,f,g)&=\int_a^{+\infty}x^{-\frac{\alpha+1}{p}+\frac{\tau_1\varepsilon}{p}}\left(\int_1^{+\infty}K(x,y)y^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dy\right)dx\\&=\int_a^{+\infty}x^{\sigma_1-\frac{\alpha+1}{p}+\frac{\tau_1\varepsilon}{p}}\left(\int_1^{+\infty}K(1,x^{\tau_1}y)y^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dy\right)dx\\&=\int_a^{+\infty}x^{\sigma_1-\frac{\alpha+1}{p}+\frac{\tau_1\varepsilon}{p}+\tau_1(\frac{\beta+1}{q}+\frac{\varepsilon}{q})-\tau_1}\left(\int_{x^{\tau_1}}^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\right)dx\\&=\int_a^{+\infty}x^{-1+\tau_1\varepsilon}\left(\int_{x^{\tau_1}}^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\right)dx\\&\geqslant\int_a^{+\infty}x^{-1+\tau_1\varepsilon}dx\int_{a^{\tau_1}}^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\\&=\frac{1}{|\tau_1|\varepsilon}a^{\tau_1\varepsilon}\int_{a^{\tau_1}}^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt.\end{aligned}$$

从而

$$\frac{1}{|\tau_1|\varepsilon}a^{\tau_1\varepsilon}\int_{a^{\tau_1}}^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\leqslant\frac{M_0}{\varepsilon}\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{p}}a^{\frac{\tau_1\varepsilon}{q}}.$$

故有

$$\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}a^{\frac{\tau_1\varepsilon}{p}}\int_{a^{\tau_1}}^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\leqslant M_0,\tag{3.6}$$

可视 ε 为一个趋于 0 的正项递减数列, 由著名的 Fatou 引理, 有

$$\begin{aligned}\int_{a^{\tau_1}}^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}}dt&=\int_{a^{\tau_1}}^{+\infty}\liminf_{\varepsilon\to0^+}K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt\\&\leqslant\liminf_{\varepsilon\to0^+}\int_{a^{\tau_1}}^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}-\frac{\varepsilon}{q}}dt.\end{aligned}$$

据此, 在 (3.6) 中令 $\varepsilon\to0^+$, 可得

$$\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}\int_{a^{\tau_1}}^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}}dt\leqslant M_0.$$

再令 $a\to+\infty$, 并注意 $\tau_1<0$, 有

$$\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}W_1\left(-\frac{\beta+1}{q}\right)=\left(\frac{1}{|\tau_1|}\right)^{\frac{1}{q}}\int_0^{+\infty}K(1,t)t^{-\frac{\beta+1}{q}}dt\leqslant M_0,$$

这与 $M_0<(\frac{1}{|\tau_1|})^{\frac{1}{q}}W_1(-\frac{\beta+1}{q})$ 矛盾.

若 $\tau_1>0$, 仍取充分小的 $\varepsilon>0$ 及足够大的 $a>0$, 令

$$f(x)=\begin{cases}x^{-(\alpha+1+\tau_1\varepsilon)/p}, & x\geqslant1,\\0, & 0<x<1,\end{cases}$$

$$g(y)=\begin{cases}y^{-(\beta+1-\varepsilon)/q}, & 0<y\leqslant a,\\0, & y>a,\end{cases}$$

则有

$$\|f\|_{p,\alpha}\|g\|_{q,\beta}=\left(\int_1^{+\infty}x^{-1-\tau_1\varepsilon}dx\right)^{\frac{1}{p}}\left(\int_0^a y^{-1+\varepsilon}dy\right)^{\frac{1}{q}}=\frac{1}{\varepsilon}a^{\frac{\varepsilon}{q}}\left(\frac{1}{\tau_1}\right)^{\frac{1}{q}},$$

$$\begin{aligned}A(K,f,g)&=\int_1^{+\infty}x^{-\frac{\alpha+1}{p}-\frac{\tau_1\varepsilon}{p}}\left(\int_0^a K(x,y)y^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dy\right)dx\\&=\int_1^{+\infty}x^{\sigma_1-\frac{\alpha+1}{p}-\frac{\tau_1\varepsilon}{p}}\left(\int_0^a K(1,x^{\tau_1}y)y^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dy\right)dx\\&=\int_1^{+\infty}x^{-1-\tau_1\varepsilon}\left(\int_0^{ax^{\tau_1}}K(1,t)t^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dt\right)dx\\&\geqslant\int_1^{+\infty}x^{-1-\tau_1\varepsilon}dx\int_0^a K(1,t)t^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dt\\&=\frac{1}{|\tau_1|\varepsilon}\int_0^a K(1,t)t^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dt.\end{aligned}$$

从而

$$\frac{1}{|\tau_1|\varepsilon}\int_0^a K(1,t)t^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dt \leqslant \frac{M_0}{\varepsilon}a^{\frac{\varepsilon}{q}}\Big(\frac{1}{|\tau_1|}\Big)^{\frac{1}{p}}.$$

故有

$$\Big(\frac{1}{|\tau_1|}\Big)^{\frac{1}{q}}\int_0^a K(1,t)t^{-\frac{\beta+1}{q}+\frac{\varepsilon}{q}}dt \leqslant M_0 a^{\frac{\varepsilon}{q}}.$$

同样地, 令 $\varepsilon\to 0^+$, 并利用 Fatou 引理, 得到

$$\Big(\frac{1}{|\tau_1|}\Big)^{\frac{1}{q}}\int_0^a K(1,t)t^{-\frac{\beta+1}{q}}dt \leqslant M_0.$$

再令 $a\to+\infty$, 有 $(\frac{1}{|\tau_1|})^{\frac{1}{q}}W_1(-\frac{\beta+1}{q})\leqslant M_0$, 这仍与 $M_0<(\frac{1}{|\tau_1|})^{\frac{1}{q}}W_1(-\frac{\beta+1}{q})$ 矛盾.

综上讨论可知 $(\frac{1}{|\tau_1|})^{\frac{1}{q}}W_1(-\frac{\beta+1}{q})=(\frac{1}{|\tau_2|})^{\frac{1}{p}}W_2(-\frac{\alpha+1}{p})$ 是 (3.1) 的最佳常数因子. □

4 超齐次核有界积分算子的构造定理

假设以 $K(x,y)$ 为核的积分算子 T 由 (1.3) 定义, 根据 Hilbert 型积分不等式的基本理论[6], Hilbert 型不等式 (3.1) 等价于关于算子 T 的不等式

$$\|T(f)\|_{p,\beta(1-p)}\leqslant M\|f\|_{p,\alpha},\quad f\in L_p^{\alpha}(0,+\infty). \tag{4.1}$$

令 $\beta(1-p)=\gamma$, 则 $\tau_1(\frac{1}{p}-\frac{\beta}{q})-(\frac{1}{q}-\frac{\alpha}{p})=\sigma_1$ 化为 $\tau_1\frac{\gamma+1}{p}+\frac{\alpha+1}{p}=\sigma_1+1$, (4.1) 化为 $\|T(f)\|_{p,\gamma}\leqslant M\|f\|_{p,\alpha}$, 于是由定理 3.1, 得到加权 Lebesgue 空间中超齐次核有界积分算子的构造定理:

定理 4.1 设 $p>1$, $\alpha,\gamma\in\mathbb{R}$, $K(x,y)$ 是具有参数 $\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$ 的超齐次非负可测函数, $\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, $0<W_2(-\frac{\alpha+1}{p})<+\infty$, 积分算子 T 由 (1.3) 定义.

(i) 当且仅当 $\tau_1\frac{\gamma+1}{p}+\frac{\alpha+1}{p}=\sigma_1+1$ 时, T 是从 $L_p^{\alpha}(0,+\infty)$ 到 $L_p^{\gamma}(0,+\infty)$ 的有界算子, 即存在常数 M 使得

$$\|T(f)\|_{p,\gamma}\leqslant M\|f\|_{p,\alpha};$$

(ii) 当 $\tau_1\frac{\gamma+1}{p}+\frac{\alpha+1}{p}=\sigma_1+1$ 时, T 的算子范数为

$$\|T\|=\Big(\frac{1}{|\tau_2|}\Big)^{\frac{1}{p}}W_2\Big(-\frac{\alpha+1}{p}\Big).$$

推论 4.1 设 $p>1$, $\lambda>0$, $\lambda_1>0$, $\lambda_2>0$, $\alpha,\gamma\in\mathbb{R}$, $p(1-\lambda\lambda_1)-1<\alpha<p-1$, 积分算子 T 为

$$T(f)(y)=\int_0^{+\infty}\frac{|\ln(x^{\lambda_1}/y^{\lambda_2})|}{(\max\{x^{\lambda_1},y^{\lambda_2}\})^{\lambda}}f(x)dx, \tag{4.2}$$

则当且仅当 $\frac{1}{\lambda_2}\frac{\gamma+1}{p}-\frac{1}{\lambda_1}\frac{\alpha+1}{p}=\lambda-\frac{1}{\lambda_1}$ 时, 算子 T 是从 $L_p^{\alpha}(0,+\infty)$ 到 $L_p^{\gamma}(0,+\infty)$ 的有界算子, 且当 T 有界时, T 的算子范数为

$$\|T\|=\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{1}{p}}\Big(\frac{\lambda_1}{(1-\frac{\alpha+1}{p})^2}+\frac{\lambda_1}{(1-\frac{\alpha+1}{p}-\lambda\lambda_1)^2}\Big).$$

证明 令

$$K(x,y)=\frac{|\ln(x^{\lambda_1}/y^{\lambda_2})|}{(\max\{x^{\lambda_1},y^{\lambda_2}\})^{\lambda}},\quad x>0,\quad y>0,$$

则 $K(x,y)$ 是具有参数 $\{-\lambda\lambda_1,-\lambda\lambda_2,-\frac{\lambda_1}{\lambda_2},-\frac{\lambda_2}{\lambda_1}\}$ 的超齐次非负函数. 因为 $\sigma_1=-\lambda\lambda_1$, $\sigma_2=-\lambda\lambda_2$, $\tau_1=-\frac{\lambda_1}{\lambda_2}$, $\tau_2=-\frac{\lambda_2}{\lambda_1}$, 故有 $\tau_1\tau_2=1$, $\sigma_1+\tau_1\sigma_2=0$, 且

$$\frac{1}{\lambda_2}\frac{\gamma+1}{p}-\frac{1}{\lambda_1}\frac{\alpha+1}{p}=\lambda-\frac{1}{\lambda_1}\Leftrightarrow\tau_1\frac{\gamma+1}{p}+\frac{\alpha+1}{p}=\sigma_1+1.$$

又因为

$$\begin{aligned}W_2\Big(-\frac{\alpha+1}{p}\Big)&=\int_0^{+\infty}K(t,1)t^{-\frac{\alpha+1}{p}}dt\\&=\int_0^{+\infty}\frac{|\ln t^{\lambda_1}|}{(\max\{t^{\lambda_1},1\})^{\lambda}}t^{-\frac{\alpha+1}{p}}dt\\&=\frac{1}{\lambda_1}\int_0^{+\infty}\frac{|\ln u|}{(\max\{u,1\})^{\lambda}}u^{-\frac{1}{\lambda_1}\frac{\alpha+1}{p}+\frac{1}{\lambda_1}-1}du\\&=-\frac{1}{\lambda_1}\int_0^1u^{\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})-1}\ln udu+\frac{1}{\lambda_1}\int_1^{+\infty}u^{\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})-\lambda-1}\ln udu,\end{aligned}$$

记 $r=\frac{1}{\lambda_1}(1-\frac{\alpha+1}{p})-1$, 则由 $p(1-\lambda\lambda_1)-1<\alpha<p-1$, 可得 $r+1>0$, $r+1-\lambda<0$, 于是

$$\begin{aligned}W_2\Big(-\frac{\alpha+1}{p}\Big)&=-\frac{1}{\lambda_1}\int_0^1u^r\ln udu+\frac{1}{\lambda_1}\int_1^{+\infty}u^{r-\lambda}\ln udu\\&=-\frac{1}{\lambda_1}\frac{1}{r+1}\Big(u^{r+1}\ln u|_0^1-\int_0^1u^rdu\Big)\\&\quad+\frac{1}{\lambda_1}\frac{1}{r+1-\lambda}\Big(u^{r+1-\lambda}\ln u|_1^{+\infty}-\int_1^{+\infty}u^{r-\lambda}du\Big)\\&=\frac{1}{\lambda_1}\Big(\frac{1}{(r+1)^2}+\frac{1}{(r+1-\lambda)^2}\Big)\\&=\frac{1}{\lambda_1}\Big(\frac{\lambda_1}{(1-\frac{\alpha+1}{p})^2}+\frac{\lambda_1}{(1-\frac{\alpha+1}{p}-\lambda\lambda_1)^2}\Big),\end{aligned}$$

从而

$$\Big(\frac{1}{|\tau_1|}\Big)^{\frac{1}{p}}W_2\Big(-\frac{\alpha+1}{p}\Big)=\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{1}{p}}\Big(\frac{\lambda_1}{(1-\frac{\alpha+1}{p})^2}+\frac{\lambda_1}{(1-\frac{\alpha+1}{p}-\lambda\lambda_1)^2}\Big).$$

根据定理 4.1, 知推论 4.1 成立. □

若在推论 4.1 中取 $\alpha=\gamma=0$, 则可得如下推论:

推论 4.2 设 $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, $\lambda>0$, $\lambda_1>0$, $\lambda_2>0$, $\lambda\lambda_1>\frac{1}{q}$, 积分算子 T 由 (4.2) 定义, 则当且仅当 $\frac{1}{\lambda_2p}+\frac{1}{\lambda_1q}=\lambda$ 时, T 是 $L_p(0,+\infty)$ 中的有界算子, 且当 T 是有界算子时, T 的算子范数为

$$\|T\|=\frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}}(\lambda_1^2q^2+\lambda_2^2p^2).$$

在推论 4.2 中再取 $\lambda=\lambda_1=\lambda_2=1$, 则可得如下推论:

推论 4.3 设 $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, 则积分算子

$$T(f)(y)=\int_0^{+\infty}\frac{|\ln(x/y)|}{\max\{x,y\}}f(x)dx$$

是 $L_p(0,+\infty)$ 中的有界算子, 且 T 的算子范数为 $\|T\|=p^2+q^2$.

致谢 对审稿人提出的意见和改进建议深表谢意.

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Construction conditions for the bounded integral operator with the super-homogeneous kernel in the weighted Lebesgue space and the formula of the operator norm

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Abstract In this paper, we introduce the concept of the super-homogeneous function, and use the super-homogeneous kernel to unify the homogeneous kernel, the generalized homogeneous kernel, and several non-homogeneous kernels. We first discuss the Hilbert-type integral inequality with the super-homogeneous kernel by using the weight function method and real analysis technique, and then study the integral operator with the super-homogeneous kernel according to the relationship between the Hilbert-type integral inequality and the integral operator with the same kernel. Finally, we obtain the construction conditions of the bounded integral operator with the super-homogeneous kernel in the weighted Lebesgue space as well as the formula for the operator norm.

Keywords super-homogeneous kernel, bounded integral operator, Hilbert-type integral inequality, weighted Lebesgue space, operator norm, construction condition

MSC(2020) 26D15

doi: 10.1360/SSM-2023-0156