

On the rainbow matching conjecture for 3-uniform hypergraphs

Jun Gao¹, Hongliang Lu², Jie Ma^{1,*} & Xingxing Yu³

¹*School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China;*

²*School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China;*

³*School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA*

Email: gj0211@mail.ustc.edu.cn, luhongliang@mail.xjtu.edu.cn, jiema@ustc.edu.cn, yu@math.gatech.edu

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Abstract Aharoni and Howard and, independently, Huang et al. (2012) proposed the following rainbow version of the Erdős matching conjecture: For positive integers n , k and m with $n \geq km$, if each of the families $F_1, \dots, F_m \subseteq \binom{[n]}{k}$ has size more than $\max\{\binom{n}{k} - \binom{n-m+1}{k}, \binom{km-1}{k}\}$, then there exist pairwise disjoint subsets $e_1, \dots, e_m$ such that $e_i \in F_i$ for all $i \in [m]$. We prove that there exists an absolute constant n_0 such that this rainbow version holds for $k = 3$ and $n \geq n_0$. We convert this rainbow matching problem to a matching problem on a special hypergraph H . We then combine several existing techniques on matchings in uniform hypergraphs: Find an absorbing matching M in H ; use a randomization process of Alon et al. (2012) to find an almost regular subgraph of $H - V(M)$; find an almost perfect matching in $H - V(M)$. To complete the process, we also need to prove a new result on matchings in 3-uniform hypergraphs, which can be viewed as a stability version of a result of Łuczak and Mieczkowska (2014) and might be of independent interest.

Keywords rainbow matching conjecture, Erdős matching conjecture, stability

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1 Introduction

For a positive integer k and a set V , let $[k] := \{1, \dots, k\}$ and

$$\binom{V}{k} := \{A \subseteq V : |A| = k\}.$$

A hypergraph H consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq 2^{V(H)}$. A hypergraph H is k -uniform if all its edges have size k and we call it a k -graph for short. Throughout this paper, we often identify $E(H)$ with H when there is no confusion and, in particular, denote by $|H|$ the number of edges in H . Given a set T of edges in H , we use $V(T)$ to define $\bigcup_{e \in T} e$. Given a vertex subset $S \subseteq V(H)$ in H , we use $H[S]$ to denote the subgraph of H induced by S , and let $H - S = H[V(H) \setminus S]$.

* Corresponding author

A *matching* in a hypergraph H is a set of pairwise disjoint edges in H . We use $\nu(H)$ to define the maximum size of a matching in H . Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of hypergraphs on the same vertex set. A set of m pairwise disjoint edges is called a *rainbow matching* for $\mathcal{F}$ if each edge is from a different F_i . If such a matching exists, then we also say that $\mathcal{F}$ *admits a rainbow matching*.

A classical problem in extremal set theory asks for the maximum number of edges in n -vertex k -graphs H with $\nu(H) < m$. Let n, k and m be positive integers with $n \geq km$. The k -graphs

$$S(n, m, k) := \binom{[n]}{k} \setminus \binom{[n] \setminus [m-1]}{k}$$

and $D(n, m, k) := \binom{[km-1]}{k}$ on the same vertex set $[n]$ do not have matchings of size m . Erdős [6] conjectured in 1965 that among all the k -graphs with no matching of size m , $S(n, m, k)$ or $D(n, m, k)$ has the maximum number of edges: Any n -vertex k -graph H with $\nu(H) < m$ contains at most

$$f(n, m, k) := \max \left\{ \binom{n}{k} - \binom{n-m+1}{k}, \binom{km-1}{k} \right\}$$

edges. This is often referred to as the *Erdős matching conjecture* in the literature, and there has been extensive research on this conjecture (see, for example, [3, 5, 8–11, 13, 22]). In particular, the special case for $k = 3$ was settled for large n by Łuczak and Mieczkowska [22] and completely resolved by Frankl [9].

The following analogous conjecture, known as the *rainbow matching conjecture*, was made by Aharoni and Howard [1] and, independently, by Huang et al. [15]. For related topics on rainbow type problems, we refer the interested readers to [16, 18, 20, 23].

Conjecture 1.1 (See [1, 15]). Let n, k and m be positive integers with $n \geq km$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of k -graphs on the same vertex set $[n]$ such that $|F_i| > f(n, m, k)$ for all $i \in [m]$. Then $\mathcal{F}$ admits a rainbow matching.

The case $k = 2$ of this conjecture is in fact a direct consequence of an earlier result of Akiyama and Frankl [2] (which was restated in [7]). The following was obtained by Huang et al. [15].

Theorem 1.2 (See [15, Theorem 3.3]). *Conjecture 1.1 holds when $n > 3k^2m$.*

Keller and Lifshitz [17] proved that Conjecture 1.1 holds when $n \geq f(m)k$ for some large constant $f(m)$ which only depends on m , and this was further improved to $n = \Omega(m \log m)k$ by Frankl and Kupavskii [12]. Both proofs use the junta method. Very recently, Lu et al. [19] showed that Conjecture 1.1 holds when $n \geq 2km$ and n is sufficiently large.

The following is our main result, which proves Conjecture 1.1 for $k = 3$ and sufficiently large n .

Theorem 1.3. *There exists an absolute constant n_0 such that the following holds for all $n \geq n_0$. For any positive integers n and m with $n \geq 3m$, let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of 3-graphs on the same vertex set $[n]$ such that $|F_i| > f(n, m, 3)$ for all $i \in [m]$. Then $\mathcal{F}$ admits a rainbow matching.*

Our proof of Theorem 1.3 uses some new ideas and combines different techniques from Alon et al. [3], Łuczak and Mieczkowska [22], and Lu et al. [21]. (For a high level description of our proof, we refer the readers to Section 2 and/or Section 7.) In the process, we prove a stability result on 3-graphs (see Lemma 4.2) that plays a crucial role in our proof and might be of independent interest: If the number of edges in an n -vertex 3-graph H with $\nu(H) < m$ is close to $f(n, m, 3)$, then H must be close to $S(n, m, 3)$ or $D(n, m, 3)$.

The rest of the paper is organized as follows. In Section 2, we introduce additional notation, and state and/or prove a few lemmas for later use. In Section 3, we deal with the families $\mathcal{F}$ in which most 3-graphs are close to the same 3-graph, i.e., $S(n, m, 3)$ or $D(n, m, 3)$. To deal with the remaining families, we need the above mentioned stability result for matchings in 3-graphs, which is done in Section 4. In Section 5, we show that there exists an absolute constant $c > 0$ such that Theorem 1.3 holds for $m > (1 - c)n/3$. The proof of Theorem 1.3 for $m \leq (1 - c)n/3$ is completed in Section 6. Finally, we complete the proof of Theorem 1.3 in Section 7.

2 Previous results and lemmas

In this section, we define saturated families and stable hypergraphs, and state several lemmas that we will use frequently. We begin with some notation. Suppose that H is a hypergraph and U and T are subsets of $V(H)$. Let

$$N_H(T) := \{A : A \subseteq V(H) \setminus T \text{ and } A \cup T \in E(H)\}$$

be the *neighborhood* of T in H , and let $d_H(T) := |N_H(T)|$. We write $d_H(v)$ for $d_H(\{v\})$. Let

$$\Delta(H) := \max_{v \in V(H)} d_H(v) \quad \text{and} \quad \Delta_2(H) := \max_{T \in \binom{V(H)}{2}} d_H(T).$$

In the case $T \subseteq U$, we often identify $d_{H[U]}(T)$ with $d_U(T)$ when there is no confusion.

It will be helpful to consider “maximal” counterexamples to Conjecture 1.1. Let n, k and m be positive integers with $n \geq km$ and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of k -graphs on the same vertex set $[n]$. We say that $\mathcal{F}$ is **saturated**, if $\mathcal{F}$ does not admit a rainbow matching, but for every $F \in \mathcal{F}$ and $e \notin F$, the new family $\mathcal{F}(e, F) := (\mathcal{F} \setminus \{F\}) \cup \{F \cup \{e\}\}$ admits a rainbow matching. The following lemma says that the vertex degrees of every k -graph in a saturated family are typically small.

Lemma 2.1. *Let n, k and m be positive integers with $n \geq km$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a saturated family of k -graphs on the same vertex set $[n]$. Then for each $v \in [n]$ and each $i \in [m]$,*

$$d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1} \quad \text{or} \quad d_{F_i}(v) = \binom{n-1}{k-1}.$$

Proof. Suppose $d_{F_i}(v) < \binom{n-1}{k-1}$, where $v \in [n]$ and $i \in [m]$. Then there exists $e \in \binom{[n]}{k} \setminus F_i$ such that $v \in e$. Since $\mathcal{F}$ is saturated, the family $\mathcal{F}(e, F_i)$ admits a rainbow matching, say $M \cup \{e\}$, with M being a rainbow matching for the family $\mathcal{F} \setminus \{F_i\}$.

If

$$d_{F_i}(v) > \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1} = \left| \binom{[n] \setminus \{v\}}{k-1} \setminus \binom{[n] \setminus (\{v\} \cup V(M))}{k-1} \right|,$$

then there exists an edge $f \in F_i$ such that $v \in f$ and $f \cap V(M) = \emptyset$. Now $M \cup \{f\}$ is a rainbow matching for $\mathcal{F}$, which leads to a contradiction. So $d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1}$. $\square$

We will remove vertices of degree $\binom{n-1}{k-1}$ and use Lemma 2.1 to produce the saturated family $\mathcal{F} = \{F_1, \dots, F_m\}$ of k -graphs such that for each $v \in V(F_i)$ and each $i \in [m]$,

$$d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1}.$$

Next, we define stable hypergraphs. Let n and k be positive integers with $n \geq k$. Let $e = \{a_1, \dots, a_k\}$ and $f = \{b_1, \dots, b_k\}$ be members of $\binom{[n]}{k}$ with $a_1 < a_2 < \dots < a_k$ and $b_1 < b_2 < \dots < b_k$. We write $e \leq f$ if $a_i \leq b_i$ for all $1 \leq i \leq k$, and $e < f$ if $e \leq f$ and $e \neq f$.

A k -graph $F \subseteq \binom{[n]}{k}$ is said to be **stable** if $e < f \in F$ implies $e \in F$. A family $\mathcal{F}$ of k -graphs on the same vertex set $[n]$ is **stable** if each k -graph in $\mathcal{F}$ is stable.

The following result of Huang et al. [15] will be used frequently, which enables us to work with stable families when proving Conjecture 1.1.

Lemma 2.2 (See [15, Lemma 2.1]). *Let n, k and m be positive integers with $n \geq km$. If the family $\{F_1, \dots, F_m\}$ of k -graphs with $V(F_i) = [n]$ for all $i \in [m]$ does not admit a rainbow matching, then there exists a stable family $\{F'_1, \dots, F'_m\}$ of k -graphs with $|F'_i| = |F_i|$ and $V(F'_i) = [n]$ for all $i \in [m]$ which still preserves this property.*

Corollary 2.3. *Let n, k and m be positive integers with $n \geq km$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of k -graphs on the vertex set $[n]$ that does not admit a rainbow matching. Then there exists a family $\mathcal{F}' = \{F'_1, \dots, F'_m\}$ of k -graphs on the same vertex set $[n]$ such that $\mathcal{F}'$ is both stable and saturated and $|F'_i| \geq |F_i|$ for $i \in [m]$.*

Proof. Let $\mathcal{F}^* = \{F_1^*, \dots, F_m^*\}$ be a family of k -graphs on the same vertex set $[n]$ such that $\mathcal{F}^*$ admits no rainbow matching, $|F_i^*| \geq |F_i|$ for $i \in [m]$, and subject to these, $\sum_{i \in [m]} |F_i^*|$ is maximum.

Then $\mathcal{F}^*$ is saturated. Now applying Lemma 2.2 to $\mathcal{F}^*$, we obtain a stable family $\mathcal{F}' = \{F_1', \dots, F_m'\}$ of k -graphs on the vertex set $[n]$ such that $\mathcal{F}'$ admits no rainbow matching, and $|F_i'| = |F_i^*|$ for $i \in [m]$. By the choice of $\mathcal{F}^*$, we see that $\mathcal{F}'$ is also saturated. $\square$

We now describe an operation that converts a rainbow matching problem to a matching problem on a single hypergraph. Let n, k, m and r be non-negative integers with $r = \lfloor n/k \rfloor - m$ and $m \geq 1$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of k -graphs on the same vertex set $[n]$, and let $\mathcal{V} = \{v_1, \dots, v_m\}$ and $\mathcal{U} = \{u_1, \dots, u_r\}$ be two disjoint sets such that $(\mathcal{V} \cup \mathcal{U}) \cap [n] = \emptyset$. We use $H(\mathcal{F})$ to define the $(k+1)$ -graph with the vertex set $[n] \cup \mathcal{V}$ and the edge set $\bigcup_{i=1}^m \{e \cup \{v_i\} : e \in F_i\}$, and use $H^*(\mathcal{F})$ to define the $(k+1)$ -graph with the vertex set $[n] \cup \mathcal{V} \cup \mathcal{U}$ and the edge set

$$E(H(\mathcal{F})) \cup \bigcup_{i=1}^r \left\{ e \cup \{u_i\} : e \in \binom{[n]}{k} \right\}.$$

If $F_1 = \dots = F_m = S(n, m, k)$ (resp. $F_1 = \dots = F_m = D(n, m, k)$), then we write $H(\mathcal{F})$ as $H_S(n, m, k)$ (resp. $H_D(n, m, k)$).

It is easy to see that $\mathcal{F}$ admits a rainbow matching if and only if $H(\mathcal{F})$ has a matching of size m , which is also if and only if $H^*(\mathcal{F})$ has a matching of size $m+r$. This allows us to access existing approaches and tools invented for matching problems. For example, we take the approach by considering whether or not the hypergraphs $H(\mathcal{F})$ in question are close to the extremal configurations $H_S(n, m, k)$ and $H_D(n, m, k)$. We will see in Section 3 that if $H(\mathcal{F})$ is close to $H_D(n, m, k)$ and $\mathcal{F}$ is stable, then $\mathcal{F}$ admits a rainbow matching.

Here, we give an easy lemma concerning a case where $H(\mathcal{F})$ is not close to $H_S(n, m, k)$, which will be used along with Lemma 2.1. Let H_1 and H_2 be two k -graphs on the same vertex set V and let ϵ be some positive real number; we say that H_2 is ϵ -close to H_1 if $|E(H_1) \setminus E(H_2)| \leq \epsilon|V|^k$.

Lemma 2.4. For any given integer $k \geq 3$, let ϵ and c be real numbers such that $0 < \epsilon \ll c \ll 1^1$. Let n and m be integers such that $n/3k^2 \leq m \leq (1-c)n/k$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of k -graphs on the vertex set $[n]$. If for every $i \in [m]$ and $v \in [n]$,

$$d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1},$$

then $H(\mathcal{F})$ is not ϵ -close to $H_S(n, m, k)$.

Proof. We note that $S(n, m, k)$ has $m-1$ vertices of degree $\binom{n-1}{k-1}$. Since for every $i \in [m]$ and $v \in [n]$,

$$d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1},$$

we have

$$|E(H_S(n, m, k)) \setminus E(H(\mathcal{F}))| \geq m \cdot (m-1) \cdot \binom{n-k(m-1)-1}{k-1} \cdot \frac{1}{k} > \frac{n^2}{10k^5} \binom{cn}{k-1} > \epsilon(n+m)^{k+1},$$

where the second inequality is due to $n/3k^2 \leq m \leq (1-c)n/k$ and the third inequality follows from $\epsilon \ll c$. This shows that $H(\mathcal{F})$ is not ϵ -close to $H_S(n, m, k)$. $\square$

To deal with the case where $H(\mathcal{F})$ is not close to $H_D(n, m, 3)$, we first find a small matching M in $H^*(\mathcal{F})$ such that M can “absorb” small vertex sets and $H^*(\mathcal{F}) - V(M)$ has an almost perfect matching. When $\mathcal{F}$ is stable, the matching M can be found very easily by the following lemma and its proof.

¹⁾ Here and throughout the rest of the paper, the notation $a \ll b$ means that a is sufficiently small compared with b which need satisfy finitely many inequalities in the proof.

Lemma 2.5. Let k be a fixed positive integer and let $0 < \gamma' \ll \gamma \ll c \ll 1$ be real numbers. Let n and m be positive integers with $n/3k^2 \leq m \leq (1-c)n/k$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a stable family of k -graphs such that $V(F_i) = [n]$ and $|F_i| > f(n, m, k)$ for all $i \in [m]$. Then for sufficiently large n , $H^*(\mathcal{F})$ has a matching M with $|M| \leq \gamma n$ such that for any set $S \subseteq V(H^*(\mathcal{F})) \setminus V(M)$ with $|S| \leq \gamma' n$ and $k|S \cap (\mathcal{V} \cup \mathcal{U})| = |S \cap [n]|$, $H^*(\mathcal{F})[V(M) \cup S]$ has a perfect matching.

Proof. Recall that $\mathcal{V} = \{v_1, \dots, v_m\}$ and $\mathcal{U} = \{u_1, \dots, u_r\}$, where $r = \lfloor n/k \rfloor - m$. Fix an integer t satisfying $\gamma' n < t < \gamma n$. Then $t < \gamma n \leq \lfloor cn/k \rfloor \leq \lfloor n/k \rfloor - m = r$. Let $s = \lceil n/3k^2 \rceil - 1$.

By Theorem 1.2 (viewing all the k -graphs as the same k -graph), since $|F_i| > f(n, m, k) \geq f(n, s, k)$ for all $i \in [m]$, every F_i has a matching of size s . Since F_i is stable, $F_i[[s]]$ is a complete k -graph. Hence,

(i) for any $i_1, i_2, \dots, i_k \leq kt \leq k\gamma n < s$ and $j \in [m]$, we have $\{v_j, i_1, i_2, \dots, i_k\} \in H^*(\mathcal{F})$.

From the definition of $H^*(\mathcal{F})$, we have

(ii) for any $i_1, i_2, \dots, i_k \in [n]$ and $j \in [r]$, $\{u_j, i_1, i_2, \dots, i_k\} \in H^*(\mathcal{F})$.

Since $t < r$, we may choose a matching M of size t in $H^*(\mathcal{F})$ with $V(M) = \{u_1, \dots, u_t\} \cup [kt]$. Note that $|M| = t \leq \gamma n$. We claim that this M is the desired matching. To see this, consider any subset S with $S \cap V(M) = \emptyset$, $|S| \leq \gamma' n$ and $k|S \cap (\mathcal{V} \cup \mathcal{U})| = |S \cap [n]|$. Let $t' = |S \cap (\mathcal{V} \cup \mathcal{U})|$. So $t' \leq \gamma' n < t$. Then by (i) and (ii), there is a perfect matching M_1 in $H^*(\mathcal{F})[S \cap (\mathcal{V} \cup \mathcal{U}) \cup [kt']]$. By (ii), there exists a perfect matching M_2 in $H^*(\mathcal{F})[(V(M) \cup S) \setminus V(M_1)]$. So $M_1 \cup M_2$ is a perfect matching in $H^*(\mathcal{F})[V(M) \cup S]$. $\square$

For the “absorbing” matching M in $H^*(\mathcal{F})$ in Lemma 2.5, we also want $H^*(\mathcal{F}) - V(M)$ to have an almost perfect matching. For this we need to use the following result of Frankl and Rödl [14].

Theorem 2.6 (See [14]). For every integer $k \geq 2$ and any real number $\sigma > 0$, there exist $\tau = \tau(k, \sigma)$ and $d_0 = d_0(k, \sigma)$ such that for every integer $n \geq D \geq d_0$ the following holds: Every n -vertex k -graph H with

$$(1 - \tau)D < \Delta_1(H) < (1 + \tau)D$$

and $\Delta_2(H) < \tau D$ contains a matching covering all but at most σn vertices.

In order to obtain a k -graph H satisfying Theorem 2.6, we use the approach from [3] by conducting two rounds of randomization on $H^*(\mathcal{F}) - V(M)$. We summarize part of the proof in [3] (more precisely, their proof of Claim 4.1) as a lemma. A *fractional matching* in a k -graph H is a function $w : E(H) \rightarrow [0, 1]$ such that for any $v \in V(H)$, $\sum_{\{e \in E(H) : v \in e\}} w(e) \leq 1$. A fractional matching is called *perfect* if $\sum_{e \in E(H)} w(e) = |V(H)|/k$.

Lemma 2.7 (See [3], retained from their proof of Claim 4.1). Let $k \geq 3$ and H be a k -graph on at most $2n$ vertices. Suppose that there are subsets $R^i \subseteq V(H)$ for $i = 1, \dots, n^{1.1}$ satisfying the following:

- (a) every vertex $v \in V(H)$ satisfies that $|\{i : v \in R^i\}| = (1 + o(1))n^{0.2}$,
- (b) every pair $\{u, v\} \subseteq V(H)$ is contained in at most two sets R^i ,
- (c) every edge $e \in H$ is contained in at most one set R^i , and
- (d) for every $i = 1, \dots, n^{1.1}$, R^i has a perfect fractional matching w^i .

Then H has a spanning subgraph H' such that

$$d_{H'}(v) = (1 + o(1))n^{0.2}$$

for all $v \in V(H')$ and $\Delta_2(H') \leq n^{0.1}$.

We will also need to control the independence number of random subgraphs of $H^*(\mathcal{F}) - V(M)$. The intuition is that when $H(\mathcal{F})$ is not close to $H_D(n, m, k)$ or $H_S(n, m, k)$, $H^*(\mathcal{F}) - V(M)$ does not have very large independence number. The following lemma in [21] was proved by Lu et al. using the container method.

Lemma 2.8 (See [21, Lemma 5.4]). Let d, ϵ' and α be positive real numbers and let k and n be positive integers. Let H be an n -vertex k -graph such that $e(H) \geq dn^k$ and $e(H[S]) \geq \epsilon' e(H)$ for all $S \subseteq V(H)$ with $|S| > \alpha n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of H uniformly at random with probability $n^{-0.9}$. Then for any positive real number $\gamma \ll \alpha$, the size of maximum independent sets in $H[R]$ is at most $(\alpha + \gamma)n^{0.1}$ with probability at least $1 - (n^{O(1)}e^{-\Omega(n^{0.1})})$.

We need an inequality on the function $f(n, m, k)$ proved by Frankl [9].

Lemma 2.9 (See [9, Proposition 5.1]). *Let n, m and k be positive integers with $n \geq km - 1$. Then*

$$f(n, m, k) \geq f(n-1, m-1, k) + \binom{n-1}{k-1}.$$

We conclude this section with the well-known Chernoff inequality.

Lemma 2.10 (Chernoff inequality [4]). *Suppose that $X_1, \dots, X_n$ are independent random variables taking values in $\{0, 1\}$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}(X)$. Then for any $0 < \delta \leq 1$,*

$$\mathbb{P}[X \geq (1 + \delta)\mu] \leq e^{-\delta^2 \mu/3} \quad \text{and} \quad \mathbb{P}[X \leq (1 - \delta)\mu] \leq e^{-\delta^2 \mu/3}. \quad (2.1)$$

In particular, if $X \sim \text{Bin}(n, p)$ and $\lambda < \frac{3}{2}np$, then

$$\mathbb{P}(|X - np| \geq \lambda) \leq e^{-\Omega(\lambda^2/np)}. \quad (2.2)$$

3 Extremal configuration $H_D(n, m, 3)$

From Lemmas 2.1 and 2.4, we see that if $\mathcal{F}$ is a saturated family of k -graphs on the vertex set $[n]$ and $H(\mathcal{F})$ is close to the extremal configuration $H_S(n, m, k)$, then there exist $F \in \mathcal{F}$ and $v \in [n]$ such that $d_F(v) = \binom{n-1}{k-1}$. Such vertices v can be removed from all the k -graphs in $\mathcal{F} \setminus \{F\}$ to obtain a smaller family $\mathcal{F}'$, so that if $\mathcal{F}'$ admits a rainbow matching, then $\mathcal{F}$ admits a rainbow matching.

In this section, we consider the case where $H(\mathcal{F})$ is close to $H_D(n, m, 3)$ and $\mathcal{F}$ is stable.

Lemma 3.1. *Let ϵ and c be real numbers such that $0 < \epsilon \ll c \leq 1$. Let n and m be positive integers such that $n/27 \leq m \leq (1-c)n/3$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a stable family of 3-graphs on the vertex set $[n]$ such that $|F_i| > f(n, m, 3)$ for all $i \in [m]$. If $H(\mathcal{F})$ is ϵ -close to $H_D(n, m, 3)$, then $\mathcal{F}$ admits a rainbow matching.*

Proof. Let $b = 6\epsilon^{1/6}n$. If F_i is $\sqrt{\epsilon}$ -close to $D(n, m, 3)$, then F_i contains a complete subgraph of size $3m - b$; for, otherwise, as F_i is stable, we have

$$|E(D(n, m, 3)) \setminus E(F_i)| \geq \binom{b}{3} > \sqrt{\epsilon}n^3,$$

which leads to a contradiction.

We claim that for any $i \in [m]$ and $j \in \{0, \dots, b\}$, $\{2j+1, 2j+2, 3m-j\} \in F_i$. To prove this claim we fix $i \in [m]$. Suppose for a contradiction that there exists an integer t with $0 \leq t \leq b$ such that $\{2t+1, 2t+2, 3m-t\} \notin F_i$. Since $|F_i| > \binom{3m-1}{3}$ and F_i is stable, we have $\{1, 2, 3m\} \in F_i$. So $t \geq 1$. We now count the edges in F_i : Let q_1 be the number of edges of F_i in $[3m-1]$, and q_2 be the number of edges of F_i not contained in $[3m-1]$. Since F_i is stable and $\{2t+1, 2t+2, 3m-t\} \notin F_i$, we see that $\{a, b, c\} \notin F_i$ when $2t+2 \leq a < b < 3m-t \leq c \leq 3m-1$. So

$$q_1 \leq \binom{3m-1}{3} - t \binom{3m-3t-3}{2}.$$

Since $\{2t+1, 2t+2, 3m-t\} \notin F_i$, we have that for any $e \in F_i$ with $e \cap ([n] \setminus [3m-1]) \neq \emptyset$, $e \cap [2t] \neq \emptyset$. This shows $q_2 \leq 2t(n-3m+1)n$. First suppose that $n \leq 7m/2$. Then we have

$$\begin{aligned} |F_i| &\leq \binom{3m-1}{3} - t \binom{3m-3t-3}{2} + 2tn(n-3m+1) \\ &\leq \binom{3m-1}{3} - t \left[\binom{3m-3t-3}{2} - 7m(m/2+1) \right] < \binom{3m-1}{3}, \end{aligned}$$

where the second inequality holds since $n \leq 7m/2$, and the last inequality holds since $t \leq b = 6\epsilon^{1/6}n \ll m$, which leads to a contradiction. So we may assume $n > 7m/2$. Let $m = \alpha n$. Then $1/27 \leq \alpha < 2/7$. We assert that

$$\binom{n}{3} - \binom{n-m+1}{3} > \binom{3m-1}{3} + 2tn^2.$$

To see this, let $f(x) = 1 - (1-x)^3 - (3x)^3$, and then

$$\frac{6}{n^3} \left(\binom{n}{3} - \binom{n-m+1}{3} - \binom{3m-1}{3} \right) = f(\alpha) + o(1).$$

Since $f'(x) = 3(1-2x-26x^2)$ is decreasing in $[1/27, 2/7]$ with $f'(1/27) > 0$ and $f'(2/7) < 0$, we have

$$f(\alpha) \geq \min\{f(1/27), f(2/7)\} = f(2/7) = \frac{2}{343}$$

for $1/27 \leq \alpha < 2/7$. This shows that

$$\binom{n}{3} - \binom{n-m+1}{3} - \binom{3m-1}{3} = \frac{f(\alpha)}{6}n^3 + o(n^3) \geq 2tn^2,$$

as asserted. Then it follows that

$$|F_i| \leq \binom{3m-1}{3} - t \binom{3m-3t-3}{2} + 2tn(n-3m+1) < \binom{3m-1}{3} + 2tn^2 < \binom{n}{3} - \binom{n-m+1}{3},$$

which leads to a contradiction as $|F_i| > f(n, m, 3)$. This finishes the proof of the claim.

Recall $\mathcal{V} = \{v_1, \dots, v_m\}$ from the definition of $H(\mathcal{F})$. By the above claim,

$$M_1 := \{\{v_i, 2i-1, 2i, 3m-i+1\} : i \in [b]\}$$

is a matching in $H(\mathcal{F})$. Without loss of generality, let $F_1, \dots, F_a$ be all the k -graphs in $\mathcal{F}$ which are not $\sqrt{\epsilon}$ -close to $D(n, m, 3)$. Since $H(\mathcal{F})$ is ϵ -close to $H_D(n, m, 3)$, we have $a \leq \sqrt{\epsilon}n < b$. Then for any $j \in [m] \setminus [b]$, since F_j is $\sqrt{\epsilon}$ -close to $D(n, m, 3)$, F_j contains a complete subgraph with size at least $3m-b$. Hence we have $\{2j-1, 2j, 3m-j+1\} \in F_j$. So $M_2 := \{\{v_j, 2j-1, 2j, 3m-j+1\} : b < j \leq m\}$ is a matching in $H(\mathcal{F})$ which is disjoint from M_1 . Then $M_1 \cup M_2$ forms a matching of size m in $H(\mathcal{F})$. So $\mathcal{F}$ admits a rainbow matching, completing the proof of Lemma 3.1. $\square$

4 A stability lemma

In this section, we prove a result for stable 3-graphs, which may be viewed as a stability version of the following result of Łuczak and Mieczkowska proved in [22].

Theorem 4.1 (See [22]). *There exists a positive integer n_1 such that for integers m and n with $n \geq n_1$ and $1 \leq m \leq n/3$, if H is an n -vertex 3-graph with $e(H) > f(n, m, 3)$, then $\nu(H) \geq m$.*

Building on the proof in [22], we prove the following lemma.

Lemma 4.2. *For any real number $\epsilon > 0$, there exists a positive integer $n_1(\epsilon)$ such that the following holds. Let m and n be integers with $n \geq n_1(\epsilon)$ and $1 \leq m \leq n/3$, and let H be a stable 3-graph on the vertex set $[n]$. If $e(H) > f(n, m, 3) - \epsilon^4 n^3$ and $\nu(H) < m$, then H is ϵ -close to $S(n, m, 3)$ or $D(n, m, 3)$.*

Proof. Suppose that $e(H) > f(n, m, 3) - \epsilon^4 n^3$ and $s := \nu(H) < m$. Let $M = \{(i_\ell, j_\ell, k_\ell) : \ell \in [s]\}$ be a largest matching in H and the partition $V(M) = I \cup J \cup K$ such that every edge $(i, j, k) \in E(M)$ with $i < j < k$ satisfies $i \in I, j \in J$ and $k \in K$. Since H is stable, we may choose $V(M)$ to be $[3s]$.

Let $V' = [n] \setminus [3s]$. For $x \in [3s]$, let $e(x)$ denote the edge in M containing x . Let

$$F_1 = \left\{ \{v\} \in \binom{[3s]}{1} : d_{V'}(v) \geq 20n \right\}, \quad F_2 = \left\{ \{v, w\} \in \binom{[3s]}{2} : e(v) \neq e(w) \text{ and } d_{V'}(v, w) \geq 20 \right\}$$

and

$$F_3 = \left\{ \{u, v, w\} \in \binom{[3s]}{3} : e(u), e(v) \text{ and } e(w) \text{ are pairwise distinct} \right\}.$$

Let $H^* = ([3s], F)$ be the hypergraph with the vertex set $[3s]$ and the edge set $F = M \cup F_1 \cup F_2 \cup F_3$.

Call an edge $e \in H$ *traceable* if $e \cap [3s] \in F$, and *untraceable* otherwise. Since M is a maximum matching in H , V' is independent in H . So the number of untraceable edges of H is bounded from above by

$$\binom{3s}{1} \cdot 20n + \left(\binom{s}{2} \binom{3}{1} \binom{3}{1} \times 19 + \binom{s}{1} \binom{3}{2} n \right) + \binom{s}{1} \binom{3}{2} \binom{3s-3}{1} \leq 32n^2 = o(n^3),$$

where we use $s < m \leq n/3$. We point out that those edges (there being $o(n^3)$ of them) will be negligible in the following proof.

Let T be a triple of edges from M . We say that T is *bad* if $V(T)$ contains three pairwise disjoint edges of H^* whose union intersects I in at most 2 vertices, and *good* otherwise. For each $i \in [3]$, let $f_i(T)$ denote the number of edges of F_i contained in $V(T)$. Note that $f_3(T) \leq 27$. The following two claims are explicit in [22].

Claim 1. There exist no three pairwise disjoint bad triples (of edges in M). Hence, there exist at most six edges in M such that each bad triple contains one of these edges.

Claim 2. Let T be a good triple.

- (i) If $f_3(T) \geq 24$, then $f_1(T) = f_2(T) = 0$.
- (ii) If $f_3(T) = 20$, then $f_1(T) \leq 1$ and $f_2(T) \leq 12$.
- (iii) If $f_3(T) \leq 19$, then $f_1(T) \leq 3$ and $f_2(T) \leq 15$. Moreover, the only triples T for which $f_3(T) = 19$, $f_2(T) = 15$ and $f_1(T) = 3$ are those in which each edge of H^* contained in $V(T)$ intersects I .
- (iv) If $f_3(T) = 21$, then $f_1(T) \leq 1$ and $f_2(T) \leq 10$.
- (v) If $22 \leq f_3(T) \leq 23$, then $f_1(T) = 0$ and $f_2(T) \leq 7$.

We remove exactly six edges from M such that the resulting matching M' only contains good triples. Since H has at most $18n^2$ edges intersecting $V(M \setminus M')$ and $32n^2$ untraceable edges, we have

$$e(H) \leq |F_1| \binom{n-3s}{2} + |F_2|(n-3s) + |F_3| + 50n^2.$$

To bound $|F_i|$, let us consider the summation of $f_i(T)$ over all $T \in \binom{M'}{3}$. Since each edge from F_i is counted exactly $\binom{(s-6)-i}{3-i}$ times in this sum, we have

$$|F_i| \binom{(s-6)-i}{3-i} = \sum_{T \in \binom{M'}{3}} f_i(T).$$

Therefore,

$$\begin{aligned} e(H) &\leq \sum_{T \in \binom{M'}{3}} \left(f_1(T) \frac{\binom{n-3s}{2}}{\binom{s-7}{2}} + f_2(T) \frac{n-3s}{s-8} + f_3(T) \right) + 50n^2 \\ &\leq \sum_{T \in \binom{M'}{3}} \left(f_1(T) \frac{(n-3s)^2}{s^2} + f_2(T) \frac{n-3s}{s} + f_3(T) \right) + O(n^2). \end{aligned}$$

Here, the last inequality is trivial when $s \leq 15$, and it holds when $s > 15$ because the difference between the above two summations is at most

$$\sum_{T \in \binom{M'}{3}} \left(f_1(T) \frac{15(n-3s)^2}{s(s^2-15s)} + f_2(T) \frac{8(n-3s)}{s(s-8)} \right) \leq \binom{s-6}{3} \left(\frac{45(n-3s)^2}{s(s^2-15s)} + \frac{120(n-3s)}{s(s-8)} \right) = O(n^2),$$

where $3s < n$, $f_1(T) \leq 3$ and $f_2(T) \leq 15$ (from Claim 2).

To further bound $e(H)$, we partition good triples T depending on $f_3(T)$ and $f_1(T)$. Let

$$T_i = \left\{ T \in \binom{M'}{3} : f_3(T) = i \right\}$$

for $i \in [27]$ and

$$X = \left\{ T \in \binom{M'}{3} : f_1(T) = 3 \right\}.$$

Consider any $T \in X$, and then T is a good triple²⁾. Since $f_1(T) = 3$, the three edges of F_1 contained in $V(T)$ are precisely the three vertices in $V(T) \cap I$, and each edge of H^* contained in $V(T)$ intersects I . Since H is stable and $V(M) = [3s]$, by using the definition of F_1 , it is not hard to see that $X \subseteq T_{19}$.

Define

$$x_1 = \sum_{i=1}^{18} |T_i| + |T_{19} \setminus X|, \quad x_2 = |T_{20}|, \quad x_3 = |T_{21}|, \quad x_4 = |T_{22}| + |T_{23}|, \quad x_5 = \sum_{i=24}^{26} |T_i|, \quad x = |X|$$

and $y = |T_{27}|$. So

$$\sum_{i=1}^5 x_i + x + y = \binom{s-6}{3}.$$

From now on, we let $t = (n - 3s)/s$. By Claim 2 and the fact $X \subseteq T_{19}$, we can derive from the above upper bound on $e(H)$ that

$$\begin{aligned} e(H) &\leq (3x + 2x_1 + x_2 + x_3)t^2 + (15x + 15x_1 + 12x_2 + 10x_3 + 7x_4)t \\ &\quad + (19x + 19x_1 + 20x_2 + 21x_3 + 23x_4 + 26x_5 + 27y) + O(n^2). \end{aligned}$$

For convenience, we write

$$f_t(x_1, x_2, x_3, x_4, x_5, x, y) = \sum_{i=1}^5 \alpha_i(t) \cdot x_i + \beta_1(t) \cdot x + \beta_2(t) \cdot y,$$

where

$$\begin{aligned} \alpha_1(t) &= 2t^2 + 15t + 19, & \alpha_2(t) &= t^2 + 12t + 20, & \alpha_3(t) &= t^2 + 10t + 21, \\ \alpha_4(t) &= 7t + 23, & \alpha_5(t) &= 26, & \beta_1(t) &= 3t^2 + 15t + 19 \quad \text{and} \quad \beta_2(t) = 27. \end{aligned}$$

Then it follows that

$$e(H) \leq f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2).$$

Next, we derive properties of the functions $\alpha_i(t)$ and $\beta_j(t)$.

Claim 3. For any $t \geq 0$, $\max\{\beta_1(t), \beta_2(t)\} \geq \max\{\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t)\} + 0.2$.

Proof. We have $\beta_2(t) = 27$. It is easy to see that for each $i \in [5]$, the functions $\alpha_i(t)$, $\beta_1(t) - \alpha_i(t)$ and $\beta_1(t)$ are increasing for $t \geq 0$. Note that $\beta_1(0.5) = 27.25$, $\alpha_2(0.5) = 26.25$, $\alpha_3(0.5) = 26.25$ and $\alpha_4(0.5) = 26.5$, and then $\max\{\beta_1(t), 27\} \geq \alpha_i(t) + 0.2$ for $t \geq 0$ and $i = 2, 3, 4$. Since $\beta_1(t) - \alpha_1(t) = t^2$ and $\alpha_1(\sqrt{0.2}) < 27 - 0.2$, we see $\max\{\beta_1(t), 27\} \geq \alpha_1(t) + 0.2$ for all $t \geq 0$. $\square$

Since

$$\beta_1(t) \binom{s-6}{3} \leq \frac{1}{2}(n-3s)^2s + \frac{5}{2}(n-3s)s^2 + \frac{19}{6}s^3 = \frac{1}{6}n^3 - \frac{1}{6}(n-s)^3,$$

we see

$$\max\{\beta_1(t), \beta_2(t)\} \binom{s-6}{3} \leq \max \left\{ \binom{n}{3} - \binom{n-s+1}{3}, \binom{3s-1}{3} \right\} + O(n^2) = f(n, s, 3) + O(n^2).$$

²⁾ Since T is good, the union of any three disjoint edges of H^* in $V(T)$ must contain the three vertices in $V(T) \cap I$.

By Claim 3 and the fact that $\sum_{i=1}^5 x_i + x + y = \binom{s-6}{3}$, we have

$$\begin{aligned} f_t(x_1, x_2, x_3, x_4, x_5, x, y) &\leq (\max\{\beta_1(t), \beta_2(t)\} - 0.2) \sum_{i=1}^5 x_i + \beta_1(t)x + \beta_2(t)y \\ &\leq \max\{\beta_1(t), \beta_2(t)\} \binom{s-6}{3} - 0.2 \sum_{i=1}^5 x_i \leq f(n, s, 3) - 0.2 \sum_{i=1}^5 x_i + O(n^2). \end{aligned} \quad (4.1)$$

Let $\cup X$ (resp. $\cup T_{27}$) denote the set of edges each of which belongs to some triple in X (resp. in T_{27}). Now we show the following claim.

Claim 4. $s > m - \epsilon n/4$, and $x > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/24}{3}$ or $y > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/12}{3}$.

Proof. If $s \leq m - \epsilon n/4$, then by (4.1) we have

$$\begin{aligned} e(H) &\leq f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2) \leq f(n, s, 3) + O(n^2) \\ &\leq f(n, m, 3) - \binom{\epsilon/4n}{3} + O(n^2) \leq f(n, m, 3) - \epsilon^4 n^3, \end{aligned}$$

which leads to a contradiction. So $s > m - \epsilon n/4$. First we see that $x + y > \binom{s-6}{3} - 10\epsilon^4 n^3$; for, otherwise, $\sum_{i=1}^5 x_i \geq 10\epsilon^4 n^3$, which together with (4.1) implies

$$e(H) \leq f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2) \leq f(n, m, 3) - 2\epsilon^4 n^3 + O(n^2) \leq f(n, m, 3) - \epsilon^4 n^3,$$

which leads to a contradiction. Now suppose that $x > \binom{\epsilon n/12}{3}$ and $y > \binom{\epsilon n/24}{3}$. Then $|\cup X| > \epsilon n/12$ and $|\cup T_{27}| > \epsilon n/24$. For any edge $e = (i, j, k) \in \cup X$ with $i < j < k$, by the previous discussion, we have $i \in F_1$. For any edge $e = (i, j, k) \in \cup T_{27}$ with $i < j < k$, by Claim 2 we see $i \notin F_1$. Thus $(\cup X) \cap (\cup T_{27}) = \emptyset$. The triples $T = \{e_1, e_2, e_3\}$ with $e_1 \in \cup X$ and $e_2, e_3 \in \cup T_{27}$ cannot satisfy both $f_3(T) = 27$ and $f_1(T) = 3$. This shows

$$x + y < \binom{s-6}{3} - |\cup X| \binom{|\cup T_{27}|}{2} \leq \binom{s-6}{3} - \frac{\epsilon n}{12} \binom{\epsilon n/24}{2},$$

contradicting that $x + y > \binom{s-6}{3} - 10\epsilon^4 n^3$. Hence, we have that either $x \leq \binom{\epsilon n/12}{3}$ or $y \leq \binom{\epsilon n/24}{3}$. $\square$

Suppose

$$x > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/24}{3}.$$

So $x > \binom{s-6}{3} - \binom{\epsilon n/12}{3}$ and thus $|\cup X| > s - 6 - \epsilon n/12$. Recall that for any $T \in X$, T is a good triple, and hence each edge of H^* contained in $V(T)$ intersects I . Hence any traceable edge which intersects $V(\cup X)$ must also intersect I . Thus, the number of edges of H not intersecting I is at most

$$|V(M') \setminus V(\cup X)| \binom{n}{2} + 50n^2 \leq \frac{\epsilon n}{4} \binom{n}{2} + 50n^2 \leq \frac{\epsilon}{4} n^3.$$

As $|I| = s \leq m - 1$,

$$|E(S(n, m, 3)) \setminus E(H)| = |E(H) \setminus E(S(n, m, 3))| + e(S(n, m, 3)) - e(H) \leq \frac{\epsilon}{4} n^3 + \epsilon^4 n^3 < \epsilon n^3.$$

So in this case, we see that H is ϵ -close to $S(n, m, 3)$.

By Claim 4, it remains to consider

$$y > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/12}{3}.$$

We claim that there exists a complete 3-graph K on more than $3m - 3\epsilon n/2$ vertices and $V(K) \subseteq V(M')$. Suppose to the contrary that $V(M')$ does not contain such a complete 3-graph K . Since

$$|V(M')| - (3m - 3\epsilon n/2) = 3(s - 6) - 3m + 3\epsilon n/2 > \frac{\epsilon n}{2}$$

and H is stable, $V(M')$ contains an independent set of size $\frac{\epsilon n}{2}$, say A . Note that if $T = \{e_1, e_2, e_3\}$ with $e_i \cap A \neq \emptyset$ for all $i \in [3]$, then $f_3(T) < 27$. Since there are at least $|A|/3 \geq \epsilon n/6$ edges in M' which intersect with A , we see that $y \leq \binom{s-6}{3} - \binom{\epsilon n/6}{3}$, which leads to a contradiction.

Then

$$|E(D(n, m, 3)) \setminus E(H)| \leq |E(D(n, m, 3)) \setminus E(K)| \leq \frac{3}{2} \epsilon n \binom{n}{2} < \epsilon n^3,$$

i.e., H is ϵ -close to $D(n, m, 3)$. This finishes the proof of Lemma 4.2. $\square$

5 Almost perfect rainbow matchings

In this section, we prove a lemma about almost perfect rainbow matchings that we will need. In fact, this result holds for families of k -graphs, for any $k \geq 3$.

Lemma 5.1. *For any given integer $k \geq 3$, there exist positive real numbers c and n_2 such that the following holds. Let n and m be integers with $n \geq km$ and $n \geq n_2$, and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a stable family of k -graphs on the same vertex set $[n]$ such that $|F_i| > \binom{km-1}{k}$ for each $i \in [m]$. If $m > (1-c)n/k$, then $\mathcal{F}$ admits a rainbow matching.*

Proof. We choose $c' = c'(k)$ and $c = c(k)$ small enough such that $0 < c \ll c' \ll 1$. Let n be sufficiently large and $n/k \geq m > (1-c)n/k$. Suppose to the contrary that $|F_i| > \binom{km-1}{k}$ for each $i \in [m]$ and $\mathcal{F}$ does not admit a rainbow matching.

By Corollary 2.3, we may additionally assume that $\mathcal{F}$ is saturated. Let U_i be the vertex set of a largest complete k -graph in F_i for $i \in [m]$. Since F_i is stable, we may choose $U_i = [U_i]$ such that $[n] \setminus U_i$ is an independent set in F_i . For each $i \in [m]$, we have $|U_i| > (1-c')km$; for, otherwise, we have the following contradiction for some $i \in [m]$:

$$|F_i| \leq \binom{n}{k} - \binom{c'km}{k} \leq \binom{n}{k} - (cn+1) \binom{n-1}{k-1} \leq \binom{n}{k} - (n-km+1) \binom{n-1}{k-1} < \binom{km-1}{k},$$

where the second inequality holds since $c \ll c' \ll 1$ and $m > (1-c)n/k$, the third inequality holds since $n - km < cn$, and the last inequality holds since

$$\binom{n}{k} - \binom{km-1}{k} = \sum_{i=1}^{n-km+1} \binom{n-i}{k-1} < (n-km+1) \binom{n-1}{k-1}.$$

Let $U = \bigcap_{i=1}^m U_i$. By the above paragraph, we see that $|U| \geq (1-c')km$. If $|U| \geq km$, then it is clear that $\mathcal{F}$ admits a rainbow matching. So we may assume that $U_m = U \subseteq [km-1]$. Because U_m is the vertex set of a largest complete k -subgraph of F_m and since F_m is stable and $|F_m| > \binom{km-1}{k}$, there exists some k -set $e \notin F_m$ such that $|e \cap U| = k-1$ and $km \in e$. Since $\mathcal{F}$ is saturated, there exists a rainbow matching M in $\mathcal{F} \setminus F_m$ such that $M \cup \{e\}$ is a rainbow matching in $\mathcal{F}(e, F_m)$. Since F_i is stable for each $i \in [m]$, we may assume that $V(M) \cup e = [km]$. Let $M' = \{e' \in M : e' \not\subseteq U\}$.

Claim 5. (a) $|M'| < c'km$,

(b) each edge of F_m is contained in U or intersects an edge of M' , and

(c) for any $v \in V(M) \setminus U$, $d_{F_m[U]}(v) \leq c'k^2m \binom{|U|}{k-2}$.

Proof. To prove (a), just observe that $|M'| \leq |V(M) \setminus U| = (km-1) - |U| < c'km$.

Suppose that (b) fails, i.e., there exists an edge $f \in F_m$ such that $f \setminus U \neq \emptyset$ and $f \cap V(M') = \emptyset$. Note that $f \cap (U \setminus V(M')) \neq \emptyset$, as $[n] \setminus U$ is independent in F_m . In particular, $|f \cap (U \setminus V(M'))| \leq k-1$. Let $|M'| = m-t$ for some $t \geq 1$. Recall that $U \cup V(M') = V(M) = [km-1]$. Hence $|U \setminus V(M')| = kt-1$,

and thus $U \setminus (V(M') \cup f)$ induces a common complete k -graph of size at least $k(t-1)$ in all F_i . Then we see that $M' \cup \{f\}$ together with a matching of size $t-1$ in $U \setminus (V(M') \cup f)$ form a rainbow matching for $\mathcal{F}$. So (b) holds.

Now we prove (c). For any $v \in V(M) \setminus U \subseteq [km]$, by the maximality of U , there exists $f \in \binom{[n]}{k} \setminus F_m$ such that $v \in f$ and $|f \cap U| = k-1$. So there exists a rainbow matching N in $\mathcal{F} \setminus F_m$ such that $N \cup \{f\}$ is a rainbow matching in $\mathcal{F}'(f, F_m)$. Since F_i is stable for $i \in [m]$, we may assume that $V(N) \cup f = [km]$. Let $N' = \{e' \in N : e' \not\subseteq U\}$. By applying (b) to N' , every edge of F_m containing v intersects $V(N')$. Since

$$V(N') \leq k|N'| \leq k(km - |U|) \leq c'k^2m,$$

there are at most $c'k^2m \binom{|U|}{k-2}$ edges e' in F_m containing v such that $e' \subseteq U \cup \{v\}$. Hence (c) holds. This proves the claim. $\square$

Note that $|e \cap U| = k-1$ and $V(M) \cup U = [km-1]$. Let q_1 be the number of edges of F_m contained in $[km-1]$, and q_2 be the number of edges of F_m with at least one vertex in $[n] \setminus [km-1]$. By (c), we have

$$q_1 \leq \binom{km-1}{k} - |V(M) \setminus U| \binom{|U|}{k-1} + |V(M) \setminus U| \cdot c'k^2m \binom{|U|}{k-2}.$$

By (b), we see $q_2 \leq |V(M')| \cdot (n - km + 1) \binom{n-2}{k-2}$. So we have

$$\begin{aligned} |F_m| &\leq \binom{km-1}{k} - |V(M) \setminus U| \left[\binom{|U|}{k-1} + c'k^2m \binom{|U|}{k-2} \right] + |V(M')| (n - km + 1) \binom{n-2}{k-2} \\ &\leq \binom{km-1}{k} - |V(M) \setminus U| \left[\binom{|U|}{k-1} + c'k^2m \binom{|U|}{k-2} \right] + k|V(M) \setminus U| (cn + 1) \binom{n-2}{k-2} \\ &= \binom{km-1}{k} - |V(M) \setminus U| \cdot \left[\binom{|U|}{k-1} - c'k^2m \binom{|U|}{k-2} - k(cn + 1) \binom{n-2}{k-2} \right] \\ &< \binom{km-1}{k}, \end{aligned}$$

where the second inequality holds since $n - km < cn$ and $|M'| \leq |V(M) \setminus U|$, and the last inequality holds since c' and c are small enough and $|U| > (1 - c')km > (1 - c')(1 - c)n$. This is a contradiction, finishing the proof of Lemma 5.1. $\square$

6 Non-extremal configurations

Note that if there exist $F \in \mathcal{F}$ and $v \in [n]$ such that $d_F(v) = \binom{n-1}{k-1}$, then v can be removed from all the k -graphs in $\mathcal{F} \setminus \{F\}$ to obtain a smaller family $\mathcal{F}'$ so that $\mathcal{F}'$ admits a rainbow matching if and only if $\mathcal{F}$ admits a rainbow matching. Hence, if such vertex does not exist in a saturated family $\mathcal{F}$, then from Lemma 2.1, we see that $d_F(v) \leq \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1}$ for all $v \in F$ and $F \in \mathcal{F}$. This leads us to the following result.

Lemma 6.1. *Given real numbers $0 < \epsilon \ll c \ll 1$, let $n \geq n(\epsilon, c)$ be a sufficiently large integer and m be an integer such that $n/27 < m < (1 - c)n/3$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a stable family of 3-graphs on the vertex set $[n]$ such that for every $i \in [m]$, $|F_i| > f(n, m, 3)$ and*

$$d_{F_i}(v) \leq \binom{n-1}{2} - \binom{n-3(m-1)-1}{2}$$

for each $v \in [n]$. If $H(\mathcal{F})$ is ϵ -close to neither $H_S(n, m, 3)$ nor $H_D(n, m, 3)$, then $\mathcal{F}$ admits a rainbow matching.

Proof. Given $0 < \epsilon \ll c \ll 1$, let n' and m' be integers such that n' is sufficiently large and $n'/27 < m' < (1 - c)n'/3$. Let $\mathcal{F} = \{F_1, \dots, F_{m'}\}$ be a family of 3-graphs on the vertex set $[n']$ such that $|F_i| > f(n', m', 3)$ and

$$d_{F_i}(v) \leq \binom{n'-1}{2} - \binom{n'-1-3(m'-1)}{2}$$

for $i \in [m']$ and $v \in [n']$. Suppose that $H(\mathcal{F})$ is not ϵ -close to $H_S(n', m', 3)$ or $H_D(n', m', 3)$. Our ultimate goal is to find a rainbow matching in $\mathcal{F}$.

Let $n' = 3m' + 3r' + s$, where $0 \leq s < 3$. Recall the definitions of $H(\mathcal{F})$ and $H^*(\mathcal{F})$ such that $V(H(\mathcal{F})) = [n'] \cup \mathcal{V}'$ and $V(H^*(\mathcal{F})) = [n'] \cup \mathcal{V}' \cup \mathcal{U}'$, where $|\mathcal{V}'| = m'$ and $|\mathcal{U}'| = r'$. By Lemma 2.5, for $0 < \gamma' \ll \gamma \ll \epsilon \ll c \ll 1$, there exists a matching M_a in $H^*(\mathcal{F})$ with $|M_a| \leq \gamma n'$ such that for any $S \subseteq V(H^*(\mathcal{F})) \setminus V(M_a)$ with $|S| \leq \gamma' n'$ and $3|S \cap (\mathcal{V}' \cup \mathcal{U}')| = |S \cap [n']|$, $H^*(\mathcal{F})[V(M_a) \cup S]$ has a perfect matching. In the rest of the proof, without loss of generality, we use the following notation:

$$H = H^*(\mathcal{F}) - V(M_a), \quad [n] = [n'] \setminus V(M_a), \\ \mathcal{V} = \mathcal{V}' \setminus V(M_a) = \{v_1, \dots, v_m\}, \quad \mathcal{U} = \mathcal{U}' \setminus V(M_a) = \{u_1, \dots, u_r\}.$$

Then $n = 3m + 3r + s$. By using the above property of the matching M_a , it now suffices for us to find an almost perfect matching in H . To find this almost perfect matching, our plan is to show that there exists an almost regular subgraph of H with bounded maximum co-degree so that Theorem 2.6 can be applied. To that end, in what follows we will use the two-round randomization technique developed in [3].

Let R be chosen from $V(H)$ by taking each vertex independently of probability $n^{-0.9}$. We take $n^{1.1}$ independent copies of R and denote them by R^i for $1 \leq i \leq n^{1.1}$. For $S \subseteq V(H)$, denote $Y_S = |\{i : S \subseteq R^i\}|$. First we have the following claim.

Claim A. With probability $1 - o(1)$, the following hold:

- (i) for every $v \in V(H)$, $Y_{\{v\}} = (1 + o(1))n^{0.2}$,
- (ii) every pair $\{u, v\} \subseteq V(H)$ is contained in at most two sets R^i , and
- (iii) every edge $e \in H$ is contained in at most one set R^i .

Proof. Note that $Y_S \sim \text{Bin}(n^{1.1}, n^{-0.9|S|})$ for any $S \subseteq V(H)$. Thus, $\mathbb{E}[Y_{\{v\}}] = n^{0.2}$ for every $v \in V(H)$. By (2.2) in Lemma 2.10, we have $\mathbb{P}(|Y_{\{v\}} - n^{0.2}| > n^{0.15}) \leq e^{-\Omega(n^{0.1})}$. By the union bound, we see that (i) holds. To prove (ii) and (iii), let

$$Z_2 = \left| \left\{ \{u, v\} \in \binom{V(H)}{2} : Y_{\{u, v\}} \geq 3 \right\} \right| \quad \text{and} \quad Z_3 = \left| \left\{ S \in \binom{V(H)}{3} : Y_S \geq 2 \right\} \right|.$$

Then

$$\mathbb{E}[Z_2] = \binom{|V(H)|}{2} \mathbb{P}(Y_{\{u, v\}} \geq 3) \leq \binom{n}{2} (n^{1.1})^3 (n^{-1.8})^3 \leq 4n^{-0.1}$$

and

$$\mathbb{E}[Z_3] \leq \binom{n}{3} (n^{1.1})^2 (n^{-2.7})^2 \leq 8n^{-0.2}.$$

By Markov's inequality, we have

$$\mathbb{P}(Z_2 = 0) > 1 - 4n^{-0.1} \quad \text{and} \quad \mathbb{P}(Z_3 = 0) > 1 - 8n^{-0.2}.$$

That implies that (ii) and (iii) hold with probability at least $1 - 4n^{-0.1}$ and $1 - 8n^{-0.2}$, respectively. $\square$

Next, we want to prove that there exists a perfect (or, rather, maximum) fractional matching in each $H[R^i]$. To do so, we define a maximal subset $R'^i \subseteq R^i$ that satisfies $R'^i \cap [n] = 3|R'^i \cap (\mathcal{V} \cup \mathcal{U})|$ as follows. If $|R^i \cap [n]| \geq 3|R^i \cap (\mathcal{V} \cup \mathcal{U})|$, we take a subset of R^i denoted by R'^i , which is chosen from R^i by deleting $|R^i \cap [n]| - 3|R^i \cap (\mathcal{V} \cup \mathcal{U})|$ vertices in $R^i \cap [n]$ independently and uniformly at random. Otherwise $|R^i \cap [n]| < 3|R^i \cap (\mathcal{V} \cup \mathcal{U})|$, we take a subset of R^i denoted by R'^i by the following step: First we delete at most 3 vertices (chosen independently and uniformly at random) in $R^i \cap [n]$ so that the number ℓ of the remaining vertices is a multiple of 3. Then we delete $|R^i \cap (\mathcal{V} \cup \mathcal{U})| - \ell/3$ vertices in $R^i \cap (\mathcal{V} \cup \mathcal{U})$ independently and uniformly at random.

For $S \subseteq V(H)$, define $Y'_S = |\{i : S \subseteq R'^i\}|$. Note that $\mathbb{E}(|R^i \cap [n]|) = n^{0.1}$, $\mathbb{E}(|R^i \cap (\mathcal{V} \cup \mathcal{U})|) = n^{0.1}/3$ and $\mathbb{E}(|R^i \cap \mathcal{V}|) = n^{-0.9}m$. For each i , let A_i be the event $||R^i \cap [n]| - n^{0.1}| < n^{0.095}$, B_i be the event $||R^i \cap (\mathcal{V} \cup \mathcal{U})| - n^{0.1}/3| < n^{0.095}$, and C_i be the event $||R^i \cap \mathcal{V}| - n^{-0.9}m| < n^{0.095}$.

Claim B. With probability $1 - o(1)$, the following hold:

- (i) $\bigwedge_i (A_i \wedge B_i \wedge C_i)$ holds,
- (ii) for every $v \in V(H)$, $Y'_{\{v\}} = (1 + o(1))n^{0.2}$,
- (iii) every pair $\{u, v\} \subseteq V(H)$ is contained in at most two sets R^i , and
- (iv) every edge $e \in H$ is contained in at most one set R^i .

Proof. Since $R^i \subseteq R^i$, it is clear from Claim A that (iii) and (iv) hold with probability $1 - o(1)$. Next, we consider (i). By (2.2) in Lemma 2.10 (with $\lambda = n^{0.095}$), for each $1 \leq i \leq n^{1.1}$, we have

$$\mathbb{P}(\overline{A_i}) \leq e^{-\Omega(n^{0.09})}, \quad \mathbb{P}(\overline{B_i}) \leq e^{-\Omega(3n^{0.09})} = e^{-\Omega(n^{0.09})} \quad \text{and} \quad \mathbb{P}(\overline{C_i}) \leq e^{-\Omega(\frac{n}{m}n^{0.09})} = e^{-\Omega(n^{0.09})}.$$

Thus by the union bound, $\mathbb{P}(\bigwedge_i (A_i \wedge B_i \wedge C_i)) = 1 - o(1)$, proving (i).

Assuming $A_i \wedge B_i \wedge C_i$, we see

$$|R^i \setminus R^i| < \max\{|R^i \cap [n]| - 3|R^i \cap (\mathcal{V} \cup \mathcal{U})|, |R^i \cap (\mathcal{V} \cup \mathcal{U})| - \lfloor |R^i \cap [n]|/3 \rfloor + 3\} < 4n^{0.095}.$$

Then by the choice of R^i , for all $v \in V(H)$, the probability $\mathbb{P}(\{v \in R^i \setminus R^i \mid (A_i \wedge B_i \wedge C_i) \wedge (v \in R^i)\})$ is at most

$$\max \left\{ \frac{|R^i \setminus R^i|}{|R^i \cap [n]|}, \frac{|R^i \setminus R^i|}{|R^i \cap (\mathcal{V} \cup \mathcal{U})|} \right\} \leq \frac{|R^i \setminus R^i|}{|R^i \cap (\mathcal{V} \cup \mathcal{U})|} < \frac{4n^{0.095}}{n^{0.1}/3 - n^{0.095}} < 13n^{-0.005}.$$

Using coupling and applying (2.2) in Lemma 2.10 to $\text{Bin}(|Y_v|, 13n^{-0.005})$ with $\lambda = 3n^{0.195}$, we have

$$\mathbb{P} \left(\left\{ Y_{\{v\}} - Y'_{\{v\}} > 16n^{0.195} \mid \bigwedge_i (A_i \wedge B_i \wedge C_i) \wedge (Y_{\{v\}} = (1 + o(1))n^{0.2}) \right\} \right) \leq e^{-\Omega(n^{0.195})}.$$

Note that with probability $1 - o(1)$, $\bigwedge_i (A_i \wedge B_i \wedge C_i)$ and $Y_{\{v\}} = (1 + o(1))n^{0.2}$ hold for all $v \in V(H)$. By the union bound, we can derive that $0 \leq Y_{\{v\}} - Y'_{\{v\}} \leq 16n^{0.195} = o(n^{0.2})$ for all $v \in V(H)$ with probability $1 - o(1)$. Hence (ii) holds with probability $1 - o(1)$. This proves Claim B. $\square$

Let $n_i = |R^i \cap [n]|$ and $m_i = |R^i \cap \mathcal{V}|$. Using Claim B(i), we see that with probability $1 - o(1)$, $m_i = (1 + o(1))mn^{-0.9} = \Theta(n^{0.1}) = \Theta(n_i)$ for all $1 \leq i \leq n^{1.1}$.

Claim C. With probability $1 - o(1)$, the following hold for all $1 \leq i \leq n^{1.1}$:

- (a) $H[R^i \setminus \mathcal{U}]$ is not $\epsilon^4/4$ -close to $H_S(n_i, m_i, 3)$ or $H_D(n_i, m_i, 3)$, and
- (b) there exists a perfect fractional matching in $H[R^i]$.

Proof. For each $T \in \binom{V(H)}{\leq 2}$, let

$$\text{Deg}^i(T) := \left| N_H(T) \cap \binom{R^i}{4 - |T|} \right|.$$

By the definition of H , we have that

- for any $v_j \in \mathcal{V}$, $d_H(v_j) \geq f(n', m', 3) - (\gamma n') \binom{n'}{2} \geq f(n, m, 3) - \gamma n^3$, and
- for any $T = \{v_j, u\}$ with $v_j \in \mathcal{V}$ and $u \in [n]$,

$$d_H(T) = d_{F_j}(u) \leq \binom{n' - 1}{2} - \binom{n' - 1 - 3(m' - 1)}{2} \leq \binom{n - 1}{2} - \binom{n - 1 - 3(m - 1)}{2} + \gamma n^2.$$

Assume that $\bigwedge_i (A_i \wedge B_i \wedge C_i)$ holds. Then $n_i = (1 + o(1))n^{0.1}$ and $m_i = (1 + o(1))mn^{-0.9}$. Since $R^i \setminus R^i = o(n_i)$, for each $T \in \binom{V(R^i)}{t}$ with $t \in [2]$, we have

$$\mathbb{E}[\text{Deg}^i(T)] = (1 + o(1))d_H(T)(n^{-0.9})^{4-t}.$$

Thus for any $v \in \mathcal{V} \cap R^i$,

$$\mathbb{E}[\text{Deg}^i(v)] \geq (1 + o(1))(f(n, m, 3) - \gamma n^3)(n^{-0.9})^3 \geq f(n_i, m_i, 3) - 2\gamma n_i^3,$$

and for any $T = \{u, v\}$ with $v \in \mathcal{V}$ and $u \in [n]$, $\mathbb{E}[\text{Deg}^i(T)]$ is at most

$$(1 + o(1)) \left[\binom{n-1}{2} - \binom{n-1-3(m-1)}{2} + \gamma n^2 \right] (n^{-0.9})^2 \leq \binom{n_i-1}{2} - \binom{n_i-1-3(m_i-1)}{2} + 2\gamma n_i^2.$$

We apply Janson's inequality (see [4, Theorem 8.7.2]) to bound the deviation of $\text{Deg}^i(T)$ for $|T| \leq 2$. Write $\text{Deg}^i(T) = \sum_{e \in N_H(T)} X_e$, where $X_e = 1$ if $e \subseteq R^i$ and $X_e = 0$ otherwise. Let $t = |T| \in \{1, 2\}$ and $p = n^{-0.9}$. Then

$$\begin{aligned} \Delta^* &= \sum_{e_i \cap e_j \neq \emptyset, e_i, e_j \in \binom{V(H)}{4-t}} \mathbb{P}(X_{e_i} = X_{e_j} = 1) \\ &\leq \sum_{\ell=1}^{4-t} p^{2(4-t)-\ell} \binom{n-t}{4-t} \binom{4-t}{\ell} \binom{n-4}{4-t-\ell} = O(n^{0.1(2(4-t)-1)}). \end{aligned}$$

By Janson's inequality, for $v \in \mathcal{V} \cap R^i$,

$$\mathbb{P}(\text{Deg}^i(v) \leq (1 - \gamma)\mathbb{E}[\text{Deg}^i(v)]) \leq e^{-\gamma^2 \mathbb{E}[\text{Deg}^i(v)] / (2 + \Delta^* / \mathbb{E}[\text{Deg}^i(v)])} \leq e^{-\Omega(n^{0.3} / (2 + n^{0.5} / n^{0.3}))} = e^{-\Omega(n^{0.1})},$$

and for the pair $\{v, u\}$ with $v \in \mathcal{V}$ and $u \in [n]$ (by considering the complement of H), we can have

$$\mathbb{P}(\text{Deg}^i(\{v, u\}) \geq (1 + \gamma)\mathbb{E}[\text{Deg}^i(\{v, u\})]) \leq e^{-\Omega(n^{0.1})}.$$

By the union bound, with probability $1 - o(1)$ we derive from above that for all $1 \leq i \leq n^{1.1}$,

- (1) for any $v \in \mathcal{V} \cap R^i$, $\text{Deg}^i(v) \geq (1 - \gamma)\mathbb{E}[\text{Deg}^i(v)] \geq f(n_i, m_i, 3) - 3\gamma n_i^3$, and
- (2) for any pair $\{u, v_j\} \subseteq R^i$ with $v_j \in \mathcal{V}$ and $u \in [n]$,

$$\text{Deg}^i(\{u, v_j\}) \leq \binom{n_i-1}{2} - \binom{n_i-1-3(m_i-1)}{2} + 3\gamma n_i^2 \leq \binom{n_i-1}{2} - \Omega(n_i^2),$$

which implies that $F_j[R^i \cap [n]]$ is not $\epsilon^3/2$ -close to $S(n_i, m_i, 3)$, since $m_i = (1 + o(1))mn^{0.9}$ and $m < (1 - c)n/3$.

This shows that $H[R^i \setminus \mathcal{U}]$ is not $\epsilon^4/4$ -close to $H_S(n_i, m_i, 3)$, where $\gamma \ll \epsilon$.

Let $\mathcal{V}_0 := \{v_i \in \mathcal{V} : F_i[[n]] \text{ is not } \epsilon\text{-close to } D(n, m, 3)\}$. We claim that $|\mathcal{V}_0| > \epsilon n$. Otherwise $|\mathcal{V}_0| \leq \epsilon n$, we have

$$|E(H_D(n', m', 3)) \setminus E(H(\mathcal{F}))| \leq \epsilon n \binom{n}{3} + (m - \epsilon n)\epsilon n^3 + \gamma(n')^4 \leq \epsilon(n')^4,$$

which leads to a contradiction as $H(\mathcal{F})$ is not ϵ -close to $H_D(n', m', 3)$. As $|\mathcal{V}_0| > \epsilon n$, with probability $1 - o(1)$ we have (by using Lemma 2.10) that

- (3) $|R^i \cap \mathcal{V}_0| \geq \frac{\epsilon n_i}{2}$ for all $1 \leq i \leq n^{1.1}$.

For $v_j \in R^i \cap \mathcal{V}_0$, we consider $F_j[[n]]$. Let G be the complement of $F_j[[n]]$. Then for any $S \subseteq V(G)$ with $|S| > 3m - \epsilon n$, we have $e(G[S]) \geq \epsilon e(G)$. Otherwise,

$$|E(D(n, m, 3)) \setminus E(F_j[[n]])| \leq \epsilon n \binom{n}{2} + \epsilon e(G) < \epsilon n^3,$$

contradicting $v_j \in \mathcal{V}_0$. By Lemma 2.8, the maximum size of the complete 3-graph in $F_j[R^i \cap [n]]$ is no more than $(3m/n - \epsilon + \gamma)n^{0.1} \leq 3m_i - \epsilon n_i/2$ with probability at least $1 - (n^{O(1)}e^{-\Omega(n^{0.1})})$. By assuming $\bigwedge_i (A_i \wedge B_i \wedge C_i)$, this implies that $F_j[R^i \cap [n]]$ is not $\epsilon^3/2$ -close to $D(n_i, m_i, 3)$. By the union bound, with probability $1 - o(1)$ we have

- (4) for all $1 \leq i \leq n^{1.1}$ and $v_j \in R^i \cap \mathcal{V}_0$, $F_j[R^i \cap [n]]$ is not $\epsilon^3/2$ -close to $D(n_i, m_i, 3)$.

By (3) and (4), we see that with probability $1 - o(1)$, $H[R^i \setminus \mathcal{U}]$ is not $\epsilon^4/4$ -close to $H_D(n_i, m_i, 3)$, proving Claim C(a).

It remains to show Claim C(b), i.e., to construct a perfect fractional matching w_i in $H[R^i]$ for each $1 \leq i \leq n^{1.1}$. Our main tool is the stability result, Lemma 4.2.

Fix some $1 \leq i \leq n^{1.1}$. We write $R^i \cap [n] = \{x_1^i, \dots, x_{n_i}^i\}$ with $x_1^i < x_2^i < \dots < x_{n_i}^i$ and define $[d]_i := \{x_1^i, x_2^i, \dots, x_d^i\}$ for any integer d . We now state two simple inequalities for later use:

$$f(x, y, 3) \geq f(x, y - a, 3) + \binom{a}{3} \quad \text{and} \quad f(x, y, 3) \geq f(x, y + a, 3) - 3ax^2 \quad (6.1)$$

hold for any positive integers x, y and a with $a < y$.

To construct a perfect fractional matching w_i in $H[R^i]$, first we consider $v_j \in R^i \cap \mathcal{V}_0$ and assign weights to the edges of $H[R^i]$ containing v_j . Using (1), by (6.1) and the fact that $\gamma \ll \epsilon \ll 1$, we have

$$|F_j[R^i \cap [n]]| = \text{Deg}^i(v_j) \geq f(n_i, m_i, 3) - 3\gamma n_i^3 \geq f(n_i, m_i + \epsilon^{20} n_i, 3) - \epsilon^{16} n_i^3.$$

By (2) and (4), $F_j[R^i \cap [n]]$ is not $\epsilon^3/2$ -close to $S(n_i, m_i, 3)$ or $D(n_i, m_i, 3)$. Since $|E(S(n_i, m_i + \epsilon^{20} n_i, 3)) \setminus E(S(n_i, m_i, 3))| \leq \epsilon^{20} n_i^3$ and $|E(D(n_i, m_i + \epsilon^{20} n_i, 3)) \setminus E(D(n_i, m_i, 3))| \leq 3\epsilon^{20} n_i^3$, we see that $F_j[R^i \cap [n]]$ is not ϵ^4 -close to $S(n_i, m_i + \epsilon^{20} n_i, 3)$ or $D(n_i, m_i + \epsilon^{20} n_i, 3)$. Then by Lemma 4.2 and the fact that F_j is stable, $F_j[R^i \cap [n]]$ contains a matching M_j with $V(M_j) = [3m_i + 3\epsilon^{20} n_i]_i$. Now we assign weights $w_i(e)$ to all the edges e of $H[R^i]$ with $v_j \in e$ as follows: If $e \setminus v_j \in M_j$, then let $w_i(e) = \frac{1}{m_i + \epsilon^{20} n_i}$, and otherwise let $w_i(e) = 0$.

Next, we consider $v_j \in R^i \cap (\mathcal{V} \setminus \mathcal{V}_0)$. By (1) and (6.1), we have

$$|F_j[R^i \cap [n]]| \geq f(n_i, m_i, 3) - 3\gamma n_i^3 \geq f(n_i, m_i - 6\gamma^{\frac{1}{3}} n_i, 3).$$

By Theorem 4.1 and the fact that F_j is stable, $F_j[R^i \cap [n]]$ contains a matching M_j with $V(M_j) = [3m_i - 18\gamma^{\frac{1}{3}} n_i]_i$. Then we assign weights $w_i(e)$ to all the edges e of $H[R^i]$ with $v_j \in e$ as follows: If $e \setminus v_j \in M_j$, then let $w_i(e) = \frac{1}{m_i - 6\gamma^{\frac{1}{3}} n_i}$, and otherwise let $w_i(e) = 0$.

Note that for every $v_j \in R^i \cap \mathcal{V}$, we have defined weights $w_i(e)$ for all the edges $e \in H[R^i]$ with $v_j \in e$, whose total weights equal one. In the remaining proof, we want to extend this function w_i to the entire $H[R^i]$ to form a perfect fractional matching. We complete this in two steps.

First, we define a perfect fractional matching w (as the *projection* of w_i) in the complete 3-graph K on the vertex set $R^i \cap [n]$. Note that a function $w : E(K) \rightarrow [0, 1]$ is a perfect fractional matching if and only if $w(v) := \sum_{v \in f \in K} w(f) = 1$ holds for every $v \in V(K)$. Initially, we define a function $w' : E(K) \rightarrow [0, 1]$ such that for each $f \in E(K)$, $w'(f) := \sum_e w_i(e)$ over all the edges $e \in H[R^i]$ with $f \subseteq e$ and $|e \cap \mathcal{V}| = 1$. Since $|\mathcal{V}_0| > \epsilon n$ and $\gamma \ll \epsilon$, it follows from the above definitions on w_i that for any $v \in R^i \cap [n]$,

$$w'(v) := \sum_{v \in f \in K} w'(f) \leq \frac{|\mathcal{V}_0|}{m_i + \epsilon^{20} n_i} + \frac{m_i - |\mathcal{V}_0|}{m_i - 6\gamma^{\frac{1}{3}} n_i} \leq \frac{\epsilon n_i}{m_i + \epsilon^{20} n_i} + \frac{m_i - \epsilon n_i}{m_i - 6\gamma^{\frac{1}{3}} n_i} < 1.$$

Since $\epsilon \ll c$, we have $3m_i + 3\epsilon^{20} n_i < n_i - 4$. So there exists a vertex set $\{a_1, a_2, a_3, a_4\}$ in K such that $w'(a_i) = 0$ for $i \in [4]$. Let K' be the 3-graph obtained from K by deleting vertices a_1, a_2, a_3 and a_4 . Starting with $w := w'$, we increase w by using the following iterations: (i) pick a vertex v in $V(K')$ with the maximum $w(v)^3$; (ii) pick any edge $f \in K'$ containing v and update $w(f) \leftarrow w(f) + 1 - w(v)$; (iii) delete all the vertices $u \in V(K')$ with $w(u) = 1$ (which must include the vertex v) from K' ; (iv) if $|V(K')| \leq 2$, then terminate; otherwise go to (i) again. This must terminate in finitely many iterations and when it terminates, we obtain a fractional matching w in K such that $w(a_i) = 0$ for $i \in [4]$ and $|V(K')| \leq 2$. So there exist two vertices b_1 and b_2 in $V(K) \setminus \{a_1, a_2, a_3, a_4\}$ such that for any vertex v in $V(K) \setminus \{a_1, a_2, a_3, a_4, b_1, b_2\}$, $w(v) = 1$. We may suppose $1 \geq w(b_1) \geq w(b_2)$. Let

$$w(a_1, a_2, b_1) = 1 - w(b_1), \quad w(a_1, a_2, b_2) = \frac{w(b_1) - w(b_2)}{2}, \quad w(a_3, a_4, b_2) = 1 - w(b_1) + \frac{w(b_1) - w(b_2)}{2}$$

and

$$w(a_1, a_2, a_3) = w(a_1, a_2, a_4) = w(a_1, a_3, a_4) = w(a_2, a_3, a_4) = \frac{w(b_1) + w(b_2)}{6}.$$

³⁾ Note that this maximum $w(v)$ is strictly less than 1.

It is easy to check that w is a perfect fractional matching in K .

Now we notice that

$$\sum_{f \in K} w'(f) = \sum_{\{e \in H[R^i] : |e \cap \mathcal{V}|=1\}} w_i(e) = |R^i \cap \mathcal{V}|$$

and

$$\sum_{f \in K} w(f) = \frac{|R^i \cap [n]|}{3} = |R^i \cap (\mathcal{V} \cup \mathcal{U})|.$$

Moreover, the neighborhood of any $u_j \in R^i \cap \mathcal{U}$ in $H[R^i]$ is the complete 3-graph K . So we can partition the total weight $\sum_{f \in K} (w(f) - w'(f)) = |R^i \cap \mathcal{U}|$ into $|R^i \cap \mathcal{U}|$ copies of 1's (say each is represented by a set E_j of edges in K), and then for each $u_j \in R^i \cap \mathcal{U}$, we assign the weight of each $f \in E_j$ to be $w_i(f \cup \{u_j\})$. One can easily check that we obtain a perfect fractional matching w_i in $H[R^i]$. This completes the proof of Claim C. $\square$

From Claims B and C, we see that the sets R^i for $1 \leq i \leq n^{1.1}$ satisfy (a)–(d) in Lemma 2.7. Then by Lemma 2.7, there exists a spanning subgraph H' of H such that for each $v \in V(H)$, $d_{H'}(v) = (1+o(1))n^{0.2}$ and $\Delta_2(H') \leq n^{0.1}$. By Theorem 2.6, H contains a matching M_b such that $S = V(H) \setminus V(M_b)$ contains at most $\gamma'n'$ vertices. Since $|S \cup M_a \cup M_b| = n' = 3r' + 3m' + s$ where $0 \leq s \leq 2$, we can delete at most s elements from S to get a subset S' such that $3|S' \cap (\mathcal{V}' \cup \mathcal{U}')| = |S' \cap [n']|$. By the setting at the beginning of the proof, Lemma 2.5 assures that $H^*(\mathcal{F})[V(M_a) \cup S']$ has a perfect matching, which together with M_b forms a matching in $H^*(\mathcal{F})$ of size $r' + m'$. Equivalently, this says that $\mathcal{F}$ admits a rainbow matching, finishing the proof of Lemma 6.1. $\square$

7 Proof of Theorem 1.3

Let n be a sufficiently large integer. Let m be a positive integer with $n \geq 3m$ and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of 3-graphs on the same vertex set $[n]$ such that $|F_i| > f(n, m, 3)$ for each $i \in [m]$. Suppose to the contrary that $\mathcal{F}$ does not admit a rainbow matching. In view of Lemma 2.2, we may assume that $\mathcal{F}$ is stable. Then by Lemma 5.1, there exists an absolute constant $c = c(3) > 0$ such that $m \leq (1-c)n/3$. By Theorem 1.2, $m \geq n/27$. Hence,

$$n/27 \leq m \leq (1-c)n/3. \quad (7.1)$$

We now apply the following algorithm. Initially, let $\mathcal{F}_0 = \mathcal{F}$, $n_0 = n$ and $m_0 = m$. We repeat the following iterations. Suppose that we have defined $\mathcal{F}_i$, which contains m_i 3-graphs on the same vertex set $[n_i]$.

Step 1. Applying Corollary 2.3 to $\mathcal{F}_i$, we obtain a family $\mathcal{F}_{i+1}$ of 3-graphs on the vertex set $[n_i]$ that is both stable and saturated, and set $n_{i+1} = n_i$ and $m_{i+1} = m_i$.

Step 2. If for any $F \in \mathcal{F}_{i+1}$ and any $v \in [n_{i+1}]$, $d_F(v) < \binom{n_{i+1}-1}{2}$, then set $t := i+1$ and output $\mathcal{F}_t$, n_t and m_t .

Step 3. If there exist $F \in \mathcal{F}_{i+1}$ and $v \in [n_{i+1}]$ such that $d_F(v) = \binom{n_{i+1}-1}{2}$, then set $n'_{i+1} = n_{i+1} - 1$, $m'_{i+1} = m_{i+1} - 1$ and $\mathcal{F}'_{i+1} := \{F' - v : F' \in \mathcal{F}_i \setminus \{F\}\}$. Relabel the vertices if necessary so that all the 3-graphs in $\mathcal{F}'_{i+1}$ have the same vertex set $[n'_{i+1}]$. Set $\mathcal{F}_i := \mathcal{F}'_{i+1}$, $n_i := n'_{i+1}$, $m_i := m'_{i+1}$ and go to Step 1.

Let $\mathcal{F}_t$ be the resulting family of 3-graphs, which contains m_t 3-graphs on the same vertex set $[n_t]$ and admits no rainbow matching. By (7.1), we see that $n_t \geq n - m > cn$ is sufficiently large. We also see from Lemma 2.9 that $|F| > f(n_t, m_t, 3)$ holds for any $F \in \mathcal{F}_t$.

By definition, we see that $\mathcal{F}_t$ is stable and saturated such that for any $F \in \mathcal{F}_t$ and $v \in V_t$, $d_F(v) < \binom{n_t-1}{2}$. On the other hand, by Lemma 2.1, it further holds that

$$d_F(v) \leq \binom{n_t-1}{2} - \binom{n_t-1-3(m_t-1)}{2} \quad \text{for any } F \in \mathcal{F}_t \quad \text{and} \quad v \in V_t.$$

Since n_t is sufficiently large, using Lemma 5.1 and Theorem 1.2 again, we may assume that

$$n_t/27 \leq m_t \leq (1-c)n_t/3.$$

Now we choose $0 < \epsilon \ll c$. Since $\mathcal{F}_t$ satisfies the above properties, by applying Lemmas 2.4, 3.1 and 6.1, we can conclude that $\mathcal{F}_t$ admits a rainbow matching. This is a contradiction, completing the proof of Theorem 1.3. $\square$

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