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A local observability analysis method for a time-varying nonlinear system and its application in the continuous self-calibration system

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Dear editor,

Employing observability analysis of a dynamic system is necessary to determine the efficiency of a Kalman filter designed to estimate the state of dynamic system, and the system state is a set of variables which can describe the motion state of the dynamic system. The observability sets a lower limit for the estimation error, and the lower the limit, the better the likelihood of obtaining an accurate estimate of the system state. In other words, if the system is not observable, it is not possible to accurately estimate the system state, even if the noise level is negligible. While the observability analysis of a constant linear system is simple, analyzing a time-varying nonlinear system is quite complicated. Goshen-Meskin and Bar-Itzhack [1] modeled the time-varying nonlinear system as the piece-wise constant system (PWCS) to analyze its observability, which is presented as a step-by-step procedure. Chen [2] introduced a concept of the local observability for the time-varying linear system, and Bartosiewicz [3] extended this concept to the nonlinear system. For the observability of the system state, singular value decomposition (SVD) [4] has been widely used to analyze the observable degree. However, this method had a theoretical limitation, namely, the dimensions of singular values corresponding to different system states are different; therefore it is unreasonable to directly compare the singular values. However, to the best of our knowledge, little research has been conducted on the observability analysis of the time-varying nonlinear system.

Based on the above observation, a local observability analysis method for the time-varying nonlinear system based on PWCS is proposed. In this method, the observability of the time-varying nonlinear system is separated into two parts, the observability of the system and observable degree of the system state. The observability of the system indicates whether the system is observable, which provides a further indication of whether the system state can be estimated using a Kalman filter. However, the observable degree is the quantitative indicator of the observability of the system state, which can indicate the estimate efficiency of the system state in the Kalman filter. First, the PWCS method was extended to the time-varying nonlinear system, and the stripped observability matrix (SOM) was derived based on the definition of local observability. The transformation results of the SOM were then used to represent the observable degree of the system state. Furthermore, this method was applied to the continuous self-calibration system, and the simulation results confirmed its rationality.

Local observability analysis of the time-varying nonlinear system. A type of time-varying nonlinear system is represented by the following state space equations:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{f}_k(\boldsymbol{x}(k)), \\ \boldsymbol{z}(k) = \boldsymbol{C}(k)\boldsymbol{x}(k), \end{cases} \tag{1}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is an n-dimensional vector; $\boldsymbol{z}(k) \in \mathbb{R}^m$ is an m-dimensional vector; $\boldsymbol{A}(k) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}(k) \in \mathbb{R}^{m\times n}$ are $n \times n$ and $m \times n$ time-varying matrices respectively; $\boldsymbol{f}_k : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear bounded mapping. The PWCS of the system can be modeled as

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}_j\boldsymbol{x}(k) + \boldsymbol{f}_j(\boldsymbol{x}(k)), \\ \boldsymbol{z}(k) = \boldsymbol{C}_j\boldsymbol{x}(k), \end{cases} \tag{2}$$

where $j = 1, 2, \ldots, r$. The local observability is defined in [2]. More specifically, a system represented by (2) is said to be "locally observable" if the system state $\boldsymbol{x}(k)$ can be determined from the knowledge of $\boldsymbol{z}(j)$ for $j = [k, k+n-1]$, which can be formulated as

$$\begin{cases} \boldsymbol{z}(k) = \boldsymbol{C}\boldsymbol{x}(k), \\ \boldsymbol{z}(k+1) = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{C}\boldsymbol{f}(\boldsymbol{x}(k)), \\ \vdots \\ \boldsymbol{z}(k+n-1) = \boldsymbol{C}\boldsymbol{A}^{n-1}\boldsymbol{x}(k) \\ \qquad\qquad + \sum\limits_{i=0}^{n-2} \boldsymbol{C}\boldsymbol{A}^{n-i-2}\boldsymbol{f}(\boldsymbol{x}(i)). \end{cases} \tag{3}$$

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Table 1 Simulation results

Error coefficients		Relative observable degrees	Convergence time (s)	Relative error (%)
k_{g13y}	Scheme 1	0.62	52	−0.0050
	Scheme 2	0.47	50	0.0042
	Scheme 3	1.44	36	−0.0021
Δ_{sx}	Scheme 1	0.0016	480	−8.88
	Scheme 2	0.51	320	−3.96
	Scheme 3	0.31	280	−6.03
k_{a0z}	Scheme 1	2.2×10^{-6}	500	−11.21
	Scheme 2	0.00055	270	2.94
	Scheme 3	0.0014	200	−1.58
Scheme 2	k_{g11x}	89.61	96	0.0040
	k_{g11y}	71.73	120	0.0016
	k_{g11z}	96.04	50	0.0009
Scheme 3	Δ_{sx}	0.31	270	−0.94
	Δ_{sy}	0.58	180	0.11
	Δ_{sz}	0.42	280	0.16

Let $\boldsymbol{Q} = [\,\boldsymbol{C}^{\mathrm{T}}\ (\boldsymbol{CA})^{\mathrm{T}}\ (\boldsymbol{CA}^2)^{\mathrm{T}}\ \cdots\ (\boldsymbol{CA}^{n-1})^{\mathrm{T}}\,]^{\mathrm{T}}$, $\boldsymbol{Z} = [\,\boldsymbol{z}(k)^{\mathrm{T}}\ \boldsymbol{z}(k+1)^{\mathrm{T}}\ \cdots\ \boldsymbol{z}(k+n-1)^{\mathrm{T}}\,]^{\mathrm{T}}$, $\boldsymbol{F}(\boldsymbol{x}) = [\,\boldsymbol{0}_{3\times1}^{\mathrm{T}}\ (\boldsymbol{Cf}(\boldsymbol{x}(k)))^{\mathrm{T}}\ \cdots]^{\mathrm{T}}$, and then Eq. (3) can be written as

$$\boldsymbol{Z} = \boldsymbol{Qx}(k) + \boldsymbol{F}(\boldsymbol{x}). \tag{4}$$

For rank($\boldsymbol{Q}$)=n, this equation has a solution in $\mathbb{R}^n$. Based on the definition of local observability, we can know that the local observability matrix of (2) is $\boldsymbol{Q}_j = [\,\boldsymbol{C}_j^{\mathrm{T}}\ (\boldsymbol{C}_j\boldsymbol{A}_j)^{\mathrm{T}}\ (\boldsymbol{C}_j\boldsymbol{A}_j^2)^{\mathrm{T}}\ \cdots\ (\boldsymbol{C}_j\boldsymbol{A}_j^{n-1})^{\mathrm{T}}\,]^{\mathrm{T}}$. We can then use the local observability matrix to compose the SOM of (1) as $\tilde{\boldsymbol{Q}}_s(r) = [\,\boldsymbol{Q}_1^{\mathrm{T}}\ \boldsymbol{Q}_2^{\mathrm{T}}\ \cdots\ \boldsymbol{Q}_r^{\mathrm{T}}\,]^{\mathrm{T}}$. Based on the PWCS theory, if rank($\tilde{\boldsymbol{Q}}_s(r)) = n$, the time-varying nonlinear system of (1) is locally observable.

Analysis of the relative observable degree of the time-varying nonlinear system. Based on the PWCS theory, the transformation results of SOM can be used to represent the observable degree of the system state. The procedures for transforming $\tilde{\boldsymbol{Q}}_s(r)$ to $\tilde{\boldsymbol{U}}_s(r)$ are as follows:

(1) $\tilde{\boldsymbol{Q}}_s(r)$ is transformed into an upper triangular matrix using the Gaussian elimination method. While performing the elementary row transformation on row i $(1 \leqslant i \leqslant n)$ of $\tilde{\boldsymbol{Q}}_s(r)$, if the absolute value of u_{ji} $(i < j \leqslant mnr)$ (which is the element of $\tilde{\boldsymbol{U}}_s(r)$) is maximal in column i, row i and row j must be exchanged before the transformation. (2) If $u_{ii} \neq 0$ $(1 \leqslant i \leqslant n)$, Gaussian elimination is performed on row n to row 1 of the upper triangular matrix to change the elements in row i $(1 \leqslant i \leqslant n)$ to 0 except u_{ii}. (3) If $u_{ii} < 0$ $(1 \leqslant i \leqslant n)$, row i is multiplied by -1 to make the value of u_{ii} positive.

Based on the definition of observability, the relative observable degree of the system state is defined in the following manner. In line i $(1 \leqslant i \leqslant n)$ of $\tilde{\boldsymbol{U}}_s(r)$, if $u_{ii} \neq 0$ and other elements are equal to 0, u_{ii} is defined as the relative observable degree of the system state x_i $(1 \leqslant i \leqslant n)$ in the state vector $\boldsymbol{X} = [\,x_1\ x_2\ \cdots\ x_n\,]^{\mathrm{T}}$. Because the dimensions of system states in $\boldsymbol{X}$ are different, the dimensions of u_{ii} $(1 \leqslant i \leqslant n)$ are also different and we cannot compare the value of u_{ii} directly. Therefore, the defined observable degree is named as the relative observable degree. However, the relative observable degree can be used to compare the observability of different system states as long as they have the same dimension or the same system state in different time segments.

Observability analysis of the continuous self-calibration system. The continuous self-calibration system is a time-varying nonlinear system that has the same form as (1) [5]; therefore the above conclusion can be used to analyze its observability. Considering that the rotation scheme affects the observability of the continuous self-calibration system, the estimation efficiency and the relative observable degree of three different rotation schemes were compared by simulation to verify the rationality of the proposed method.

Scheme 1. (1) Rotate 1°/s around axis Z for π beginning with $\alpha = 0, \gamma = 0$; (2) Rotate 1°/s around axis X for $\pi/2$ beginning with $\alpha = 0, \gamma = \pi$; (3) Rotate 1°/s around axis Z for π beginning with $\alpha = \pi/2, \gamma = \pi$; (4) Rotate 1°/s around axis X for $\pi/2$ beginning with $\alpha = \pi/2, \gamma = 0$.

Scheme 2. (1) Rotate 1°/s around axis Z for $9\pi/4$ beginning with $\alpha = 0, \gamma = 0$; (2) Rotate 1°/s around axis X for $7\pi/3$ beginning with $\alpha = 0, \gamma = \pi/4$; (3) Rotate 1°/s around axis Z for $11\pi/8$ beginning with $\alpha = \pi/3, \gamma = \pi/4$; (4) Rotate 1°/s around axis X for $8\pi/3$ beginning with $\alpha = \pi/3, \gamma = \pi$.

Scheme 3. (1) Rotate 1°/s around axis Z for $9\pi/4$ beginning with $\alpha = 0, \gamma = 0$; (2) Rotate 1°/s around axis X for $3\pi/2$ beginning with $\alpha = 0, \gamma = \pi/4$; (3) Rotate 1°/s around axis Z for $9\pi/4$ beginning with $\alpha = -\pi/2, \gamma = \pi/4$; (4) Rotate 1°/s around axis X for $9\pi/4$ beginning with $\alpha = -\pi/2, \gamma = \pi/2$; (5) Rotate 1°/s around axis Z for $3\pi/2$ beginning with $\alpha = -\pi/4, \gamma = \pi/2$; (6) Rotate 1°/s around axis X for $9\pi/4$ beginning with $\alpha = -\pi/4, \gamma = 0$.

In Table 1, k_{g13y}, Δ_{sx} and k_{a0z} are the error coefficients in the continuous self-calibration system, which are regarded as the system states to be estimated. The estimation efficiency of the Kalman filter includes two fundamental aspects, estimation precision and estimation speed. In this study, the estimation precision and estimation speed are represented by the relative error of the error coefficients and the convergence time associated with the Kalman filter. By comparing the convergence time and relative error with their relative observable degrees in Table 1, it is possible to see that the larger the relative observable degree, the faster the convergence speed and the lower the relative error, which proves that the observability obtained by our proposed method can be used to indicate the estimation efficiency of the Kalman filter.

Conclusion. We proposed a local observability analysis method based on PWCS to analyze the observability of the time-varying nonlinear system. Based on the definition of local observability and the PWCS theory, SOM and observable condition of a type of time-varying nonlinear system were obtained. For the observability of the system state, the relative observable degree of the system state was defined and represented by the transformation results of the SOM. Using this method, the observability of the continuous self-calibration system with three rotation schemes was analyzed, and the results revealed that the larger the relative observable degree, the faster the convergence speed and the lower the relative error, thus proving the rationality of the proposed observability analysis method.

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Supporting information Appendix A. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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