Chinese Annals of History of Science and Technology **2** (2), 072–085 (2018)
doi: 10.3724/SP.J.1461.2018.02072

The Chinese Version of the Pythagorean Theorem and the Case of the Missing Diagram: A Memoir[1]

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Abstract: For a long time, it has been something of a mystery why, in Joseph Needham's third volume of *Science and Civilisation in China*, a translation by Arnold Koslow of a proof of the Pythagorean Theorem was published, together with an inappropriate diagram indicating how that proof proceeded. The story of the origin of the translation, accompanied by the irrelevant diagram, is here recounted by the author of both items, along with the translation accompanied by the appropriate missing diagram for the proof, which was intended for publication but never made it.
Keywords: ancient proof of Chinese Pythagorean theorem, Joseph Needham translation, the missing diagram, *Zhoubi suanjing*

1 Background

In 1959, the third volume of Joseph Needham's magisterial series *Science and Civilisation in China* appeared. The monumental volume 3 was devoted to "Mathematics and the Sciences of the Heavens and the Earth," and among its eight hundred or so pages, there appeared early on a new translation of what Needham and his collaborator, the mathematically astute Dr. Wang Ling 王鈴, called the Pythagorean theorem (Needham 1959, 22–23).

Anyone familiar with the scope of Needham's project will not be surprised to find even within the compass of this third volume both a thorough study of the relation of this mathematical result to subsequent Chinese traditions and mensuration, and the comparative study of geometrical considerations with other

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1 Dedicated with much fondness and indebtedness to the memory of Joseph Needham, Wang Ling 王鈴, and Lu Gwei-Djen 魯桂珍. I am no less indebted to the acuity, patience, and encouragement of Professors Karine Chemla and Geoffrey Lloyd in the telling and the defense of this tale of the missing diagram. Special thanks to Robert Rodriguez for critical technical support. Whatever inaccuracies and infelicities there remain, do so, despite their best advice.

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traditions.

Nevertheless, there are two questions prompted by their new translation: (1) What exactly is the point of the new translation and how is the result it proposes related to prior and continuing practical problems that were involved for the most part in surveying? (2) What is special about the new translation?

We regard the second question as central, and our essay will be devoted to it.

The first question concerns the provenance of that part of the text in the *Zhoubi suanjing* 周髀算經 (Mathematical classic of Zhou's gnomon) that discusses the Pythagorean theorem. Here too Needham has much to say, but the dating, though ancient (probably the third century BCE), is still problematic. The problem of whether this text depends upon an earlier Chinese source or was transmitted ultimately by some Indian, Egyptian, Babylonian, or even a Stone Age source is moot. Nevertheless, Needham's volume 3 still covered much of the then known comparative accounts to be found in Indian, Babylonian, and Greek sources.

We now turn to the first question, the answer to which has a crucial impact on the second question. The point of the Needham translation was not simply the provision of a statement of the Pythagorean theorem, whether in geometrical or algebraic terms; it was to provide not only a statement of the theorem, but to provide a proof of it as well. As Needham expressed it:

> That Chinese geometry was always entirely empirical and non-demonstrative is a statement which (as we have seen, Needham 1956, 94) cannot be made in an unqualified way. The Chinese proof of the Pythagoras theorem was indeed a proof. (Needham 1956, 103)

The translation of the passage in the *Zhoubi suanjing* was deemed to be vital because it presented a mathematical statement and an argument; in short, a mathematical proof. The implication of such a translation was that the result was not a record of practical measurement, and the text was certainly not an incoherent babble. It was a confirmation and endorsement for Needham of the view that at an early age the Chinese had a mastery of geometrical reasoning and argument.

There was, however, a certain mystery that was associated with this proposed translation. The new translation was accompanied by a traditional diagram from a later period (Figure 1), and the translation and this diagram appeared together in widely known papers and books (Swetz and Kao 1977; Van der Waerden 1983; Lam and Shen 1984). Nevertheless, it was apparent that the associated diagram was irrelevant to the text of the argument. On the one hand, one could argue that the diagram concerned the Pythagorean theorem (the prominent triangle in the diagram was a 3-4-5-right triangle); it was the translation that was "off." On the other hand, the translation in the Needham volume seemed to be laying out a mathematical argument, but the diagram did not represent that argument. And there was also the view that the Chinese text itself was incoherent.[3]

2 New translation; wrong diagram

In the remainder of this study, I would like to say something about the circumstances of the translation and the presence of the irrelevant diagram. Joseph Needham and Wang Ling attributed the translation to me. I was at the time a first-year research student at King's College who had come with a three-year fellowship from America to write my doctoral thesis in the philosophy of science with Richard Braithwaite, and to continue my interest in the Chinese history of science with Needham, while continuing my formal training in Chinese with Professor E. Pulleyblank and Dr. Piet van der Loon. After a while, Joseph gave me a copy of the proofs of volume 2 of *Science and Civilisation*, which I returned with some minor comments (Needham 1956). Those were soon to go off to press (and appeared two years later, in 1956). He then gave me the proofs of volume 3. I found problems with their translation, and wrote up a new version and a new diagram to go with it. I showed them to Joseph, saying that I had a different translation of the text. He said that I should discuss it with Wang Ling. We spent a good part of the afternoon going over it, and when we returned, he said to Joseph: "His translation is better than ours." So Joseph took a pen, crossed out the old translation in the proof pages, and typed my version on a slip of paper that he pasted into the proofs, and added the footnote generously giving me the credit. He said that I ought to publish it on my own, as it would be lost in the vastness of volume 3. More importantly, he also said that unfortunately he could not replace the old diagram with the new one. The reason was that Cambridge University Press would charge him for drawing the new diagram, and he had run short of money to pay for it. There was, he said to us, a subvention from the Bollingen Foundation for such changes, but the money that was left in the account would not be sufficient, and he could not cover it on his own. And so the new translation was printed with the old diagram (vol. 3 appeared in 1959). None of us were happy about this compromise, but Wang Ling liked the translation; Joseph was extraordinarily generous in his praise, and it would appear in this magnificent volume. I felt at the time that people could easily reconstruct the diagram from the translation.

3 The missing diagram

We would now like to describe the missing diagram that might help to resolve some of the controversy over the proof. Obviously, for historical reasons, I am not about to

3 Forcefully represented by Cullen 1996, 87.

change the translation that was agreed to by Joseph, Wang Ling, and myself. The point in the present note is to resurrect, if that is the proper word, the diagram that was intended to accompany the translation. The result of this restoration of course has to be judged on its merit, but we shall include some "afterthoughts" about the result.

We begin with some context: the passage in the *Zhoubi* in which the proof was given, which is worth quoting at length:

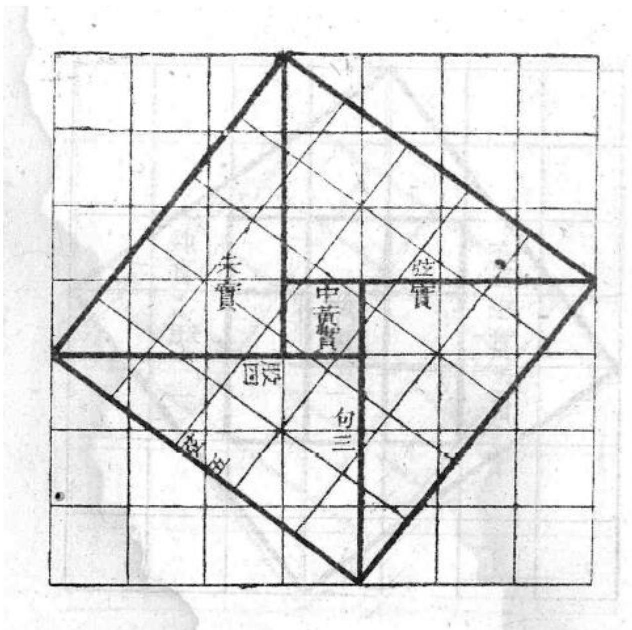


Figure 1

(1) Of old, Chou Kung addressed Shang Kao, saying, 'I have heard that the Grand Prefect (Shang Kao) is versed in the art of numbering. May I venture to enquire how Fu-Hsi anciently established the degrees of the celestial sphere? There are no steps by which one may ascend the heavens, and the earth is not measurable with a foot-rule. I should like to ask you what was the origin of these numbers?'[4]

(2) Shang Kao replied, 'The art of numbering proceeds from the circle (*yuan* 圓) and the square (*fang* 方). The circle is derived from the square and the square from the rectangle (lit. T-square, or carpenter's square; *chü* 矩).[5]

(3) The rectangle originates from (the fact that) 9 × 9 = 81 (i.e. the multiplication table or the properties of numbers as such).[6]

4 "昔者周公問于商高曰：'竊聞乎大夫善數也。請問古者包犧立周天曆度。夫天不可階而升，地不可得尺寸而度，請問數安從出？'"

5 "商高曰：'數之法出於圓方，圓出於方，方出於矩。'"

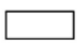
6 "'矩出於九九八十一。'"

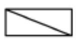
> (4) Thus, let us cut a rectangle (diagonally), and make the width (*kou* 句) 3 (units) wide, and the length (*ku* 股) 4 (units) long. The diagonal (*ching* 徑) between the (two) corners will then be 5 (units) long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a (square) plate. Thus the (four) outer half-rectangles of width 3, length 4, and diagonal 5, together make (*te chhêng* 得成) two rectangles (of area 24); then (when this is subtracted from the square plate of area 49) the remainder (*chang* 長) is of area 25. This (process) is called "piling up the rectangles" (*chi chü* 積矩).[7]
> (5) The methods used by Yü the Great in governing the world were derived from these numbers.'[8] (Needham 1959, 22–23)

There follows a number of further observations not of immediate relevance to the translation of the theorem, followed by the closing with a very enthusiastic summary:

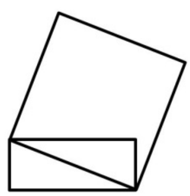
> (11) He who understands the earth is a wise man, and he who understands the heavens is a sage. Knowledge is derived from the straight line. The straight line is derived from the right angle. And the combination of the right angle with numbers is what guides and rules the ten thousand things.[9]
> (12) Chou Kung exclaimed 'Excellent indeed!'[10] (Needham 1959, 23)

Needham, Wang Ling, and I thought that there was a detailed proof conveyed in the crucial paragraph (4), provided it is accompanied by the appropriate geometric diagram. Before we explain the proof, let us begin with a running algorithmic description which ends with the final diagram used in the proof:

(i) Take a rectangle  and with a diagonal, obtain the

(ii) half-rectangle  Then place a square on the diagonal of the half-rectangle,

(iii)



7 “‘故折矩，以為句廣三，股脩四，徑隅五。既方其外，半之一矩。環而共盤，得成三四五。兩矩共長二十有五，是謂積矩。’”

8 “‘故禹之所以治天下者，此數之所生也。’”

9 “‘是故知地者智，知天者聖。智出於句，句出於矩。夫矩之於數，其裁制萬物，惟所為耳。’”

10 “周公曰：‘善哉！’”

The author is grateful to John Moffett, librarian of the Needham Research Institute, for having confirmed that the edition of the *Zhoubi suanjing* that Joseph Needham and Wang Ling used for their initial translation of the passage under discussion here was taken from the 1898 edition of *Gujin suanxue congshu* 古今算學叢書 (Collected works of ancient and modern mathematics), vol. 1:2a–3a, and he generously supplied a copy of the text where the passage reproduced here may be found (Anonymous [n.d.] 1898).

(iv) Then, using the half-rectangle "left outside," surround or pile up the half-rectangles around the figure in (iii). Thus we have in (v) the missing diagram intended to accompany the translation in Needham's volume 3.

(v)

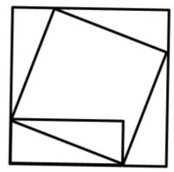


Figure 2

All the half-rectangles are equal, and form an outer square.[11] With (v), the final stage of the algorithm, we have the diagram that supports the proof. This was the diagram which Wang Ling and I discussed, and which Needham approved.

The stages (i) to (v), culminating in the diagram, have of course an intimate connection with the heart of the translation given in (4). That translation describes the construction in stages, the last of which is the final diagram. We think that (4) contains both a description of the course of the construction and the crucial proof (beginning with the penultimate sentence, "Thus the [four] outer rectangles. . .").

The construction is general of course, beginning as it does with an arbitrary rectangle, while the translation in (4) discusses only the special case of the 3-4-5 triangle. Of course, our proposed geometric construction is a conjecture, just as the later obviously irrelevant diagrams in various commentaries were conjecture.[12]

The translation in volume 3 has been described by C. Cullen as one "in which the text is made comprehensible by rewriting it with major additions" (Cullen 1996, 87, footnote 93). The most reasonable answer to such a charge is to look at our translation with some of the emphatic and explanatory parenthetical additions removed.[13] That is:

11 The construction assumes that the half-triangles are congruent and that the outer plate is indeed a square with straight sides. These requirements are presupposed, but they are not proved from other assumptions, nor are they put forward as requirements on what figures can be constructed. Although there may be different routes to the final diagram, that doesn't diminish the result that the diagram of (v) is the basis of an argument for the Pythagorean theorem.

12 It is understandable that Needham might be charged with the historiographic error of projecting the diagram of a later commentary into the opening section of the *Zhoubi*. In her thorough and very penetrating study of the matter, with most of which I find myself in agreement, K. Chemla (2005, 138, footnote 46), as well as C. Cullen, seems to think that this charge is warranted. It is an understandable view given the text that was published, but in light of the episode of the displaced diagram, the charge is not accurate.

13 It should be stressed that we are not at all suggesting (4') as a replacement of the Needham text (4). The latter has to stand as it is. I am suggesting that certain apparent additions in (4) can be dropped and the proof is still evident, provided our proposed geometric diagram is used.

(4') Thus let us cut a rectangle, and make the width (*kou* 句) 3 wide, and the length (*ku* 股) 4 long. The diagonal (*ching* 徑) between the corners will then be 5 long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a plate. Thus the outer half-rectangles of width 3, length 4, and diagonal 5, together make (*te chhêng* 得成) two rectangles of area 24; then the remainder (*chang* 長) is of area 25. This is called "piling up the rectangles" (*chi chü* 積矩).

The remaining parenthetical expressions refer to specific words in the text that are also indicated in the footnotes to the translation published in Needham's volume 3. There have been some serious questions about the way the words 徑 (*ching*) and 長 (*chang*) have been translated as "diagonal" and "remainder," respectively. Though at the time of the composition of the *Zhoubi* there may not have even been a word for triangles, in light of the general meanings for *ching* (by-way, a short-cut, diameter, direct, and straight), it seemed reasonable to translate it in this context as "diagonal." The case for reading *chang* as "remainder" is a bit unusual given the most familiar meanings that it has, but one of the meanings that it can have in some phrases is that of surplus.[14]

Aside from C. Cullen's dissent from the conclusion that the *Zhoubi* contains a proof, there have been other discussions in addition to the one in Needham's volume 3, such as those of K. Chemla and Qu Anjing 曲安京, that also attempt to provide an explicit proof. Unfortunately, those proofs are significantly different from each other. There is some agreement concerning the talk about the rotating of pieces (what we called the circumscribing by half-rectangles) but not uniform agreement of the shape of the pieces rotated. There does not seem to be agreement over the initial starting point of the construction of a diagram. Is it the division of a rectangle with sides 3 and 4 by a diagonal, or the placing together of two squares (one with side 3, the other with side 4)? That perhaps raises an interesting difference over the meaning of the key phrase "*ji ju*" (Needham: "*chi chü*"), the piling up of rectangles. Is it the piling up of the squares (one with side 3, the other with side 4, next to each other); is it the piling up of the rectangles (two of them each with sides 3 and 4), as the present version has it; or is it the piling up (or accumulation) of trisquares, as Cullen would have it?

At this point we can only remind the reader that the intent of the present note is to provide a fuller record of the translation and proper diagram for the version in Needham's volume 3. It seems to me that an evaluation of the features, positive and negative, of each proposal is going to be lengthy and difficult, and beyond the scope of this note, whose aim is just to provide a more complete record.

4 The proof

With the determination of the geometric figure (v), the argument for the special case of

14 As for the example in the phrase *chang wu* 長物, it means surplus or things left over.

a 3-4-5 right triangle falls easily into place. Thus, once the square on the diagonal is determined as 25, the length of the diagonal by root extraction is of course 5.

This is but one case, but any other case of a Pythagorean triple,[15] say (5, 12, 13), could be covered by exactly the same argument given for the case of (3, 4, 5):

> (4*) Thus let us cut a rectangle, and make the width (*kou* 句) 5, and the length (*ku* 股) 12. The diagonal (*ching* 徑) between the corners will then be 13 long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a plate. Thus the outer half-rectangles of width 5, length 12, and diagonal 13, together make (*te chhêng* 得成) two rectangles of area 120; then the remainder (*chang* 長) is of area 169. This is called "piling up the rectangles" (*chi chü* 積矩).

This example, if it had occurred in the text, would have been some evidence that the writer understood the result to be a general one that covered Pythagorean triples. Alas there are no other illustrations of this result that have come down to us as part of the text. The question then is how general is the scope of the proof.

In this connection it is worth noting that the usual understanding of the Pythagorean theorem covers more than just Pythagorean triples of natural numbers. It covers *all* lengths, not just those that are positive integral multiples of some given unit. The best-known case (in the Western tradition) is not a Pythagorean triple. It is the case where the two legs of the right triangle are 1 and the hypotenuse is the square root of 2. Here the Chinese argument, if it were applied, would yield the right answer: the two legs of the right triangle are 1, and the diagonal has a length whose square is 2.

The mathematical result that

> If one has a rectangle whose sides are length m and n, and forms the half-rectangle by drawing a diagonal, then the area of the square on the diagonal is the sum of the area of the completed square $[(n + m)^2]$ minus the area of the four surrounding half-rectangles $[2mn]$.

is enough to cover the all the cases of Pythagorean triples. That coverage alone entitles the *Zhoubi* proof to be counted as a version of the Pythagorean theorem.[16]

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15 A Pythagorean triple is any triple of natural numbers (m, n, p) for which the sum of the squares of n and m equals the square of p. There are infinitely many of them.

16 One has to be careful about what exactly is meant by the Pythagorean theorem, and the kind of generality that it involves. The Greek version involving sums of squares is supposed to hold for any right triangles whose legs can have arbitrary length. They needn't be integral multiples of some unit. Our diagram (v) does not make any assumption about the lengths of the sides of the rectangle. However, the example of the *Zhoubi* uses 3-4-5, and all the special cases of Pythagorean triples are covered by the Chinese proof. We think that the Chinese version is general in that it covers all these cases. That is, all those cases which assume that the initial rectangle has sides that are integral multiples. There is no evidence in the extant text that the Chinese version had the wider generality that extends to those cases where the initial rectangles had sides of any length whatever. However, although our diagram looks like it supports a proof for the wider generality, we hasten to add that there is no indication of that fact in the present text.

Another version of the Pythagorean theorem, the Euclidean one, says that the area of the square of the hypotenuse is the sum of the areas of the sides, where the sides of the triangle can be of any arbitrary length and are not restricted to natural numbers. The two versions are not equivalent. We have so far argued that the *Zhoubi* has a proof of one version of the Pythagorean theorem. Nevertheless, it is not at all evident from the text that at the time of its composition, they knew of the other more general Euclidean version that is more familiar to us.

The translation provides a proof, if we have correctly identified the missing diagram. That is what Needham, Wang Ling, and I had in mind. If any addition to the text was needed to provide cogency, it was not a character here and there that was missing. What was missing was a diagram to which the text could properly refer. Our proposal for the missing diagram is of course speculation. We wish there were a fuller version of this text somewhere with this diagram.

There is another speculation that does not bear directly upon the present translation but that is worth considering. We know that at a later time, Chinese mathematicians were aware of the more general Euclidean version of the Pythagorean theorem according to which the area of the square on the hypotenuse is the sum of the areas of the squares on the sides.[17] *We* know that the two versions are not equivalent. However, most of us know the more general version by deploying elementary algebra (of course we know that $(r + s)^2$ minus $2rs$ equals $(r^2 + s^2)$) where r and s can be any real numbers and not restricted to only natural numbers. The interesting question is whether the Chinese mathematicians concerned with the composition of the *Zhoubi* also knew of the more general version by using a geometrical argument.

It is evident that even at the time of the proof in the *Zhoubi*, a geometric proof of the more general version could have been given which involved nothing more complicated than would have been available at that time. One had only to consider the diagram (Figure 3), in which the large square is the same as the outer square of our diagram (Figure 2, (v)), and the rectangles in the lower left and the upper right corners are the same as the rectangle with which the construction began (Figure 2, (i)).

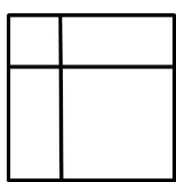

Figure 3

17 This version of the Pythagorean theorem seems to have been known in the later *Jiuzhang suanshu* 九章算術 (Nine chapters on mathematical procedures). This is made evident in the illuminating article of Lam Lay-Yong and Shen Kangsheng 沈康生 (1984) and the penetrating and insightful article of Chemla (2005).

Now we know from the proposed diagram for our translation that the area of the square on the diagonal is the area of this square, less the area of the two rectangles. However, if the two rectangles are removed from the outer plate (Figure 3), then what remains is the area of two squares: one in the upper left-hand corner and the other in the lower right corner which are the areas of the squares on the two sides. This geometrical proof shows that the more general version of the Pythagorean theorem is indeed provable geometrically, in a way that would have been easily understood we believe by ancient Chinese mathematicians.

This very simple geometrical proof of the equivalence of the two forms of the Pythagorean theorem relies heavily on our missing diagram, and I think that this connection supports the claim that the result in the *Zhoubi* is indeed equivalent to a second version of the Pythagorean theorem which relates the square on the diagonal to the sum of the squares on the sides. Nevertheless, there is no such proof of the second version in the *Zhoubi* itself.[18]

If the proposal for the missing diagram is correct,[19] that would settle the problem of the diagram. However, it would not settle the further claim that Needham, Wang Ling, and I endorsed: that the translation together with the diagram constitutes a proof of (a version) of the Pythagorean theorem. This seemed evident without making any apologies or reservations for the way in which the argument was presented. It was something of a pleasant surprise then to find the very same diagrams (Figures 4 and 5) deployed in a contemporary proof of the theorem.

In his recent book, *Mathematics: A Very Short Introduction*, Professor Timothy Gowers, the distinguished Cambridge University mathematician and Fields medalist, wrote of the Pythagorean theorem that "It has several proofs, but one stands out as particularly short and easy to understand. Indeed, it needs little more than the following two diagrams" (Gowers 2002, 48–49). Gowers then continues with the proof, noting that:

> the squares that I have labelled *A*, *B*, and *C* have sides of length *a*, *b*, and *c* respectively, and therefore areas a^2, b^2, and c^2. Since moving the four triangles does not change their area or cause them to overlap, the area of the part of the big square that they do not cover is the same in both diagrams. But on the left this area is $a^2 + b^2$ and on the right is c^2.

18 It is worth noting that Cullen, in his discussion of the much later commentary of Zhao Shuang on the *Zhoubi*, used the very same diagram of our Figure 3 to explicate a passage (Cullen 1996, 213, Figure 24).

19 We, like most commentators on this text, assume that there was a diagram accompanying the text, which is missing. That assumption may be wrong. There might have been an oral tradition that supplemented the text, or there might have been a mechanical device that came with the manual, as it were.

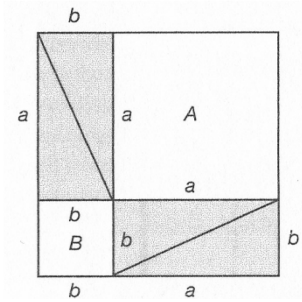

Figure 4

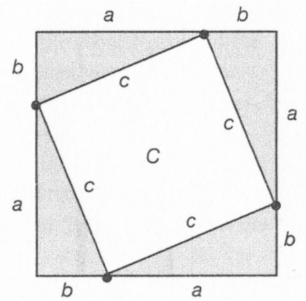

Figure 5

It is patent that this proof not only used the same diagram (Figure 5)[20] as the one we proposed, but there are elements of the proof that are the same: On the one hand there is the reference to the moving of the triangles in the Gowers version, and on the other there is our reference to circumscribing, by half-rectangles, the square on the diagonal in the *Zhoubi.* There is also the similarity of the reference in the Gowers proof to "area of the part of the big square that they [the four triangles] do not cover," and what we believe to be a similar reference to "the remainder" when the *Zhoubi* concludes its proof with this statement:

> Thus the outer half-rectangles of width 3, length 4, and diagonal 5, together make (*te chhêng* 得成) two rectangles of area 24; then the remainder (*chang* 長) is of area 25.

Here we understand "the remainder" in the *Zhoubi* text to refer to the region that remains when the regions of the outer half-rectangles are discounted from the plate.

5 Is it a proof? Some afterthoughts

Needham, Wang Ling, and I certainly thought that the diagram and the translation together constituted a proof. But in such matters, there may be reservations to the translation and to the diagram. We consider a few:

One interesting reaction to this translation is that in the opening sentence of our translation of (4) (and the slightly modified version (4')) the text seems to assume the very result that is supposed to be demonstrated.

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20 In his extremely interesting paper, "On Hypotenuse Diagrams in Ancient China," Qu Anjing 曲安京 also uses the very same diagram as the one proposed here. His translation of the critical passage of the *Zhoubi* differs significantly from ours in that he generates the final diagram by beginning with a diagram (Qu 1997, 198, Fig. 4a) like the one in our Figure 3, rather than with the diagram that cuts a rectangle with a diagonal.

> Thus let us cut a rectangle (diagonally), and make the width (*kou* 勾) 3 (units) wide, and the length (*ku* 股) 4 (units) long. The diagonal (*ching* 徑) between the (two) corners will then be 5 (units) long.

Other translations use "therefore" instead of the initial "thus." Either translation taken literally as recording the conclusion of an argument would be disastrous for any proof. In contrast, however, we understand the opening sentence of (4) as a statement of what is to be shown rather than as an assumption of its proof. The text of (4) then proceeds to construct the appropriate diagram, and the closing remaining sentences provide the argument for the conclusion. The sense is better conveyed by going somewhat against the syntax, to say something like this "Let us thus cut a rectangle. . . ." The sense of the sentence "The diagonal (*ching*) between the (two) corners will then be five (units) long" indicates the result that will be obtained. That is also indicated in the translation by the use of the future tense.

A second reaction calls attention to an ambiguity in our claim that the translation provides a proof of a version of the Pythagorean theorem. The concept of a mathematical proof, it is claimed, involves an index that refers to the standards of proof that are relevant. That, if true, introduces a systematic ambiguity in the notion of proof: If it is an ancient proof that we claim to have translated, then that argument or proof ought to meet the ancient standards. If what we offered is a proof by the standards of contemporary mathematics but not by ancient Chinese standards, then the result should not be presented as an ancient achievement. Of course, if it is a proof by both standards, then there is no clarification or disclaimer needed. I assume that these standards, if they exist, are revealed in the appropriate mathematical practice. Even so, it is extremely difficult, if not impossible, to determine what contemporary mathematical standards there are, let alone those of ancient China. In the contemporary case there are many practices, some of which are in contrast to each other. In the case of ancient practices, the task is made even more difficult by the lack of a decent store of examples of proofs accepted by the writers of the *Zhoubi*. It is a nice idea to think that a mathematical argument meets a specific standard of proof if and only if that argument conforms to some specific mathematical practice of the time, but we think that is not even true. So, in the absence of definite conditions which one can appeal to, the prospect for shedding any light on whether or not the argument in the *Zhoubi* is a proof, by their standards, is dim.

In the light of these difficulties that arise for a concept of mathematical proof which is pegged to standards, a more modest minimal position using the concept of a mathematical argument might fare better. One could claim that there is an argument for a definite conclusion in the *Zhoubi* and an argument for a conclusion in Gowers's book. Second, that the two conclusions are different, but equivalent. Third, that the two arguments for those conclusions are basically the same. One could then say that by a proof of a mathematical statement, one means an argument for that statement. Using this weaker concept of proof, it seems reasonable to me that the writers of the *Zhoubi* had a proof of a version of the Pythagorean theorem. Of course, that is not the end of the matter. Someone could try to argue that the concept of a mathematical argument should in turn also be pegged to appropriate standards. That view would lead to a complete skepticism in the attribution of proofs to those who had them before our time. A persistence along these lines would lead to expunging most of the historical record of past mathematical achievement.

In fact, there is a deep circularity involved. Suppose that in order to determine whether the argument in the *Zhoubi* was a proof of the Pythagorean theorem, one had to certify that the argument met the appropriate standards. If the most natural way of determining that was to certify that the argument was in conformity with the then current mathematical practice concerning proofs, then the prospects are hopeless. Not only because of the lack of data, but for a more general reason: For a natural way of understanding the mathematical practice concerning proofs is to study some of those proofs. But to study the proofs, we would have to use examples that were in conformity with that practice. The inquiry would never get off the ground.

Finally, a word about translations of proofs. Not all proofs have an underlying strategy that is evident to the reader. It's nice when, given the conclusion that is desired, one sees why the proof begins where it does, and the strategy guiding the subsequent steps is evident. In talking about mathematical proofs, it is important to remind ourselves that proofs don't always consist in formal or even informal systematic arguments so structured that they result in conclusions or theorems. Often in mathematical practice there is a setting of a mathematical problem or task, and then there is the solution or solutions that solve the problem. There is no conflict here between proving a mathematical theorem and solving a mathematical problem. The problem or task can be the production of an argument, rather than a construction. One natural way to look at the present passage in the *Zhoubi* is whether presenting a solution to a task or problem: given the sides of a rectangle, to determine the length of the diagonal.

In that case it is natural to start with a rectangle and cut it from corner to corner. The strategy then is to determine the length of the diagonal by determining the area of the square on it (an effective strategy provided that one already knows how to determine square roots). The remaining task in this strategy is the provision of a way to determine the area of the square. We think that the Needham translation starts the proof at the natural place, with a *rectangle* and a diagonal whose length is to be determined. The subsequent text reveals how the general strategy is carried through. Such proofs where the strategy is evident are prized. It would be nice to know that in this case we have such a proof. However, our present aim has not been to evaluate or pick and choose among various proofs and translations that have been proposed. It is the preliminary one of supplying the diagram that was intended to accompany the text of the translation in Needham's volume 3, but never made it.

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