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论 文

Kähler-Finsler 流形到 Riemann 流形的调和映照

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摘要 本文首先给出 Kähler-Finsler 流形到 Riemann 流形光滑映照的 $\bar{\partial}$-能量泛函的第一变分公式, 利用 Jost 和 Yau 关于 Hermite 调和映照的存在定理, 得到 Kähler-Finsler 流形到 Riemann 流形调和映照的存在定理. 其次, 给出 Kähler-Finsler 流形到 Riemann 流形调和映照的第二变分公式, 作为应用, 证明目标 Riemann 流形具有非正复截面曲率时, 调和映照是稳定的. 最后, 讨论 Kähler-Finsler 流形到 Kähler 流形调和映照的第二变分公式, 目标 Kähler 流形的曲率张量是强非正时, 调和映照是稳定的.

关键词 调和映照 $\bar{\partial}$-能量泛函 第二变分 Kähler-Finsler 流形

MSC (2020) 主题分类 53C43, 53C60, 58E20

1 引言

Bochner [2] 首先在 Riemann 几何中引进了调和映照的概念. 如果两个 Riemann 流形之间的光滑映照是能量泛函的一阶变分的临界点, 则称为调和映照. 因此调和映照是相应的 Euler-Lagrange 方程的解. Eells 和 Sampson [8] 开创了 Riemann 几何中调和映照理论. Lichnerowicz [23] 利用 Eells 和 Sampson 的调和映照理论研究了 Kähler 流形之间的全纯映照. Siu 和 Yau [39] 利用调和映照证明了 Frankel 猜想. Siu [36–38] 得到了 Kähler 流形之间调和映照的复解析性. Sampson [33, 34] 进一步发展了 Siu 的理论, 将其推广到 Kähler 流形到 Riemann 流形的调和映照上. Riemann 几何和 Kähler 几何中调和映照的著作可参见文献 [6, 7, 14, 15, 18, 48, 55] 等.

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当出发流形为非 Kähler 流形时, 全纯映照或者反全纯映照未必是调和映照. Jost 和 Yau [17] 定义了 Hermite 流形到 Riemann 流形的 Hermite 调和映照. 假设 M 是一个 Hermite 流形, 在局部坐标系下, 其度量记为 $(h_{\alpha\bar{\beta}}(z))$, N 是一个 Riemann 流形, 在局部坐标系下, 其度量记为 $(g_{ij}(y))$, 并且 Christoffel 记号记为 γ^i_{jk}. 如果 M 到 N 的光滑映照 f 满足方程

$$h^{\bar{\beta}\alpha}\left(\frac{\partial^2 f^i}{\partial z^\alpha \partial \bar{z}^\beta}+\gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial \bar{z}^\beta}\right)=0, \tag{1.1}$$

则 Jost 和 Yau 称其为 Hermite 调和映照. 显然, 当出发流形为 Kähler 流形时, Hermite 调和映照就是调和映照. 在目标 Riemann 流形具有非正截面曲率的条件下, Jost 和 Yau [17] 证明了 Hermite 调和映照总是存在的. 进一步地, 如果 f 满足方程

$$\frac{\partial^2 f^i}{\partial z^\alpha \partial \bar{z}^\beta}+\gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial \bar{z}^\beta}=0, \tag{1.2}$$

则称其为拟调和映照 (参见文献 [16]). 当然, 拟调和映照一定是调和映照. 关于拟调和映照的刚性结果可参见文献 [25,30,31,41–43] 等.

Chern [4] 指出, Finsler 几何就是没有二次型限制的 Riemann 几何. 从这个角度来讲, 复 Finsler 几何就是没有 Hermite 二次型限制的 Hermite 几何. Chern [4] 指出, Finsler 几何可能在复域上是最有用的, 因为每一个有边或无边的复流形都存在 Carathéodory 拟度量和 Kobayashi 拟度量. 在适当 (虽然有点严格) 的条件下, 它们都是 C^2 度量, 最重要的是, 它们自然都是 Finsler 度量. 复 Finsler 几何是极其美丽的. 陈省身 [5] 还指出, 将调和积分理论推广到 Finsler 情形, 这将是微分几何研究的一块新园地, 预料前景无限. 因此在 Finsler 几何中发展调和映照理论是一个很自然的问题. 目前实 Finsler 几何中调和映照理论已经取得巨大进展, 如文献 [11,12,22,26,27,35] 等. Nishikawa [28,29] 研究了 Riemann 曲面到复 Finsler 流形的调和映照. 考虑到目标流形是复 Finsler 流形时能量密度函数的奇性, Han 和 Shen [10] 利用射影切丛上的体积测度, 得到了复 Finsler 流形到 Hermite 流形的光滑映照的 $\bar{\partial}$-能量泛函和 ∂-能量泛函第一变分公式. Han 和 Shen [10] 还得到了 Kähler-Finsler 流形到 Kähler 流形调和映照的存在性定理.

本文研究 Kähler-Finsler 流形到 Riemann 流形的调和映照. 首先, 给出 Kähler-Finsler 流形到 Riemann 流形的光滑映照的 $\bar{\partial}$-能量的第一变分公式.

定理 1.1 令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,g) 是一个紧致 Riemann 流形. 如果 f 是 M 到 N 的光滑映照, 则 $\bar{\partial}$-能量泛函的第一变分公式为

$$\frac{\partial}{\partial t}E''(f_t)\bigg|_{t=0}=-2\int_{\mathbb{P}\tilde{M}}G^{\bar{\beta}\alpha}f^j_{\alpha\,|\,\bar{\beta}}g_{ij}V^i d\mu_{\mathbb{P}\tilde{M}}, \tag{1.3}$$

其中, $f^j_{\alpha\,|\,\bar{\beta}}$ 的定义参见 (3.11), V^i 的定义参见 (3.18).

将 $\bar{\partial}$-能量泛函的临界点定义为调和映照. 进一步地, 如果 f 满足方程

$$G^{\bar{\beta}\alpha}f^j_{\alpha\,|\,\bar{\beta}}=0, \tag{1.4}$$

则称 f 是 Kähler-Finsler 流形 (M,G) 到 Riemann 流形 (N,g) 的强调和映照.

利用 Jost 和 Yau [17] 关于 Hermite 调和映照的存在性定理, 可得如下定理.

定理 1.2 令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,g) 是一个具有非正截面曲率的紧致 Riemann 流形. 假设 $f_0: M\to N$ 是一个连续但不同伦于 M 到 N 的闭测地线的映照, 则存在一个 M 到 N 的同伦于 f_0 的调和映照 f.

第 4 节给出 Kähler-Finsler 流形到 Riemann 流形调和映照的第二变分公式, 作为应用, 还考虑了 Kähler-Finsler 流形到 Riemann 流形的调和映照的稳定性.

定理 1.3　令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,g) 是一个紧致 Riemann 流形. 假设 $f:(M,G)\to(N,g)$ 是一个调和映照, 则 f 的 $\bar{\partial}$-能量泛函的第二变分公式为

$$\begin{aligned}\left.\frac{\partial^2 E''(f_t)}{\partial t\partial\bar{t}}\right|_{t=0} &= \int_{\mathbb{P}\tilde{M}}\langle D'V,D'V\rangle d\mu_{\mathbb{P}\tilde{M}}+\int_{\mathbb{P}\tilde{M}}\langle D''V,D''V\rangle d\mu_{\mathbb{P}\tilde{M}}\\ &\quad -\int_{\mathbb{P}\tilde{M}}G^{\bar{\beta}\alpha}[R(\partial_{\bar{\beta}}f,V,\partial_\alpha f,\overline{V})+R(\partial_\alpha f,V,\partial_{\bar{\beta}}f,\overline{V})]d\mu_{\mathbb{P}\tilde{M}}, \end{aligned}\tag{1.5}$$

其中, V 的定义参见 (3.18), $D'V$ 和 $D''V$ 的定义参见 (4.1).

定理 1.4　令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,g) 是一个紧致 Riemann 流形. 如果 (N,g) 具有非正的复截面曲率, 则调和映照 $f:(M,G)\to(N,g)$ 是稳定的.

最后, 给出 Kähler-Finsler 流形到 Kähler 流形的调和映照的第二变分公式.

定理 1.5　令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,h) 是一个紧致 Kähler 流形. 假设 $f:(M,G)\to(N,h)$ 是一个调和映照, 则 f 的 ∂-能量泛函和 $\bar{\partial}$-能量泛函的第二变分公式分别为

$$\begin{aligned}\left.\frac{\partial^2 E'(f_t)}{\partial t\partial\bar{t}}\right|_{t=0} &= \int_{\mathbb{P}\tilde{M}}[\langle D'V',D'V'\rangle+\langle D'\overline{V''},D'\overline{V''}\rangle]d\mu_{\mathbb{P}\tilde{M}}\\ &\quad -\int_{\mathbb{P}\tilde{M}}G^{\bar{\beta}\alpha}[R(\partial_\alpha f,\overline{\partial_\beta f},V',\overline{V'})+R(\partial_\alpha f,\overline{\partial_\beta f},\overline{V''},V'')]d\mu_{\mathbb{P}\tilde{M}}\\ &\quad +2\mathrm{Re}\int_{\mathbb{P}\tilde{M}}G^{\bar{\beta}\alpha}R(V',\overline{\partial_\beta f},\overline{V''},\overline{\partial_{\bar{\alpha}} f})d\mu_{\mathbb{P}\tilde{M}}, \end{aligned}\tag{1.6}$$

$$\begin{aligned}\left.\frac{\partial^2 E''(f_t)}{\partial t\partial\bar{t}}\right|_{t=0} &= \int_{\mathbb{P}\tilde{M}}[\langle D''V',D''V'\rangle+\langle D''\overline{V''},D''\overline{V''}\rangle]d\mu_{\mathbb{P}\tilde{M}}\\ &\quad -\int_{\mathbb{P}\tilde{M}}G^{\bar{\beta}\alpha}[R(\partial_{\bar{\beta}} f,\overline{\partial_{\bar{\alpha}} f},V',\overline{V'})+R(\partial_{\bar{\beta}} f,\overline{\partial_{\bar{\alpha}} f},\overline{V''},V'')]d\mu_{\mathbb{P}\tilde{M}}\\ &\quad +2\mathrm{Re}\int_{\mathbb{P}\tilde{M}}G^{\bar{\beta}\alpha}R(\partial_{\bar{\beta}} f,V'',\partial_\alpha f,\overline{V'})d\mu_{\mathbb{P}\tilde{M}}, \end{aligned}\tag{1.7}$$

其中, V' 和 V'' 的定义参见 (5.10), $D'V'$、$D'\overline{V''}$、$D''V'$ 和 $D''\overline{V''}$ 的定义参见 (5.11).

再利用 Han 和 Shen [10] 的结果和上述定理, 得到如下定理.

定理 1.6　令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,h) 是一个紧致 Kähler 流形. 如果 (N,h) 的曲率张量是强非正的 (或者 $(N,\mathrm{Re}h)$ 具有非正的复截面曲率), 则调和映照 $f:(M,G)\to(N,h)$ 是稳定的. 特别地, 令 $\mathscr{D}$ 为某一类典型域 D 的紧商, 则 (M,G) 到 $(\mathscr{D},ds^2_{\mathscr{D}})$ 的调和映照是稳定的, 其中 $ds^2_{\mathscr{D}}$ 是由 D 上的不变 Kähler 度量诱导的 $\mathscr{D}$ 上的 Kähler 度量.

2　预备知识

假设 M 是一个 m 维复流形, 其局部全纯坐标记为 $z=(z^1,z^2,\ldots,z^m)$. 记 $\pi:T^{1,0}M\to M$ 为 M 的全纯切丛, 则局部全纯坐标系下, $T^{1,0}M$ 中任意一个全纯向量可以写作 $v=v^\alpha\frac{\partial}{\partial z^\alpha}$. 因此, $(z,v)=(z^1,\ldots,z^m,v^1,\ldots,v^m)$ 是 $T^{1,0}M$ 的局部全纯坐标. 记 $\tilde{M}=T^{1,0}M\backslash\{0\}$ 为孔状全纯切丛.

定义 2.1 [1]　假设 M 是一个 m 维复流形, $G:T^{1,0}M\to[0,+\infty)$ 是一个连续函数. 如果以下 3 条成立:

(a) G 在 $\tilde{M}$ 上光滑;

(b) 对于所有 $(z,v)\in\tilde{M}$, 都有 $G(z,v)>0$;

(c) $G(z,\zeta v)=|\zeta|^2G(z,v)$ 对所有 $(z,v)\in T^{1,0}M$ 和 $\zeta\in\mathbb{C}$ 成立,

则称 G 为复流形 M 上的一个复 Finsler 度量. 称赋予复 Finsler 度量的复流形为复 Finsler 流形.

为了简化记号, 将 $\overline{z^\beta}$ 和 $\overline{v^\beta}$ 分别记作 $\bar{z}^\beta$ 和 $\overline{v}^\beta$, G 关于 v 的偏导数记为

$$G_\alpha=\dot{\partial}_\alpha G=\frac{\partial G}{\partial v^\alpha},\quad G_{\bar\alpha}=\dot{\partial}_{\bar\alpha}G=\frac{\partial G}{\partial\overline{v}^\alpha},\quad G_{\alpha\bar\beta}=\dot{\partial}_\alpha\dot{\partial}_{\bar\beta}G=\frac{\partial^2G}{\partial v^\alpha\partial\overline{v}^\beta},$$

G 关于 z 的偏导数记作分号后加指标的形式, 如

$$G_{;\alpha}=\partial_\alpha G=\frac{\partial G}{\partial z^\alpha},\quad G_{;\alpha\bar\beta}=\partial_\alpha\partial_{\bar\beta}G=\frac{\partial^2G}{\partial z^\alpha\partial\bar{z}^\beta},\quad G_{\alpha;\bar\beta}=\dot{\partial}_\alpha\partial_{\bar\beta}G=\frac{\partial^2G}{\partial\bar{z}^\beta\partial v^\alpha}.$$

定义 2.2 [1] 假设 G 是一个复 Finsler 度量. 如果 G 的 Levi 矩阵 $(G_{\alpha\bar\beta})$ 在 $\tilde{M}$ 上恒正定, 则称 G 是强拟凸的. 称赋予强拟凸的复 Finsler 度量的复流形为强拟凸的复 Finsler 流形.

假设 (M,G) 是一个强拟凸的复 Finsler 流形. 记

$$\varGamma^\alpha_{;\mu}=G^{\bar\lambda\alpha}G_{\bar\lambda;\mu}\tag{2.1}$$

为 (M,G) 的复非线性联络系数, 其中 $(G^{\bar\lambda\alpha})=(G_{\alpha\bar\lambda})^{-1}$. 令

$$\delta_\mu=\partial_\mu-\varGamma^\alpha_{;\mu}\dot{\partial}_\alpha,\quad \delta v^\alpha=dv^\alpha+\varGamma^\alpha_{;\beta}dz^\beta,\tag{2.2}$$

则 $\tilde{M}$ 的全纯切丛 $T^{1,0}\tilde{M}$ 可以分解成由 $\{\delta_\mu\}$ 张成的水平丛 $\mathcal{H}$ 和由 $\{\dot{\partial}_\alpha\}$ 张成的垂直丛 $\mathcal{V}$ 的直和. $\{\delta_\mu,\dot{\partial}_\alpha\}$ 的对偶标架为 $\{dz^\mu,\delta v^\alpha\}$.

由 G 的强拟凸性, 可以在垂直丛 $\mathcal{V}$ 上定义一个 Hermite 内积 $\langle\cdot,\cdot\rangle$. Chern-Finsler 联络 $D:\mathcal{X}(\mathcal{V})\to\mathcal{X}(T^*_{\mathbb{C}}\tilde{M}\otimes\mathcal{V})$ 是唯一好的与 Hermite 内积 $\langle\cdot,\cdot\rangle$ 相容的复垂直联络. 可以将 Hermite 内积 $\langle\cdot,\cdot\rangle$ 和 Chern-Finsler 联络 D 通过 Chern-Finsler 联络的复水平映射延拓到水平丛 $\mathcal{H}$ 上. Chern-Finsler 联络是 Kobayashi [19] 首先引进的, 联络 1-形式为

$$\omega^\alpha_\beta=G^{\bar\lambda\alpha}\partial G_{\beta\bar\lambda}=\varGamma^\alpha_{\beta;\mu}dz^\mu+\varGamma^\alpha_{\beta\gamma}\delta v^\gamma,\tag{2.3}$$

其中,

$$\varGamma^\alpha_{\beta;\mu}=G^{\bar\lambda\alpha}\delta_\mu(G_{\beta\bar\lambda}),\quad \varGamma^\alpha_{\beta\gamma}=G^{\bar\lambda\alpha}G_{\beta\bar\lambda\gamma}.\tag{2.4}$$

Rund [32] 首先引进了强拟凸复 Finsler 度量 G 的复 Rund 联络, 复 Rund 联络和 Chern-Finsler 联络具有相同非线性联络系数和复水平丛. 记 $\hat{D}:\mathcal{X}(\mathcal{V})\to\mathcal{X}(T^*_{\mathbb{C}}\tilde{M}\otimes\mathcal{V})$ 为复 Rund 联络. 复 Rund 联络的联络 1-形式为

$$\hat\omega^\alpha_\beta=\varGamma^\alpha_{\beta;\mu}dz^\mu.\tag{2.5}$$

因此, $\hat\omega^\alpha_\beta$ 就是 Chern-Finsler 联络的联络 1-形式 ω^α_β 的水平部分. 文献 [40] 已经证明了复 Rund 联络 $\hat{D}$ 是一个好的复垂直联络并且和 Hermite 内积 $\langle\cdot,\cdot\rangle$ 具有水平相容性, 即

$$H\langle Z,W\rangle=\langle\nabla_HZ,W\rangle+\langle Z,\nabla_{\overline{H}}W\rangle,\quad\forall\,H\in\mathcal{V},\quad\forall\,Z,W\in\mathcal{V}.$$

定义 2.3 [1]　令 G 是复流形 M 上的一个强拟凸的复 Finsler 度量. 若 $\varGamma^{\alpha}_{\beta;\mu} = \varGamma^{\alpha}_{\mu;\beta}$, 则称 G 是强 Kähler 的; 若 $(\varGamma^{\alpha}_{\beta;\mu} - \varGamma^{\alpha}_{\mu;\beta})v^{\beta} = 0$, 则称 G 是 Kähler 的; 若 $G_{\alpha}(\varGamma^{\alpha}_{\beta;\mu} - \varGamma^{\alpha}_{\mu;\beta})v^{\beta} = 0$, 则称 G 是弱 Kähler 的.

Chen 和 Shen [3] 证明了 Kähler-Finsler 度量实际上是强 Kähler 的. 因此, Kähler 度量在复 Finsler 几何中的推广只有 Kähler-Finsler 度量和弱 Kähler-Finsler 度量这两类. 文献 [44–47, 58] 构造了许多 Kähler-Finsler 度量.

由于 G 的强拟凸性, 所以在射影切丛 $\mathbb{P}\tilde{M} = \tilde{M}/\mathbb{C}^*$ 上可以诱导一个 Hermite 度量

$$\tilde{G} = G_{\alpha\bar{\beta}} dz^{\alpha} \otimes d\bar{z}^{\beta} + (\log G)_{\alpha\bar{\beta}} \delta v^{\alpha} \otimes \delta\bar{v}^{\beta}.$$

记 $\omega_{\mathcal{V}} = \sqrt{-1}(\log G)_{\alpha\bar{\beta}} \delta v^{\alpha} \wedge \delta\bar{v}^{\beta}$ 和 $\omega_{\mathcal{H}} = \sqrt{-1} G_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta}$, 则 $\mathbb{P}\tilde{M}$ 上有一个不变的体积形式 (参见文献 [59])

$$d\mu_{\mathbb{P}\tilde{M}} = \frac{\omega_{\mathcal{V}}^{m-1}}{(m-1)!} \wedge \frac{\omega_{\mathcal{H}}^{m}}{m!}. \tag{2.6}$$

为了简单起见, 记 $d\sigma$ 为 $d\mu_{\mathbb{P}\tilde{M}}$ 的垂直部分, 则

$$d\sigma = \frac{\omega_{\mathcal{V}}^{m-1}}{(m-1)!}. \tag{2.7}$$

所以

$$d\mu_{\mathbb{P}\tilde{M}} = \det(G_{\alpha\bar{\beta}}) d\sigma \wedge d\nu, \tag{2.8}$$

其中 $d\nu = (\sqrt{-1}\sum_{\alpha=1}^{m} dz^{\alpha} \wedge d\bar{z}^{\alpha})^{m}$. 令

$$d\mu_M = \left(\int_{\mathbb{P}_z\tilde{M}} \det(G_{\alpha\bar{\beta}}) d\sigma \right) d\nu. \tag{2.9}$$

文献 [60] 已经证明了 $d\mu_M$ 是 M 上的一个不变体积形式.

令 $X = X^{\alpha}\delta_{\alpha}$ 为 $\mathbb{P}\tilde{M}$ 上一个水平向量场. 定义 X 关于 $d\mu_{\mathbb{P}\tilde{M}}$ 的散度为

$$(\mathrm{div} X) d\mu_{\mathbb{P}\tilde{M}} = \mathcal{L}_X d\mu_{\mathbb{P}\tilde{M}}, \tag{2.10}$$

其中 $\mathcal{L}_X$ 是沿 X 关于体积形式 $d\mu_{\mathbb{P}\tilde{M}}$ 的 Lie 导数. 记 $\mathfrak{i}(X)$ 为内乘算子. 由文献 [57] 可知

$$(\mathrm{div} X) d\mu_{\mathbb{P}\tilde{M}} = d(\mathfrak{i}(X) d\mu_{\mathbb{P}\tilde{M}}), \tag{2.11}$$

$$\mathrm{div} X = \delta_{\alpha}(X^{\alpha}) + X^{\alpha} \sum_{\gamma=1}^{m} \varGamma^{\gamma}_{\gamma;\alpha}. \tag{2.12}$$

对于任意的 $f \in C^{\infty}(\mathbb{P}\tilde{M})$, 都有

$$\mathrm{div}(fX) = X(f) + f \mathrm{div} X. \tag{2.13}$$

3 第一变分公式

令 N 是一个 n 维实流形, 其局部坐标记为 $y=(y^1,y^2,\dots,y^n)$. 假设 $g=g_{ij}(y)dy^idy^j$ 是 N 上的一个光滑 Riemann 度量. 记 ∇ 为 (N,g) 上的 Levi-Civita 联络, 其联络系数记为

$$\gamma_{ij}^k=\frac{1}{2}g^{kl}\bigg(\frac{\partial g_{il}}{\partial y^j}+\frac{\partial g_{jl}}{\partial y^i}-\frac{\partial g_{ij}}{\partial y^l}\bigg). \tag{3.1}$$

Levi-Civita 联络 ∇ 的曲率算子 $\mathcal{R}$ 定义为

$$\mathcal{R}(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z.$$

令 $\mathcal{R}(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k}=R_{kij}^l\frac{\partial}{\partial y^l}$, 则

$$R_{kij}^l=\frac{\partial\gamma_{kj}^l}{\partial y^i}-\frac{\partial\gamma_{ki}^l}{\partial y^j}+\gamma_{kj}^h\gamma_{hi}^l-\gamma_{ki}^h\gamma_{hj}^l. \tag{3.2}$$

Levi-Civita 联络 ∇ 的 Riemann 曲率张量记为

$$R_{ijkl}=g_{ih}R_{jkl}^h.$$

在局部坐标系下,

$$R_{ijkl}=\frac{1}{2}\bigg(\frac{\partial^2g_{il}}{\partial y^j\partial y^k}+\frac{\partial^2g_{jk}}{\partial y^i\partial y^l}-\frac{\partial^2g_{ik}}{\partial y^j\partial y^l}-\frac{\partial^2g_{jl}}{\partial y^i\partial y^k}\bigg)+g^{st}([jk,s][il,t]-[jl,s][ik,t]), \tag{3.3}$$

其中

$$[jk,s]=\frac{1}{2}\bigg(\frac{\partial g_{js}}{\partial y^k}+\frac{\partial g_{ks}}{\partial y^j}-\frac{\partial g_{jk}}{\partial y^s}\bigg). \tag{3.4}$$

Riemann 曲率算子 R 定义为

$$R(X,Y,Z,W)=g(X,\mathcal{R}(Z,W)Y).$$

定义由两个线性无关的切向量 $X=X^i\frac{\partial}{\partial y^i}$ 和 $Y=Y^i\frac{\partial}{\partial y^i}\in TN$ 张成的 2-平面 $\Pi(X,Y)$ 上的截面曲率为

$$K(X,Y)=\frac{R_{ijkl}X^iY^jX^kY^l}{(g_{ij}g_{kl}-g_{il}g_{jk})X^iX^jY^kY^l}. \tag{3.5}$$

本文将度量 g 对称、复线性延拓到复化的切丛 $T_{\mathbb{C}}N$ 上, 并将曲率算子 R 复线性化.

定义 3.1 [13] 对于任意的非零复切向量 $Z=Z^i\frac{\partial}{\partial y^i}$ 和 $W=W^j\frac{\partial}{\partial y^j}$, 若

$$R(Z,W,\overline{Z},\overline{W})=R_{ijkl}Z^iW^j\overline{Z}^k\overline{W}^l\leqslant 0\quad(\text{或者 }<0), \tag{3.6}$$

则称 (N,g) 具有非正 (或者负) 的复截面曲率.

非正复截面曲率对应于 Sampson [34] 所定义的 Hermite 负曲率. 令 (M,G) 是一个紧致强拟凸复 Finsler 流形, (N,g) 是一个 Riemann 流形. 假设 f 是 M 到 N 的光滑映照, 在局部坐标系下可将 f 表示为

$$y^i=f^i(z^1,\dots,z^m,\bar{z}^1,\dots,\bar{z}^m),\quad i=1,\dots,n.$$

存在一个诱导的取值在 $f^{-1}T_{\mathbb{C}}N$ 上的 1-形式 df. 而 df 可分解成 (1, 0)-形式 ∂f 和 (0, 1)-形式 $\bar{\partial}f$, 其中,

$$\partial f : T^{1,0}M \to T_{\mathbb{C}}N, \quad \bar{\partial} f : T^{0,1}M \to T_{\mathbb{C}}N.$$

局部地,

$$\partial f = f^i_\alpha dz^\alpha \otimes \frac{\partial}{\partial y^i}, \quad \bar{\partial} f = f^i_{\bar{\beta}} d\bar{z}^\beta \otimes \frac{\partial}{\partial y^i},$$

其中 $f^i_\alpha = \frac{\partial f^i}{\partial z^\alpha}$, $f^i_{\bar{\beta}} = \frac{\partial f^i}{\partial \bar{z}^\beta}$. 显然, $\bar{\partial} f = \overline{\partial f}$.

由于基本张量 $G_{\alpha\bar{\beta}}$ 对切方向 v 的齐次性, 所以自然可以将 f 看作射影切丛 $\mathbb{P}\tilde{M}$ 到 N 的映照, 因此 df 可以看作 $T_{\mathbb{C}}\mathbb{P}\tilde{M}$ 到 $T_{\mathbb{C}}N$ 的映照, 此时将 $f^{-1}T_{\mathbb{C}}N$ 看作 $\mathbb{P}\tilde{M}$ 上的拉回丛, 将 ∂f^i 和 $\bar{\partial} f^i$ 看作 $\mathbb{P}\tilde{M}$ 上的水平形式. 利用复 Rund 联络 $\hat{D}$ 和拉回丛 $f^{-1}T_{\mathbb{C}}N$ 上的 Levi-Civita 联络 $\nabla^{f^{-1}}$, 定义

$$(\tilde{D}(df))(X, Y) = \nabla^{f^{-1}}_X df(Y) - df(\hat{D}_Y X), \quad X, Y \in \mathcal{X}(\mathcal{H}_{\mathbb{C}}). \tag{3.7}$$

直接计算可得

$$\tilde{D}(\partial f) = (f^i_{\alpha\,|\,\beta} dz^\alpha \otimes dz^\beta + f^i_{\bar{\beta}\,|\,\alpha} d\bar{z}^\beta \otimes dz^\alpha) \otimes \frac{\partial}{\partial y^i}, \tag{3.8}$$

$$\tilde{D}(\bar{\partial} f) = (f^i_{\alpha\,|\,\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + f^i_{\bar{\alpha}\,|\,\bar{\beta}} d\bar{z}^\alpha \otimes d\bar{z}^\beta) \otimes \frac{\partial}{\partial y^i}, \tag{3.9}$$

其中,

$$f^i_{\alpha\,|\,\beta} = f^i_{\alpha\beta} - \Gamma^\mu_{\alpha;\beta} f^i_\mu + \gamma^i_{jk} f^j_\beta f^k_\alpha, \tag{3.10}$$

$$f^i_{\alpha\,|\,\bar{\beta}} = f^i_{\bar{\beta}\,|\,\alpha} = f^i_{\alpha\bar{\beta}} + \gamma^i_{jk} f^j_\alpha f^k_{\bar{\beta}}, \tag{3.11}$$

$$f^i_{\bar{\alpha}\,|\,\bar{\beta}} = f^i_{\bar{\alpha}\bar{\beta}} - \Gamma^{\bar{\mu}}_{\bar{\alpha};\bar{\beta}} f^i_{\bar{\mu}} + \gamma^i_{jk} f^j_{\bar{\beta}} f^k_{\bar{\alpha}}. \tag{3.12}$$

显然, $f^i_{\bar{\alpha}\,|\,\bar{\beta}} = \overline{f^i_{\alpha\,|\,\beta}}$. 因此, $\tilde{D}(\partial f) = \overline{\tilde{D}(\bar{\partial} f)}$. 当 G 是 Kähler-Finsler 度量时, $f^i_{\alpha\,|\,\beta} = f^i_{\beta\,|\,\alpha}$.

定义映照 f 的 $\bar{\partial}$-能量密度为

$$e''(f) = \langle \bar{\partial} f, \bar{\partial} f \rangle = G^{\bar{\beta}\alpha}(z, v) f^j_{\bar{\beta}} \overline{f^i_{\bar{\alpha}}} g_{ij}(f(z)), \tag{3.13}$$

这里 $\langle \cdot, \cdot \rangle$ 表示 $\mathbb{P}\tilde{M}$ 上取值于 $f^{-1}T_{\mathbb{C}}N$ 的复水平外微分形式的逐点内积. 则映照 f 的 $\bar{\partial}$-能量泛函定义为

$$E''(f) = \int_{\mathbb{P}\tilde{M}} e''(f) d\mu_{\mathbb{P}\tilde{M}}. \tag{3.14}$$

类似地, 也可以定义映照 f 的 ∂-能量密度

$$e'(f) = \langle \partial f, \partial f \rangle = G^{\bar{\beta}\alpha}(z, v) f^i_\alpha \overline{f^j_\beta} g_{ij}(f(z))$$

和 ∂-能量泛函

$$E'(f) = \int_{\mathbb{P}\tilde{M}} e'(f) d\mu_{\mathbb{P}\tilde{M}}.$$

则映照 f 的 d-能量密度和 d-能量泛函分别记作 $e(f) = e'(f) + e''(f)$ 和 $E(f) = E'(f) + E''(f)$.

由于 (N, g) 是一个 Riemann 流形, 于是有

$$E'(f) = E''(f) = \frac{1}{2} E(f). \tag{3.15}$$

因此 $K(f)=E'(f)-E''(f)$ 是同伦不变量.

考虑 $f=f_0$ 的一个光滑变分, 即一族光滑映照

$$f_t: M\to N,\quad t\in\Delta_\varepsilon=\{t\in\mathbb{C}:|t|<\varepsilon\}.$$

则 f_t 的 $\bar\partial$-能量泛函变分为

$$\frac{\partial}{\partial t}E''(f_t)\Big|_{t=0}=\int_{\mathbb{P}\tilde M}\frac{\partial}{\partial t}e''(f_t)\Big|_{t=0}d\mu_{\mathbb{P}\tilde M}. \tag{3.16}$$

将 $\{f_t\}$ 在拉回丛 $f_t^{-1}T_{\mathbb{C}}N$ 上诱导的向量场记为

$$V_t=\partial f_t\left(\frac{\partial}{\partial t}\right)=\frac{\partial f_t^i}{\partial t}\frac{\partial}{\partial y^i}. \tag{3.17}$$

当 $t=0$ 时, 记

$$V=V_0=\left(\frac{\partial f_t^i}{\partial t}\Big|_{t=0}\right)\frac{\partial}{\partial y^i}=:V^i\frac{\partial}{\partial y^i}. \tag{3.18}$$

从现在起, 在无特殊说明的情形下总是假设 G 是紧致复流形 M 上的 Kähler-Finsler 度量.

定理 1.1 的证明 注意到 $\{V^i\}$ 与 $\{v^\alpha\}$ 无关, 直接计算可得

$$\frac{\partial e''(f_t)}{\partial t}\Big|_{t=0}=G^{\bar\beta\alpha}\left[\delta_\alpha(V^i)f^j_{\bar\beta}g_{ij}+f^i_\alpha\delta_{\bar\beta}(V^j)g_{ij}+f^i_\alpha f^j_{\bar\beta}\frac{\partial g_{ij}}{\partial y^k}V^k\right].$$

记 $X=G^{\bar\beta\alpha}f^j_{\bar\beta}g_{ij}V^i\delta_\alpha$, 则

$$G^{\bar\beta\alpha}\delta_\alpha(V^i)f^j_{\bar\beta}g_{ij}=\delta_\alpha(X^\alpha)+X^\alpha\sum_{\gamma=1}^m\varGamma^\gamma_{\alpha;\gamma}-G^{\bar\beta\alpha}\left(f^j_{\alpha\bar\beta}g_{ij}+f^j_{\bar\beta}\frac{\partial g_{ij}}{\partial y^k}f^k_\alpha\right)V^i.$$

由 (2.11) 和 (2.12) 可得

$$\int_{\mathbb{P}\tilde M}G^{\bar\beta\alpha}\delta_\alpha(V^i)f^j_{\bar\beta}g_{ij}d\mu_{\mathbb{P}\tilde M}=-\int_{\mathbb{P}\tilde M}G^{\bar\beta\alpha}\left(f^j_{\alpha\bar\beta}+f^k_\alpha f^l_{\bar\beta}\frac{\partial g_{pl}}{\partial y^k}g^{pj}\right)g_{ij}V^id\mu_{\mathbb{P}\tilde M}. \tag{3.19}$$

利用上述方法, 则有

$$\int_{\mathbb{P}\tilde M}G^{\bar\beta\alpha}f^j_\alpha\delta_{\bar\beta}(V^i)g_{ij}d\mu_{\mathbb{P}\tilde M}=-\int_{\mathbb{P}\tilde M}G^{\bar\beta\alpha}\left(f^j_{\alpha\bar\beta}+f^k_\alpha f^l_{\bar\beta}\frac{\partial g_{kp}}{\partial y^l}g^{jp}\right)g_{ij}V^id\mu_{\mathbb{P}\tilde M}. \tag{3.20}$$

另外, 因为

$$G^{\bar\beta\alpha}f^k_\alpha f^l_{\bar\beta}\frac{\partial g_{kl}}{\partial y^i}V^i=G^{\bar\beta\alpha}f^k_\alpha f^l_{\bar\beta}\frac{\partial g_{kl}}{\partial y^p}g^{jp}g_{ij}V^i,$$

所以

$$\begin{aligned}\frac{\partial}{\partial t}E''(f_t)\Big|_{t=0}&=\int_{\mathbb{P}\tilde M}G^{\bar\beta\alpha}\left[-2f^j_{\alpha\bar\beta}+f^k_\alpha f^l_{\bar\beta}\left(\frac{\partial g_{kl}}{\partial y^p}-\frac{\partial g_{pl}}{\partial y^k}-\frac{\partial g_{kp}}{\partial y^l}\right)g^{pj}\right]g_{ij}V^id\mu_{\mathbb{P}\tilde M}\\&=-2\int_{\mathbb{P}\tilde M}G^{\bar\beta\alpha}(f^j_{\alpha\bar\beta}+\gamma^j_{kl}f^k_\alpha f^l_{\bar\beta})g_{ij}V^id\mu_{\mathbb{P}\tilde M}.\end{aligned}$$

结论得证. □

将 Chern-Finsler 联络的水平挠率张量记作 $S^{\alpha}_{\beta\gamma}=\frac{1}{2}(\varGamma^{\alpha}_{\beta;\gamma}-\varGamma^{\alpha}_{\gamma;\beta})$, 将指标 α 和 γ 缩并可以得到如下形式的张量[20]

$$S_{\beta}=\sum_{\gamma=1}^{m}S^{\gamma}_{\beta\gamma}=\sum_{\gamma=1}^{m}\frac{1}{2}(\varGamma^{\gamma}_{\beta;\gamma}-\varGamma^{\gamma}_{\gamma;\beta}).$$

为了简单起见, 将 $\overline{S_{\beta}}$ 简记为 $S_{\bar{\beta}}$.

注 3.1 当 (M,G) 是一般的强拟凸复 Finsler 流形时, 利用定理 1.1 的证明过程, 可得

$$\frac{\partial}{\partial t}E''(f_t)\bigg|_{t=0}=-2\int_{\mathbb{P}\tilde{M}}G^{\bar{\beta}\alpha}(f^{j}_{\alpha\,|\,\bar{\beta}}-S_{\alpha}f^{j}_{\bar{\beta}}-S_{\bar{\beta}}f^{j}_{\alpha})g_{ij}V^{i}d\mu_{\mathbb{P}\tilde{M}}. \tag{3.21}$$

因此, 定理 1.1 中将 Kähler-Finsler 流形的条件改成平衡复 Finsler 流形[21] (等价于 $S_{\alpha}=0$), (1.3) 仍然成立.

定义 f 的张力场为

$$\tau(f)=G^{\bar{\beta}\alpha}f^{i}_{\alpha\,|\,\bar{\beta}}\frac{\partial}{\partial y^{i}}, \tag{3.22}$$

这是拉回丛 $f^{-1}T_{\mathbb{C}}N$ 上的一个截影.

定义 3.2 假设 $f:(M,G)\to(N,g)$ 是一个光滑映照. 如果 f 是 $\bar{\partial}$-能量泛函的第一变分的临界点, 则称 f 是调和映照; 如果 $\tau(f)$ 消失, 则称 f 是强调和映照.

由 (1.3) 可以看出, f 是调和映照当且仅当

$$\int_{\mathbb{P}\tilde{M}}g(\tau(f),V)d\mu_{\mathbb{P}\tilde{M}}=0. \tag{3.23}$$

接下来考虑 Kähler-Finsler 流形到 Riemann 流形的调和映照的存在性.

引理 3.1[10] 假设 (M,G) 是一个紧致强拟凸的复 Finsler 流形, 则对于任意的函数 $f:\mathbb{P}\tilde{M}\to\mathbb{R}$ 都有

$$\int_{\mathbb{P}\tilde{M}}fd\mu_{\mathbb{P}\tilde{M}}=\int_{M}d\nu\int_{\mathbb{P}_z\tilde{M}}f\det(G_{\kappa\bar{\nu}})d\sigma.$$

引理 3.2[10] 假设 (M,G) 是一个强拟凸的复 Finsler 流形. 令

$$\gamma^{\bar{\beta}\alpha}(z):=\frac{\int_{\mathbb{P}_z\tilde{M}}G^{\bar{\beta}\alpha}(z,v)\det(G_{\kappa\bar{\nu}})d\sigma}{\int_{\mathbb{P}_z\tilde{M}}\det(G_{\kappa\bar{\nu}})d\sigma},$$

则 $ds^2=\gamma_{\alpha\bar{\beta}}dz^{\alpha}d\bar{z}^{\beta}$ 是 M 上的一个 Hermite 度量, 其中 $(\gamma_{\alpha\bar{\beta}})=(\gamma^{\bar{\beta}\alpha})^{-1}$.

由引理 3.1 和 3.2 可知, f 是调和映照当且仅当

$$\int_{M}\gamma^{\bar{\beta}\alpha}f^{j}_{\alpha\,|\,\bar{\beta}}g_{ij}V^{i}d\mu_{M}=0. \tag{3.24}$$

由于向量场 V 是任意的, 所以 f 是调和映照当且仅当

$$\gamma^{\bar{\beta}\alpha}f^{j}_{\alpha\,|\,\bar{\beta}}=0. \tag{3.25}$$

因此, f 是 (M,G) 到 (N,g) 的调和映照当且仅当 f 是 (M,ds^2) 到 (N,g) 的 Hermite 调和映照. 利用 Jost 和 Yau[17] 关于 Hermite 调和映照的存在性结果, 可得到定理 1.2 和如下结论.

定理 3.1 令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,g) 是一个具有非正截面曲率的紧致 Riemann 流形. 假设 $f_0:M\to N$ 是一个光滑映照并且 Euler 示性类 $\mathscr{E}(f_0^{*}TN)\neq0$, 则存在一个 M 到 N 的同伦于 f_0 的调和映照 f.

注 3.2 Jost 和 Yau[17] 关于 Hermite 调和映照存在性的其他结果, 对紧致 Kähler-Finsler 流形 (M,G) 到具有非正截面曲率的紧致 Riemann 流形 (N,g) 的调和映照同样成立. 这里不再一一列举.

4 第二变分公式

本节给出 Kähler-Finsler 流形到 Riemann 流形的调和映照的 $\bar{\partial}$-能量泛函的第二变分公式. 如果 $\bar{\partial}$-能量泛函的第二变分公式是非负的, 则称调和映照是稳定的.

为了方便起见, 引进记号: $\partial_a f = f_a^i \frac{\partial}{\partial y^i}$, $D_a V^i = \frac{\partial V^i}{\partial z^a} + \gamma_{jk}^i V^j f_a^k$, 其中 $a = \alpha, \bar{\alpha}$, $1 \leqslant \alpha \leqslant m$,

$$D'V = (D_\alpha V^i) dz^\alpha \otimes \frac{\partial}{\partial y^i}, \quad D''V = (D_{\bar{\beta}} V^i) d\bar{z}^\beta \otimes \frac{\partial}{\partial y^i}. \tag{4.1}$$

可以直接验证 $D'V + D''V = \nabla^{f^{-1}} V$.

定理 1.3 的证明 为了简单起见, 在不引起混淆的情形下将 f_t 简写为 f. 由 (3.13) 可得

$$\frac{\partial}{\partial t} e''(f_t) = G^{\bar{\beta}\alpha} \bigg(\frac{\partial f_\alpha^i}{\partial t} f_{\bar{\beta}}^j g_{ij} + f_\alpha^i \frac{\partial f_{\bar{\beta}}^j}{\partial t} g_{ij} + f_\alpha^i f_{\bar{\beta}}^j \frac{\partial g_{ij}}{\partial y^k} \frac{\partial f^k}{\partial t} \bigg).$$

将上式对 $\bar{t}$ 求偏导, 则有

$$\begin{aligned} \frac{\partial^2 e''(f)}{\partial t \partial \bar{t}} = {} & G^{\bar{\beta}\alpha} \bigg(\frac{\partial f_\alpha^i}{\partial t} \frac{\partial f_{\bar{\beta}}^j}{\partial \bar{t}} g_{ij} + \frac{\partial f_\alpha^i}{\partial \bar{t}} \frac{\partial f_{\bar{\beta}}^j}{\partial t} g_{ij} + f_\alpha^i f_{\bar{\beta}}^j \frac{\partial^2 g_{ij}}{\partial y^k \partial y^l} \frac{\partial f^k}{\partial t} \frac{\partial f^l}{\partial \bar{t}} + f_\alpha^i f_{\bar{\beta}}^j \frac{\partial g_{ij}}{\partial y^k} \frac{\partial^2 f^k}{\partial t \partial \bar{t}} \bigg) \\ & + 2\mathrm{Re} \bigg(G^{\bar{\beta}\alpha} \frac{\partial^2 f_\alpha^i}{\partial t \partial \bar{t}} f_{\bar{\beta}}^j g_{ij} + G^{\bar{\beta}\alpha} \frac{\partial f_\alpha^i}{\partial \bar{t}} \frac{\partial f^k}{\partial t} f_{\bar{\beta}}^j \frac{\partial g_{ij}}{\partial y^k} + G^{\bar{\beta}\alpha} \frac{\partial f_\alpha^i}{\partial t} \frac{\partial f^k}{\partial \bar{t}} f_{\bar{\beta}}^j \frac{\partial g_{ij}}{\partial y^k} \bigg). \end{aligned} \tag{4.2}$$

因为

$$\begin{aligned} & \frac{\partial g_{ij}}{\partial y^k} = g_{hj} \gamma_{ik}^h + g_{ih} \gamma_{jk}^h, \\ & \frac{\partial^2 g_{ij}}{\partial y^k \partial y^l} = (g_{rj} \gamma_{hl}^r + g_{hr} \gamma_{jl}^r) \gamma_{ik}^h + g_{hj} \frac{\partial \gamma_{ik}^h}{\partial y^l} + (g_{rh} \gamma_{il}^r + g_{ir} \gamma_{hl}^r) \gamma_{jk}^h + g_{ih} \frac{\partial \gamma_{jk}^h}{\partial y^l}, \end{aligned}$$

所以 (4.2) 变为

$$\begin{aligned} \frac{\partial^2 e''(f)}{\partial t \partial \bar{t}} = {} & G^{\bar{\beta}\alpha} \bigg(\frac{\partial f_\alpha^i}{\partial t} \frac{\partial f_{\bar{\beta}}^j}{\partial \bar{t}} + \frac{\partial f_\alpha^i}{\partial \bar{t}} \frac{\partial f_{\bar{\beta}}^j}{\partial t} \bigg) g_{ij} + G^{\bar{\beta}\alpha} f_\alpha^i f_{\bar{\beta}}^j g_{hr} (\gamma_{jl}^r \gamma_{ik}^h + \gamma_{il}^r \gamma_{jk}^h) \frac{\partial f^k}{\partial t} \frac{\partial f^l}{\partial \bar{t}} \\ & + G^{\bar{\beta}\alpha} f_\alpha^i f_{\bar{\beta}}^j \bigg(g_{rj} \gamma_{hl}^r \gamma_{ik}^h + g_{hj} \frac{\partial \gamma_{ik}^h}{\partial y^l} + g_{ir} \gamma_{hl}^r \gamma_{jk}^h + g_{ih} \frac{\partial \gamma_{jk}^h}{\partial y^l} \bigg) \frac{\partial f^k}{\partial t} \frac{\partial f^l}{\partial \bar{t}} \\ & + G^{\bar{\beta}\alpha} f_\alpha^i f_{\bar{\beta}}^j (g_{hj} \gamma_{ik}^h + g_{ih} \gamma_{jk}^h) \frac{\partial^2 f^k}{\partial t \partial \bar{t}} \\ & + 2\mathrm{Re} \bigg[G^{\bar{\beta}\alpha} \bigg(\frac{\partial^2 f_\alpha^i}{\partial t \partial \bar{t}} + \frac{\partial f_\alpha^r}{\partial \bar{t}} \frac{\partial f^h}{\partial t} \gamma_{rh}^i + \frac{\partial f_\alpha^h}{\partial t} \frac{\partial f^r}{\partial \bar{t}} \gamma_{hr}^i \bigg) f_{\bar{\beta}}^j g_{ij} \bigg] \\ & + 2\mathrm{Re} \bigg[G^{\bar{\beta}\alpha} \bigg(\frac{\partial f_\alpha^i}{\partial \bar{t}} \frac{\partial f^k}{\partial t} + \frac{\partial f_\alpha^i}{\partial t} \frac{\partial f^k}{\partial \bar{t}} \bigg) \gamma_{lk}^j f_{\bar{\beta}}^l g_{ij} \bigg]. \end{aligned} \tag{4.3}$$

类似于文献 [39], 引进下列记号:

$$\frac{D}{\partial \bar{t}} \frac{\partial f^i}{\partial t} = \frac{\partial^2 f^i}{\partial t \partial \bar{t}} + \gamma_{hr}^i \frac{\partial f^h}{\partial t} \frac{\partial f^r}{\partial \bar{t}}, \quad \frac{D}{\partial t}(f_\alpha^i) = \frac{\partial f_\alpha^i}{\partial t} + \gamma_{hk}^i \frac{\partial f^h}{\partial t} f_\alpha^k, \quad \frac{D}{\partial t}(f_{\bar{\beta}}^i) = \frac{\partial f_{\bar{\beta}}^i}{\partial t} + \gamma_{hl}^i \frac{\partial f^h}{\partial t} f_{\bar{\beta}}^l.$$

令 $X = G^{\bar{\beta}\alpha} \frac{D}{\partial \bar{t}} \frac{\partial f^i}{\partial t} f_{\bar{\beta}}^j g_{ij} \delta_\alpha$, 则

$$\mathrm{div}(X) = G^{\bar{\beta}\alpha} \bigg[\frac{\partial^2 f_\alpha^i}{\partial t \partial \bar{t}} + \frac{\partial \gamma_{hr}^i}{\partial y^k} f_\alpha^k \frac{\partial f^h}{\partial t} \frac{\partial f^r}{\partial \bar{t}} + \gamma_{hr}^i \bigg(\frac{\partial f_\alpha^h}{\partial t} \frac{\partial f^r}{\partial \bar{t}} + \frac{\partial f^h}{\partial t} \frac{\partial f_\alpha^r}{\partial \bar{t}} \bigg) \bigg] f_{\bar{\beta}}^j g_{ij}$$

$$+G^{\bar{\beta}\alpha}\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}f^j_{\alpha\,|\,\bar{\beta}}g_{ij}+G^{\bar{\beta}\alpha}\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}g_{hj}\gamma^h_{ik}f^j_{\bar{\beta}}f^k_{\alpha}. \tag{4.4}$$

因为

$$2\mathrm{Re}\bigg[G^{\bar{\beta}\alpha}g_{ij}\frac{\partial\gamma^i_{hr}}{\partial y^k}f^k_{\alpha}f^j_{\bar{\beta}}\frac{\partial f^h}{\partial t}\frac{\partial f^r}{\partial\bar{t}}\bigg]=G^{\bar{\beta}\alpha}\bigg(g_{ij}\frac{\partial\gamma^i_{hr}}{\partial y^k}+g_{ik}\frac{\partial\gamma^i_{hr}}{\partial y^j}\bigg)f^k_{\alpha}f^j_{\bar{\beta}}\frac{\partial f^h}{\partial t}\frac{\partial f^r}{\partial\bar{t}},$$

所以

$$\begin{aligned}&G^{\bar{\beta}\alpha}f^i_{\alpha}f^j_{\bar{\beta}}\bigg(g_{rj}\gamma^r_{hl}\gamma^h_{ik}+g_{hj}\frac{\partial\gamma^h_{ik}}{\partial y^l}+g_{ir}\gamma^r_{hl}\gamma^h_{jk}+g_{ih}\frac{\partial\gamma^h_{jk}}{\partial y^l}\bigg)\frac{\partial f^k}{\partial t}\frac{\partial f^l}{\partial\bar{t}}-2\mathrm{Re}\bigg[G^{\bar{\beta}\alpha}g_{ij}\frac{\partial\gamma^i_{hr}}{\partial y^k}f^k_{\alpha}f^j_{\bar{\beta}}\frac{\partial f^h}{\partial t}\frac{\partial f^r}{\partial\bar{t}}\bigg]\\&\quad=G^{\bar{\beta}\alpha}[(g_{jh}\gamma^p_{kl}\gamma^h_{pi}+g_{ih}\gamma^p_{kl}\gamma^h_{pj})-(R_{jkil}+R_{ikjl})]f^i_{\alpha}f^j_{\bar{\beta}}\frac{\partial f^k}{\partial t}\frac{\partial f^l}{\partial\bar{t}}.\end{aligned}$$

注意到

$$\begin{aligned}&2\mathrm{Re}\bigg[G^{\bar{\beta}\alpha}\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}g_{hj}\gamma^h_{ik}f^j_{\bar{\beta}}f^k_{\alpha}\bigg]\\&\quad=G^{\bar{\beta}\alpha}f^i_{\alpha}f^j_{\bar{\beta}}\bigg[(g_{hj}\gamma^h_{ik}+g_{ih}\gamma^h_{jk})\frac{\partial^2 f^k}{\partial t\partial\bar{t}}+(g_{ih}\gamma^p_{kl}\gamma^h_{pj}+g_{jh}\gamma^p_{kl}\gamma^h_{pi})\frac{\partial f^k}{\partial t}\frac{\partial f^l}{\partial\bar{t}}\bigg],\end{aligned}$$

并且

$$\begin{aligned}&\bigg|\frac{D}{\partial t}(f^i_{\alpha})dz^{\alpha}\otimes\frac{\partial}{\partial y^i}\bigg|^2+\bigg|\frac{D}{\partial t}(f^i_{\bar{\beta}})d\bar{z}^{\beta}\otimes\frac{\partial}{\partial y^i}\bigg|^2\\&\quad=G^{\bar{\beta}\alpha}\bigg(\frac{\partial f^i_{\alpha}}{\partial t}\frac{\partial f^j_{\bar{\beta}}}{\partial\bar{t}}+\frac{\partial f^i_{\alpha}}{\partial\bar{t}}\frac{\partial f^j_{\bar{\beta}}}{\partial t}\bigg)g_{ij}+G^{\bar{\beta}\alpha}\frac{\partial f^h}{\partial t}\frac{\partial f^r}{\partial\bar{t}}(\gamma^i_{hk}\gamma^j_{rl}+\gamma^i_{rk}\gamma^j_{hl})f^k_{\alpha}f^l_{\bar{\beta}}g_{ij}\\&\qquad+2\mathrm{Re}\bigg[G^{\bar{\beta}\alpha}\bigg(\frac{\partial f^i_{\alpha}}{\partial t}\frac{\partial f^r}{\partial\bar{t}}\gamma^j_{rl}+\frac{\partial f^i_{\alpha}}{\partial\bar{t}}\frac{\partial f^h}{\partial t}\gamma^j_{hl}\bigg)f^l_{\bar{\beta}}g_{ij}\bigg],\end{aligned}$$

则有

$$\begin{aligned}\frac{\partial^2 e''(f)}{\partial t\partial\bar{t}}-2\mathrm{Rediv}(X)=&\bigg|\frac{D}{\partial t}(f^i_{\alpha})dz^{\alpha}\otimes\frac{\partial}{\partial y^i}\bigg|^2+\bigg|\frac{D}{\partial t}(f^i_{\bar{\beta}})d\bar{z}^{\beta}\otimes\frac{\partial}{\partial y^i}\bigg|^2\\&-R_{ikjl}G^{\bar{\beta}\alpha}(f^i_{\alpha}f^j_{\bar{\beta}}+f^i_{\bar{\beta}}f^j_{\alpha})\frac{\partial f^k}{\partial t}\frac{\partial f^l}{\partial\bar{t}}-2\mathrm{Re}\bigg[\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}G^{\bar{\beta}\alpha}f^j_{\alpha\,|\,\bar{\beta}}g_{ij}\bigg].\end{aligned}\tag{4.5}$$

另外,

$$(D_{\alpha}V^i)dz^{\alpha}+(D_{\bar{\beta}}V^i)d\bar{z}^{\beta}=\bigg(\frac{D}{\partial t}(f^i_{\alpha})dz^{\alpha}+\frac{D}{\partial t}(f^i_{\bar{\beta}})d\bar{z}^{\beta}\bigg)\bigg|_{t=0},$$

$$\int_{\mathbb{P}\tilde{M}}\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}G^{\bar{\beta}\alpha}f^j_{\alpha\,|\,\bar{\beta}}g_{ij}d\mu_{\mathbb{P}\tilde{M}}=\int_M\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}g_{ij}f^j_{\alpha\,|\,\bar{\beta}}\gamma^{\bar{\beta}\alpha}d\mu_M,$$

而且 f 调和当且仅当

$$\gamma^{\bar{\beta}\alpha}f^j_{\alpha\,|\,\bar{\beta}}=0.$$

将 (4.5) 等号两边同时积分, 则可以得到第二变分 (1.5). □

接下来, 考虑 Kähler-Finsler 流形到 Riemann 流形的调和映照的稳定性.

定理 1.4 的证明　由于 $(\gamma^{\bar{\beta}\alpha})$ 是正定的, 因此存在非奇异的方阵 $(P^{\beta\alpha})$ 使得 $\gamma^{\bar{\beta}\alpha}=\sum_{\lambda=1}^m\overline{P^{\lambda\beta}}P^{\lambda\alpha}$. 由引理 3.1 和 3.2, 可以将 (1.5) 中的曲率项写成

$$\int_{\mathbb{P}\tilde{M}}R_{ikjl}G^{\bar{\beta}\alpha}(f^i_{\alpha}f^j_{\bar{\beta}}+f^i_{\bar{\beta}}f^j_{\alpha})V^k\overline{V}^l d\mu_{\mathbb{P}\tilde{M}}$$

$$= \int_M R_{ikjl}\gamma^{\bar{\beta}\alpha}(f^i_\alpha f^j_{\bar{\beta}} + f^i_{\bar{\beta}} f^j_\alpha)V^k\overline{V}^l d\mu_M$$
$$= \sum_{\lambda=1}^m \int_M R_{ikjl}[(P^{\lambda\alpha} f^i_\alpha)(\overline{P^{\lambda\beta} f^j_\beta}) + (\overline{P^{\lambda\beta} f^i_\beta})(P^{\lambda\alpha} f^j_\alpha)]V^k\overline{V}^l d\mu_M.$$

因此当 (N,g) 具有非正的复截面曲率时, 则有 $\frac{\partial^2 E''(f_t)}{\partial t\partial \bar{t}}|_{t=0} \geqslant 0$, 调和映照 f 是稳定的. □

已知当 Riemann 流形 (N,g) 具有常负截面曲率时, 则 (N,g) 具有负的复截面曲率. 如果 Riemann 流形 (N,g) 的截面曲率位于 -1 和 $-\delta$ $(0<\delta<1)$ 之间, 则称 (N,g) 为负 δ-夹型流形. Yau 和 Zheng[56] 证明了当 $\delta \geqslant \frac{1}{4}$ 时负 δ-夹型流形具有非正的复截面曲率. 因此, 根据定理 1.4 可得到以下结论.

命题 4.1 令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,g) 是一个具有负常截面曲率的紧致 Riemann 流形, 则 (M,G) 到 (N,g) 的调和映照是稳定的.

命题 4.2 令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,g) 是一个 $\delta \geqslant \frac{1}{4}$ 的负 δ-夹型紧致 Riemann 流形, 则 (M,G) 到 (N,g) 的调和映照是稳定的.

5 目标流形为 Kähler 流形的情形

若 N 是一个 n 维复流形, 其局部全纯坐标系记为 $w=(w^1,w^2,\dots,w^n)$, 其中 $w^i = y^i + \sqrt{-1}y^{n+i}$. 假设 $h = h_{i\bar{j}}(w)dw^i d\bar{w}^j$ 是 N 上的一个 Hermite 度量. 将 h 的 Chern 联络 ∇ 系数记为 $\Gamma^i_{jk} = h^{\bar{l}i}\partial_k(h_{j\bar{l}})$, 全纯曲率张量记为

$$R_{i\bar{j}k\bar{l}} = -\partial_k\partial_{\bar{l}}(h_{i\bar{j}}) + h^{\bar{s}t}\partial_k(h_{i\bar{s}})\partial_{\bar{l}}(h_{t\bar{j}}).$$

如果 f 是 M 到 N 的光滑映照, 其 $\bar{\partial}$-能量密度和 $\bar{\partial}$-能量泛函分别定义为

$$e''(f) = \langle \bar{\partial} f, \bar{\partial} f\rangle = G^{\bar{\beta}\alpha}(z,v) f^i_{\bar{\beta}} f^{\bar{j}}_\alpha h_{i\bar{j}}(f(z)), \tag{5.1}$$
$$E''(f) = \int_{\mathbb{P}\tilde{M}} e''(f) d\mu_{\mathbb{P}\tilde{M}}, \tag{5.2}$$

其中 $\langle\cdot,\cdot\rangle$ 表示 $\mathbb{P}\tilde{M}$ 上取值于 $f^{-1}T^{1,0}N$ 的复水平外微分形式的逐点内积. 同样也可以定义映照 f 的 ∂-能量密度和 ∂-能量泛函:

$$e'(f) = \langle \partial f, \partial f\rangle = G^{\bar{\beta}\alpha}(z,v) f^i_\alpha f^{\bar{j}}_{\bar{\beta}} h_{i\bar{j}}(f(z)), \tag{5.3}$$
$$E'(f) = \int_{\mathbb{P}\tilde{M}} e'(f) d\mu_{\mathbb{P}\tilde{M}}. \tag{5.4}$$

如果记 $F = (\mathrm{Re} f, \mathrm{Im} f)$, 则 F 是复流形 M 到实流形 $N_{\mathbb{R}}$ 的光滑映照. 通过直接计算得

$$e''(F) = e'(F) = \frac{1}{2}(e'(f) + e''(f)). \tag{5.5}$$

在不引起混淆的情形下, 仍记

$$f^i_{\alpha\,|\,\bar{\beta}} = f^i_{\alpha\bar{\beta}} + \Gamma^i_{jk} f^j_\alpha f^k_{\bar{\beta}}. \tag{5.6}$$

Han 和 Shen[10] 将复 Finsler 流形到 Hermite 流形的调和映照定义为 $\bar{\partial}$-能量泛函的第一变分的临界点, 并且证明了当 (M,G) 是一个紧致 Kähler-Finsler 流形且 (N,h) 是 Kähler 流形时, 映照 f 的第一变分满足

$$\frac{\partial E'(f_t)}{\partial t} = \frac{\partial E''(f_t)}{\partial t} = -\int_{\mathbb{P}\tilde{M}} G^{\bar{\beta}\alpha}\left[(f^i_t)_{\alpha\,|\,\bar{\beta}}\frac{\partial \bar{f}^j_t}{\partial t} + \overline{(f^j_t)_{\beta\,|\,\bar{\alpha}}}\frac{\partial f^i_t}{\partial t}\right] h_{i\bar{j}} d\mu_{\mathbb{P}\tilde{M}}. \tag{5.7}$$

Han 和 Shen [10] 利用引理 3.1 和 3.2, 得到 f 是 Kähler-Finsler 流形 (M,G) 到 Kähler 流形 (N,h) 的调和映照当且仅当

$$\gamma^{\bar{\beta}\alpha} f^i_{\alpha\,|\,\bar{\beta}} = 0. \tag{5.8}$$

在局部坐标系下, 也可以直接验证 (5.7) 中积分项的被积部分等于

$$2G^{\bar{\beta}\alpha}(F^{\mathrm{i}}_t)_{\alpha\,|\,\bar{\beta}} \frac{\partial F^{\mathrm{j}}_t}{\partial t} g_{\mathrm{ij}},$$

其中, $F_t = (\mathrm{Re} f_t, \mathrm{Im} f_t)$, 指标 i 和 j 均是 1 到 $2n$, $g = \mathrm{Re} h$ 是相应的 Riemann 度量. 于是, 可得到以下结论.

命题 5.1 令 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,h) 是一个紧致 Kähler 流形. 假设 f 是 M 到 N 的光滑映照, 则 f 是 (M,G) 到 (N,h) 的调和映照当且仅当 $F = (\mathrm{Re} f, \mathrm{Im} f)$ 是 (M,G) 到 Riemann 流形 $(N, \mathrm{Re} h)$ 的调和映照.

接下来考虑 Kähler-Finsler 流形到 Kähler 流形的调和映照的能量泛函的二阶变分. 假设 (M,G) 是一个紧致 Kähler-Finsler 流形, (N,h) 是一个紧致 Kähler 流形.

由变分 f_t 诱导的拉回丛 $f_t^{-1}T_{\mathbb{C}}N$ 上的向量场记为

$$V_t = df_t\left(\frac{\partial}{\partial t}\right) = \partial f_t\left(\frac{\partial}{\partial t}\right) + \partial \bar{f}_t\left(\frac{\partial}{\partial t}\right) =: V'_t + V''_t, \tag{5.9}$$

其中,

$$V'_t = \frac{\partial f^i_t}{\partial t}\frac{\partial}{\partial w^i}, \quad V''_t = \frac{\partial \bar{f}^i_t}{\partial t}\frac{\partial}{\partial \bar{w}^i}.$$

当 $t=0$ 时, 记

$$V' = V'_0 = \left(\frac{\partial f^i_t}{\partial t}\bigg|_{t=0}\right)\frac{\partial}{\partial w^i}, \quad V'' = V''_0 = \left(\frac{\partial \bar{f}^i_t}{\partial t}\bigg|_{t=0}\right)\frac{\partial}{\partial \bar{w}^i}. \tag{5.10}$$

今后记 $V'^{\bar{i}} = \overline{V'^i}$, $V''^i = \overline{V''^{\bar{i}}}$, 并且引进下列记号: $\partial_a f = f^i_\alpha \frac{\partial}{\partial w^i}$, $\partial_a \bar{f} = f^{\bar{i}}_a \frac{\partial}{\partial \bar{w}^i}$,

$$D_a U^i = \frac{\partial U^i}{\partial z^a} + \Gamma^i_{jk} f^j_a U^k, \quad U^i = V'^i,\, V''^i,$$
$$D_a U^{\bar{i}} = \frac{\partial U^{\bar{i}}}{\partial z^a} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}} f^{\bar{j}}_a U^{\bar{k}}, \quad U^{\bar{i}} = V'^{\bar{i}},\, V''^{\bar{i}},$$

其中, $a = \alpha, \bar{\alpha}$, $1 \leqslant \alpha \leqslant m$, $1 \leqslant i \leqslant n$. 显然, $D_\alpha U^i = \overline{D_{\bar{\alpha}} U^{\bar{i}}}$, $D_{\bar{\alpha}} U^i = \overline{D_\alpha U^{\bar{i}}}$. 当 $U = V', \overline{V''}$ 时, 记

$$D'U = (D_\alpha U^i) dz^\alpha \otimes \frac{\partial}{\partial w^i}, \quad D''U = (D_{\bar{\beta}} U^i) d\bar{z}^\beta \otimes \frac{\partial}{\partial w^i}. \tag{5.11}$$

可以直接验证 $D'V' + D''V' = \nabla^{f^{-1}} V'$ 和 $D'\overline{V''} + D''\overline{V''} = \nabla^{f^{-1}} \overline{V''}$.

定理 1.5 的证明 这里只证明 (1.6), 而 (1.7) 可采用相同的方法得到. 为了简单起见, 在不引起混淆的情形下将 f_t 简写为 f. 直接计算可得

$$\frac{\partial}{\partial t} e'(f_t) = G^{\bar{\beta}\alpha}\left(\frac{\partial f^i_\alpha}{\partial t} f^{\bar{j}}_{\bar{\beta}} + f^i_\alpha \frac{\partial f^{\bar{j}}_{\bar{\beta}}}{\partial t}\right) h_{i\bar{j}} + G^{\bar{\beta}\alpha} f^i_\alpha f^{\bar{j}}_{\bar{\beta}} \left(\frac{\partial h_{i\bar{j}}}{\partial w^k}\frac{\partial f^k}{\partial t} + \frac{\partial h_{i\bar{j}}}{\partial \bar{w}^l}\frac{\partial f^{\bar{l}}}{\partial t}\right).$$

将上式对 $\bar{t}$ 求偏导, 则有

$$\frac{\partial^2 e'(f_t)}{\partial t \partial \bar{t}} = G^{\bar{\beta}\alpha}\left[\left(\frac{\partial f^i_\alpha}{\partial t}\frac{\partial f^{\bar{j}}_{\bar{\beta}}}{\partial \bar{t}} + \frac{\partial f^i_\alpha}{\partial \bar{t}}\frac{\partial f^{\bar{j}}_{\bar{\beta}}}{\partial t}\right) h_{i\bar{j}} + f^i_\alpha f^{\bar{j}}_{\bar{\beta}} \frac{\partial^2 h_{i\bar{j}}}{\partial w^k \partial \bar{w}^l}\left(\frac{\partial f^k}{\partial t}\frac{\partial f^{\bar{l}}}{\partial \bar{t}} + \frac{\partial f^k}{\partial \bar{t}}\frac{\partial f^{\bar{l}}}{\partial t}\right)\right]$$

$$
\begin{aligned}
&+2\mathrm{Re}\bigg(G^{\bar{\beta}\alpha}f^i_\alpha f^{\bar{j}}_{\bar{\beta}}\frac{\partial h_{i\bar{j}}}{\partial w^k}\frac{\partial^2 f^k}{\partial t\partial\bar{t}}+G^{\bar{\beta}\alpha}\frac{\partial^2 f^i_\alpha}{\partial t\partial\bar{t}}f^{\bar{j}}_{\bar{\beta}}h_{i\bar{j}}+G^{\bar{\beta}\alpha}f^i_\alpha f^{\bar{j}}_{\bar{\beta}}\frac{\partial^2 h_{i\bar{j}}}{\partial w^k\partial w^l}\frac{\partial f^k}{\partial t}\frac{\partial f^l}{\partial\bar{t}}\bigg)\\
&+2\mathrm{Re}\bigg[G^{\bar{\beta}\alpha}\bigg(\frac{\partial f^i_\alpha}{\partial\bar{t}}f^{\bar{j}}_{\bar{\beta}}+f^i_\alpha\frac{\partial f^{\bar{j}}_{\bar{\beta}}}{\partial\bar{t}}\bigg)\bigg(\frac{\partial h_{i\bar{j}}}{\partial w^k}\frac{\partial f^k}{\partial t}+\frac{\partial h_{i\bar{j}}}{\partial\bar{w}^l}\frac{\partial f^{\bar{l}}}{\partial t}\bigg)\bigg].
\end{aligned}\tag{5.12}
$$

引进下列记号:

$$
\begin{aligned}
\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}&=\frac{\partial^2 f^i}{\partial t\partial\bar{t}}+\Gamma^i_{hr}\frac{\partial f^h}{\partial t}\frac{\partial f^r}{\partial\bar{t}},\\
\frac{D}{\partial t}(f^i_\alpha)&=\frac{\partial f^i_\alpha}{\partial t}+\Gamma^i_{hk}\frac{\partial f^h}{\partial t}f^k_\alpha,\\
\frac{D}{\partial\bar{t}}(f^i_\alpha)&=\frac{\partial f^i_\alpha}{\partial\bar{t}}+\Gamma^i_{hl}\frac{\partial f^h}{\partial\bar{t}}f^l_\alpha.
\end{aligned}
$$

令 $X=G^{\bar{\beta}\alpha}\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}f^{\bar{j}}_{\bar{\beta}}h_{i\bar{j}}\delta_\alpha$, 则

$$
\begin{aligned}
\mathrm{div}(X)=G^{\bar{\beta}\alpha}&\bigg[\frac{\partial^2 f^i_\alpha}{\partial t\partial\bar{t}}+\bigg(\frac{\partial\Gamma^i_{hr}}{\partial w^k}f^k_\alpha+\frac{\partial\Gamma^i_{hr}}{\partial\bar{w}^l}f^{\bar{l}}_\alpha\bigg)\frac{\partial f^h}{\partial t}\frac{\partial f^r}{\partial\bar{t}}+\Gamma^i_{hr}\bigg(\frac{\partial f^h_\alpha}{\partial t}\frac{\partial f^r}{\partial\bar{t}}+\frac{\partial f^h}{\partial t}\frac{\partial f^r_\alpha}{\partial\bar{t}}\bigg)\bigg]f^{\bar{j}}_{\bar{\beta}}h_{i\bar{j}}\\
&+G^{\bar{\beta}\alpha}\bigg(\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}\bigg)\bigg(\frac{\partial^2\bar{f}^j}{\partial z^\alpha\partial\bar{z}^\beta}+\Gamma^{\bar{j}}_{\bar{q}\bar{l}}f^{\bar{q}}_\alpha f^{\bar{l}}_{\bar{\beta}}\bigg)h_{i\bar{j}}+G^{\bar{\beta}\alpha}\bigg(\frac{D}{\partial\bar{t}}\frac{\partial f^p}{\partial t}\bigg)\Gamma^i_{pk}f^k_\alpha f^{\bar{j}}_{\bar{\beta}}h_{i\bar{j}}.
\end{aligned}\tag{5.13}
$$

因为

$$
\begin{aligned}
f^i_\alpha f^{\bar{j}}_{\bar{\beta}}\frac{\partial^2 h_{i\bar{j}}}{\partial w^k\partial\bar{w}^l}&=f^i_\alpha f^{\bar{j}}_{\bar{\beta}}(\Gamma^p_{ik}\Gamma^{\bar{q}}_{\bar{j}\bar{l}}h_{p\bar{q}}-R_{i\bar{j}k\bar{l}})=f^p_\alpha\Gamma^i_{pk}f^{\bar{q}}_{\bar{\beta}}\Gamma^{\bar{j}}_{\bar{q}\bar{l}}h_{i\bar{j}}-R_{i\bar{j}k\bar{l}}f^i_\alpha f^{\bar{j}}_{\bar{\beta}},\\
f^i_\alpha f^{\bar{j}}_{\bar{\beta}}\frac{\partial^2 h_{i\bar{j}}}{\partial w^k\partial w^l}&=f^i_\alpha f^{\bar{j}}_{\bar{\beta}}\bigg(\frac{\partial\Gamma^p_{ik}}{\partial w^l}+\Gamma^s_{ik}\Gamma^p_{sl}\bigg)h_{p\bar{j}}=f^p_\alpha f^{\bar{j}}_{\bar{\beta}}\bigg(\frac{\partial\Gamma^i_{pk}}{\partial w^l}+\Gamma^s_{pk}\Gamma^i_{sl}\bigg)h_{i\bar{j}},
\end{aligned}
$$

所以

$$
\begin{aligned}
&\frac{\partial^2 e'(f_t)}{\partial t\partial\bar{t}}-2\mathrm{Re}\,\mathrm{div}(X)\\
&\quad=G^{\bar{\beta}\alpha}\bigg[\frac{D}{\partial t}(f^i_\alpha)\overline{\frac{D}{\partial t}(f^j_\beta)}+\frac{D}{\partial\bar{t}}(f^i_\alpha)\overline{\frac{D}{\partial\bar{t}}(f^j_\beta)}\bigg]h_{i\bar{j}}-R_{i\bar{j}k\bar{l}}G^{\bar{\beta}\alpha}f^i_\alpha f^{\bar{j}}_{\bar{\beta}}\bigg(\frac{\partial f^k}{\partial t}\overline{\frac{\partial f^l}{\partial t}}+\frac{\partial f^k}{\partial\bar{t}}\overline{\frac{\partial f^l}{\partial\bar{t}}}\bigg)\\
&\qquad+2\mathrm{Re}\bigg[G^{\bar{\beta}\alpha}R_{i\bar{j}k\bar{l}}f^{\bar{j}}_{\bar{\beta}}f^{\bar{l}}_\alpha\frac{\partial f^i}{\partial t}\frac{\partial f^k}{\partial\bar{t}}\bigg]-2\mathrm{Re}\bigg[G^{\bar{\beta}\alpha}\bigg(\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}\bigg)\overline{f^j_{\beta\,|\,\bar{\alpha}}}h_{i\bar{j}}\bigg].
\end{aligned}\tag{5.14}
$$

由于 f 调和, 所以

$$
\mathrm{Re}\int_{\mathbb{P}\tilde{M}}G^{\bar{\beta}\alpha}\bigg(\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}\bigg)\overline{f^j_{\beta\,|\,\bar{\alpha}}}h_{i\bar{j}}d\mu_{\mathbb{P}\tilde{M}}=\mathrm{Re}\int_M\bigg(\frac{D}{\partial\bar{t}}\frac{\partial f^i}{\partial t}\bigg)h_{i\bar{j}}\overline{[\gamma^{\bar{\alpha}\beta}f^j_{\beta\,|\,\bar{\alpha}}]}d\mu_M=0.
$$

将 (5.14) 等号两边同时积分, 则可以得到第二变分 (1.6). □

注 5.1 韩敬伟 [9] 给出了复 Finsler 流形到 Hermite 流形的调和映照的 $\bar{\partial}$-能量泛函的第二变分公式, (1.7) 在文献 [9] 中也被明确给出.

不难看出,

$$
\begin{aligned}
V&=\partial f_t\bigg(\frac{\partial}{\partial t}\bigg)\bigg|_{t=0}+\partial\bar{f}_t\bigg(\frac{\partial}{\partial t}\bigg)\bigg|_{t=0}=\partial F_t\bigg(\frac{\partial}{\partial t}\bigg)\bigg|_{t=0},\\
\partial_a F&=\partial_a f+\partial_a\bar{f}=f^i_a\frac{\partial}{\partial w^i}+f^{\bar{i}}_a\frac{\partial}{\partial\bar{w}^i},\quad a=\alpha,\bar{\alpha}.
\end{aligned}
$$

记 $\gamma^{\mathfrak{i}}_{\mathfrak{j}\mathfrak{l}}$ 是 Riemann 度量 $g=\mathrm{Re}h$ 的 Levi-Civita 联络的联络系数. 在 Kähler 条件下, 则有

$$\gamma^i_{jk}=-\gamma^i_{j+n\ k+n}=\gamma^{i+n}_{j\ k+n}=\mathrm{Re}(\Gamma^i_{jk}),\tag{5.15}$$

$$\gamma^{i+n}_{jk}=-\gamma^i_{j+n\ k}=-\gamma^{i+n}_{j+n\ k+n}=\mathrm{Im}(\Gamma^i_{jk}).\tag{5.16}$$

可直接计算出

$$\begin{aligned}
\gamma^i_{\mathfrak{j}\mathfrak{l}}V^{\mathfrak{j}}F^{\mathfrak{l}}_\alpha&=\frac{1}{2}(\Gamma^i_{jk}V'^jf^k_\alpha+\Gamma^{\bar{i}}_{\bar{j}\bar{k}}V''^{\bar{j}}f^{\bar{k}}_\alpha),\\
\gamma^{i+n}_{\mathfrak{j}\mathfrak{l}}V^{\mathfrak{j}}F^{\mathfrak{l}}_\alpha&=\frac{1}{2\sqrt{-1}}(\Gamma^i_{jk}V'^jf^k_\alpha-\Gamma^{\bar{i}}_{\bar{j}\bar{k}}V''^{\bar{j}}f^{\bar{k}}_\alpha),\\
\gamma^i_{\mathfrak{j}\mathfrak{l}}V^{\mathfrak{j}}F^{\mathfrak{l}}_{\bar\beta}&=\frac{1}{2}(\Gamma^i_{jk}V'^jf^k_{\bar\beta}+\Gamma^{\bar{i}}_{\bar{j}\bar{k}}V''^{\bar{j}}f^{\bar{k}}_{\bar\beta}),\\
\gamma^{i+n}_{\mathfrak{j}\mathfrak{l}}V^{\mathfrak{j}}F^{\mathfrak{l}}_{\bar\beta}&=\frac{1}{2\sqrt{-1}}(\Gamma^i_{jk}V'^jf^k_{\bar\beta}-\Gamma^{\bar{i}}_{\bar{j}\bar{k}}V''^{\bar{j}}f^{\bar{k}}_{\bar\beta}),
\end{aligned}$$

进一步可得

$$\begin{aligned}
D_aV^i&=\frac{1}{2}(D_aV'^i+D_aV''^{\bar{i}}),\\
D_aV^{i+n}&=\frac{1}{2\sqrt{-1}}(D_aV'^i-D_aV''^{\bar{i}}),
\end{aligned}$$

其中 $a=\alpha,\bar\alpha$. 于是,

$$\langle D'V,D'V\rangle=\frac{1}{2}[\langle D'V',D'V'\rangle+\langle D'\overline{V''},D'\overline{V''}\rangle],\tag{5.17}$$

$$\langle D''V,D''V\rangle=\frac{1}{2}[\langle D''V',D''V'\rangle+\langle D''\overline{V''},D''\overline{V''}\rangle].\tag{5.18}$$

将 Riemann 度量 $g=\mathrm{Re}h$ 复化, 则

$$g\left(\frac{\partial}{\partial w^i},\frac{\partial}{\partial w^j}\right)=g\left(\frac{\partial}{\partial\bar{w}^i},\frac{\partial}{\partial\bar{w}^j}\right)=0,\quad g\left(\frac{\partial}{\partial w^i},\frac{\partial}{\partial\bar{w}^j}\right)=\overline{g\left(\frac{\partial}{\partial\bar{w}^i},\frac{\partial}{\partial w^j}\right)}=\frac{1}{2}h_{i\bar{j}}.$$

因为

$$\begin{aligned}
&G^{\bar\beta\alpha}R_{i\bar{j}k\bar{l}}(f^i_\alpha f^{\bar{j}}_{\bar\beta}V'^kV'^{\bar{l}}+f^i_{\bar\beta}f^{\bar{j}}_\alpha V''^kV''^{\bar{l}})-2\mathrm{Re}[G^{\bar\beta\alpha}f^{\bar{j}}_{\bar\beta}f^{\bar{l}}_\alpha R_{i\bar{j}k\bar{l}}V'^iV''^k]\\
&\quad=G^{\bar\beta\alpha}R_{i\bar{j}k\bar{l}}(f^i_\alpha V'^{\bar{j}}-f^{\bar{j}}_\alpha V''^i)(\overline{f^l_\beta V'^{\bar{k}}-f^{\bar{k}}_\beta V''^l}),\\
&G^{\bar\beta\alpha}R_{i\bar{j}k\bar{l}}(f^{\bar{j}}_\alpha f^i_{\bar\beta}V'^kV'^{\bar{l}}+f^i_\alpha f^{\bar{j}}_{\bar\beta}V''^kV''^{\bar{l}})-2\mathrm{Re}[G^{\bar\beta\alpha}f^i_{\bar\beta}f^k_\alpha R_{i\bar{j}k\bar{l}}V''^{\bar{j}}V'^{\bar{l}}]\\
&\quad=G^{\bar\beta\alpha}R_{i\bar{j}k\bar{l}}(f^i_\alpha V''^{\bar{j}}-f^{\bar{j}}_\alpha V'^i)(\overline{f^l_\beta V''^{\bar{k}}-f^{\bar{k}}_\beta V'^l}),
\end{aligned}$$

所以

$$G^{\bar\beta\alpha}R(\partial_{\bar\beta}F,V,\partial_\alpha F,\overline{V})=\frac{1}{2}G^{\bar\beta\alpha}R_{i\bar{j}k\bar{l}}(f^i_\alpha V'^{\bar{j}}-f^{\bar{j}}_\alpha V''^i)(\overline{f^l_\beta V'^{\bar{k}}-f^{\bar{k}}_\beta V''^l}),\tag{5.19}$$

$$G^{\bar\beta\alpha}R(\partial_\alpha F,V,\partial_{\bar\beta}F,\overline{V})=\frac{1}{2}G^{\bar\beta\alpha}R_{i\bar{j}k\bar{l}}(f^i_\alpha V''^{\bar{j}}-f^{\bar{j}}_\alpha V'^i)(\overline{f^l_\beta V''^{\bar{k}}-f^{\bar{k}}_\beta V'^l}).\tag{5.20}$$

在局部坐标下, 本文验证了

$$\left.\frac{\partial^2E'(f_t)}{\partial t\partial\bar{t}}\right|_{t=0}+\left.\frac{\partial^2E''(f_t)}{\partial t\partial\bar{t}}\right|_{t=0}=2\left.\frac{\partial^2E''(F_t)}{\partial t\partial\bar{t}}\right|_{t=0}.\tag{5.21}$$

由 (5.5) 和 (5.7), 可以看出调和映照 f 的第二变分公式满足

$$\frac{\partial^2 E'(f_t)}{\partial t\partial\bar{t}}\bigg|_{t=0}=\frac{\partial^2 E''(f_t)}{\partial t\partial\bar{t}}\bigg|_{t=0}=\frac{\partial^2 E''(F_t)}{\partial t\partial\bar{t}}\bigg|_{t=0}.$$

于是调和映照 f 的 $\bar{\partial}$-能量泛函的第二变分公式可以写成

$$\begin{aligned}\frac{\partial^2 E''(f_t)}{\partial t\partial\bar{t}}\bigg|_{t=0}&=\int_{\mathbb{P}\tilde{M}}[\langle D'V,D'V\rangle+\langle D''V,D''V\rangle]d\mu_{\mathbb{P}\tilde{M}}\\&\quad-\int_{\mathbb{P}\tilde{M}}G^{\bar{\beta}\alpha}[R(\partial_{\bar{\beta}}F,V,\partial_\alpha F,\overline{V})+R(\partial_\alpha F,V,\partial_{\bar{\beta}}F,\overline{V})]d\mu_{\mathbb{P}\tilde{M}}.\end{aligned}\tag{5.22}$$

Siu [36] 称 Kähler 流形 (N,h) 的曲率张量 $R_{i\bar{j}k\bar{l}}$ 是强负的 (或者强非正的), 如果对于任意的复数 A^i、B^j、C^i 和 D^j, 当至少有一对指标 (i,j) 使得 $A^i\overline{B}^j-C^i\overline{D}^j\neq 0$ 时有

$$R_{i\bar{j}k\bar{l}}(A^i\overline{B}^j-C^i\overline{D}^j)\overline{(A^l\overline{B}^k-C^l\overline{D}^k)}<0\quad(\text{或者 }\leqslant 0).$$

Siu [36, 37] 分别证明了经典的四类典型域和两类例外典型域的不变 Kähler 度量的曲率张量都是强非正的. 经典的四类典型域分别记为

$$\begin{aligned}D^I_{r,s}&=\{Z\in\mathbb{C}^{r\times s}:I_r-Z\overline{Z}^{\mathrm{T}}>0\},\\D^{II}_p&=\{Z\in\mathbb{C}^{p\times p}:I_p-Z\overline{Z}^{\mathrm{T}}>0,Z=Z^{\mathrm{T}}\},\\D^{III}_q&=\{Z\in\mathbb{C}^{q\times q}:I_q-Z\overline{Z}^{\mathrm{T}}>0,Z=-Z^{\mathrm{T}}\},\\D^{IV}_n&=\{z\in\mathbb{C}^n:1+|zz^{\mathrm{T}}|^2-2|z|^2>0,|zz^{\mathrm{T}}|<1\},\end{aligned}$$

它们的不变 Kähler 度量的势函数分别为

$$\begin{aligned}&\Phi_{D^I_{r,s}}=\log\det(I_r-Z\overline{Z}^{\mathrm{T}})^{-1},\quad\Phi_{D^{II}_p}=\log\det(I_p-Z\overline{Z}^{\mathrm{T}})^{-1},\\&\Phi_{D^{III}_q}=\log\det(I_q-Z\overline{Z}^{\mathrm{T}})^{-1},\quad\Phi_{D^{IV}_n}=\log(1+|zz^{\mathrm{T}}|^2-2|z|^2)^{-1}.\end{aligned}$$

注意到这里不变 Kähler 度量和 Bergman 度量相差一个常数. 两类例外典型域的实现比较复杂, 可参见文献 [51–54]. 事实上, 两类例外典型域的无界域形式都属于文献 [49] 中的正规 Siegel 域 (旧称 N-Siegel 域), 它们的 Bergman 核函数在 Xu [50] 中已经出现, 只是未单独列出.

文献 [24] 证明了一个 Kähler 流形 (N,h) 的曲率张量 $R_{i\bar{j}k\bar{l}}$ 是强非正的等价于 $(N,\mathrm{Re}h)$ 具有非正的复截面曲率. 因此, 由定理 1.4 和命题 5.1 可得到定理 1.6.

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Harmonic maps from Kähler-Finsler manifolds to Riemannian manifolds

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Abstract In this paper, we first derive the first variation formula of the $\bar{\partial}$-energy functional for a smooth map from a Kähler-Finsler manifold to a Riemannian manifold. By applying the existence theorem proved by Jost and Yau for Hermitian harmonic maps, we obtain an existence theorem for harmonic maps from Kähler-Finsler manifolds to Riemannian manifolds. Next, we derive the second variation formula of harmonic maps from a Kähler-Finsler manifold to a Riemannian manifold. As an application, we prove that harmonic maps are stable if the target Riemannian manifold has non-positive complex sectional curvature. At last, we discuss the second variation formulas of harmonic maps from a Kähler-Finsler manifold to a Kähler manifold and prove that harmonic maps are stable if curvature tensors of the target Kähler manifold are strongly non-positive.

Keywords **harmonic map, $\bar{\partial}$-energy functional, the second variation, Kähler-Finsler manifold**

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