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Fast Laplace transform methods for the PDE system of Parisian and Parasian option pricing

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Abstract This paper develops a fast Laplace transform method for solving the complex PDE system arising from Parisian and Parasian option pricing. The value functions of the options are governed by a system of partial differential equations (PDEs) of two and three dimensions. Applying the Laplace transform to the PDEs with respect to the calendar time to maturity leads to a coupled system consisting of an ordinary differential equation (ODE) and a 2-dimensional partial differential equation (2d-PDE). The solution to this ODE is found analytically on a specific parabola contour that is used in the fast Laplace inversion, whereas the solution to the 2d-PDE is approximated by solving 1-dimensional integro-differential equations. The Laplace inversion is realized by the fast contour integral methods. Numerical results confirm that the Laplace transform methods have the exponential convergence rates and are more efficient than the implicit finite difference methods, Monte Carlo methods and moving window methods.

Keywords Parisian option, Parasian option, coupled PDE, Laplace transform method, convergence rate

MSC(2020) 35R35, 91G20, 91G60, 91G80

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1 Introduction

Parisian and Parasian options are two special kinds of barrier options for which the knock-in or knock-out feature is only activated if the underlying remains continually in breach of the barrier $\bar{S}$ for a pre-specified time period $\bar{T}$. In Parisian and Parasian option pricing systems there is a separate "clock" set up to record the total time, denoted by ξ, in which the underlying has passed the barrier $\bar{S}$. How the clock is reset is the main difference between Parisian and Parasian options. If the "clock" is reset to zero at each time the underlying price crosses the barrier, we refer to it as a continuous Parisian option, or simply Parisian option. If one adds the time spent below or above the barrier without resetting the accumulated time to zero each time the underlying crosses the barrier, we refer to it as a cumulative-Parisian option, or simply Parasian option.

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In this paper, we study the Parisian and Parasian options only on the up-and-out call with the strike price K and maturity T. It will be straightforward to extend the studies to the up-and-in call, down-and-out call, down-and-in call, and the corresponding put options. Under the risk-neutral measure, the stock price $S_{\tilde{t}}$ is assumed to follow the geometric Brownian motion

$$dS_{\tilde{t}} = (r-\delta)S_{\tilde{t}}d\tilde{t} + \sigma S_{\tilde{t}}dW_{\tilde{t}},$$

where $W_{\tilde{t}}$ is the standard Brownian motion with respect to the calendar time $\tilde{t}$, $\sigma > 0$ the volatility, $r > 0$ the risk-free interest rate and $\delta > 0$ the continuous dividend rate. For the case of an up-and-out barrier, the "clock" time ξ is defined by

$$\begin{cases} \xi = 0, \quad d\xi = 0, \quad S \leqslant \bar{S}, \\ d\xi = d\tilde{t}, \quad S > \bar{S}, \end{cases}$$

where $\bar{S}$ is the pre-set barrier of the underlying price. The prices of Parisian and Parasian options depend on the underlying price S, the time to maturity t (i.e., $t = T - \tilde{t}$ with T representing the expiry time and $\tilde{t}$ the current time) and the barrier time ξ.

Let

$$x = \log(S/\bar{S}), \quad \bar{x} = \log(\bar{S}/\bar{S}) = 0$$

and $\bar{T}$ be the pre-specified trigger time. The domains Ω_1 and Ω_2 are defined as

$$\Omega_1 := \{(t,x) \mid \bar{T} < t \leqslant T,\, -\infty < x < \bar{x}\},$$
$$\Omega_2 := \{(t,x,\xi) \mid \bar{T} - \xi < t \leqslant T,\, 0 \leqslant \xi < \bar{T},\, \bar{x} < x < \infty\}.$$

As shown in Figure 1, the region Ω_1 refers to the plane QKEX and Ω_2 the cuboid KGME-DLBJ. For $(t,x) \in \Omega_1$, the value function $V_1(t,x)$ of the Parisian option is governed by a 2-dimensional PDE. In this case, the barrier time keeps $\xi = 0$. For the other case where $(t,x,\xi) \in \Omega_2$, the value function $V_2(t,x,\xi)$ of the Parisian option price is governed by a 3d-PDE. The main difficulty of solving the problem is that the coupled PDEs have different dimensions and the connection conditions are very complicated. To be more precise, the PDEs for pricing the Parisian option are given by (see [14])

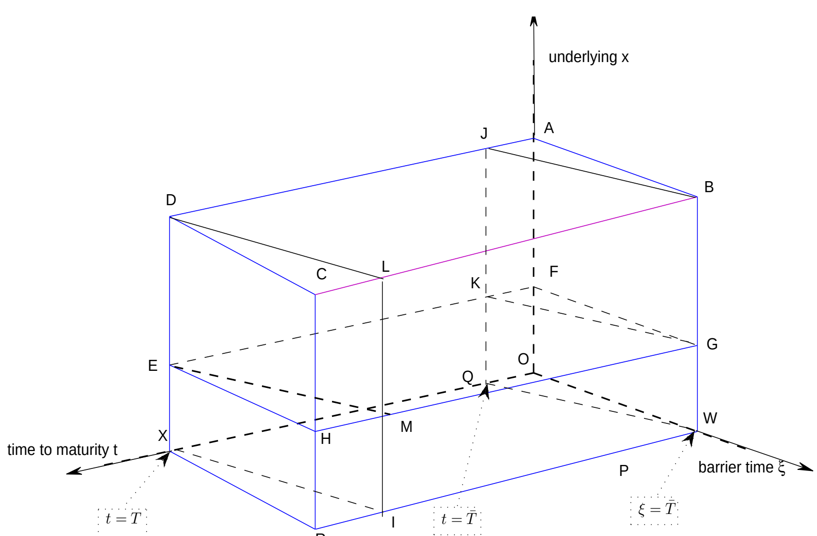


Figure 1 (Color online) Pricing domain of the Parisian up-and-out option

$$\begin{cases} \dfrac{\partial V_1}{\partial t} = \mathcal{A}V_1, & (t,x) \in \Omega_1, \\ V_1(t,x) = V_{BS}(t, \bar{S}\mathrm{e}^x), & 0 \leqslant t \leqslant \bar{T}, \quad -\infty < x < \bar{x}, \\ V_1(t,\bar{x}) = V_2(t,\bar{x},0), & \bar{T} < t \leqslant T, \\ \lim\limits_{x\uparrow\bar{x}} \dfrac{\partial V_1(t,x)}{\partial x} = \lim\limits_{x\downarrow\bar{x}} \dfrac{\partial V_2(t,x,0)}{\partial x}, & \bar{T} < t \leqslant T, \\ \lim\limits_{x\downarrow-\infty} V_1(t,x) = 0, & \bar{T} < t \leqslant T \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dfrac{\partial V_2}{\partial t} = \dfrac{\partial V_2}{\partial \xi} + \mathcal{A}V_2, & (t,x,\xi) \in \Omega_2, \\ V_2(\bar{T}-\xi, x, \xi) = V_{BS}(\bar{T}-\xi, \bar{S}\mathrm{e}^x), & \xi \in [0,\bar{T}), \quad \bar{x} < x < \infty, \\ V_2(t,\bar{x},\xi) = V_2(t,\bar{x},0) = V_1(t,\bar{x}), & \bar{T} < t \leqslant T, \quad 0 \leqslant \xi < \bar{T}, \\ \lim\limits_{x\downarrow\bar{x}} \dfrac{\partial V_2(t,x,0)}{\partial x} = \lim\limits_{x\uparrow\bar{x}} \dfrac{\partial V_1(t,x)}{\partial x}, & \bar{T} < t \leqslant T, \\ V_2(t,x,\bar{T}) = 0, & 0 < t \leqslant T, \quad \bar{x} < x < \infty, \\ \lim\limits_{x\uparrow\infty} V_2(t,x,\xi) = 0, & \bar{T}-\xi < t \leqslant T, \quad 0 \leqslant \xi < \bar{T}, \end{cases} \tag{1.2}$$

where the linear differential operator is defined by

$$\mathcal{A} = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} - r\mathrm{I},$$

and I denotes the identity operator. The value of European call option has analytical Black-Scholes formula

$$V_{BS}(t,S) = S\mathrm{e}^{-\delta t} N(d_+(t,S)) - \mathrm{e}^{-rt} K N(d_-(t,S)),$$

where

$$d_\pm(t,S) = \frac{1}{\sigma\sqrt{t}}\left[\log\frac{S}{K} + \left(r - \delta \pm \frac{1}{2}\sigma^2\right)t\right], \quad N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} \mathrm{e}^{-\frac{z^2}{2}} dz.$$

Contrary to the "resetting" feature of the continuous Parisian options, Parasian options are introduced with no reset of the "clock" time ξ: the knock-in or knock-out is only activated if the cumulative time spent beyond or below $\bar{x}$ exceeds some prescribed time $\bar{T}$. According to Zhu and Chen [14], the pricing domain of the knock-out-call Parasian option is divided into two sub-domains

$$\begin{aligned} (t,x,\xi) \in \widetilde{\Omega}_1 &:= \{(t,x,\xi) \mid \bar{T}-\xi < t \leqslant T, -\infty < x < \bar{x}, 0 \leqslant \xi < \bar{T}\}, \\ (t,x,\xi) \in \widetilde{\Omega}_2 &:= \{(t,x,\xi) \mid \bar{T}-\xi < t \leqslant T, \bar{x} < x < \infty, 0 \leqslant \xi < \bar{T}\}. \end{aligned}$$

As shown in Figure 1, the region $\widetilde{\Omega}_1$ refers to the cubic QWIX-EMGK and $\widetilde{\Omega}_2$ the cuboid KGME-DLBJ. The value functions $V_1(t,x,\xi)$ and $V_2(t,x,\xi)$ for Parasian options satisfy

$$\begin{cases} \dfrac{\partial V_1}{\partial t} = \mathcal{A}V_1, & (t,x,\xi) \in \widetilde{\Omega}_1, \\ V_1(t,x,\bar{T}-t) = V_{BS}(t, \bar{S}\mathrm{e}^x), & 0 \leqslant t \leqslant \bar{T}, \quad -\infty < x < \bar{x}, \\ V_1(t,\bar{x},\xi) = V_2(t,\bar{x},\xi), & \bar{T} < t \leqslant T, \quad 0 \leqslant \xi < \bar{T}, \\ \lim\limits_{x\uparrow\bar{x}} \dfrac{\partial V_1(t,x,\xi)}{\partial x} = \lim\limits_{x\downarrow\bar{x}} \dfrac{\partial V_2(t,x,\xi)}{\partial x}, & \bar{T} < t \leqslant T, \quad 0 \leqslant \xi < \bar{T}, \\ V_1(t,x,\bar{T}) = 0, & \bar{T} < t \leqslant T, \quad -\infty < x < \bar{x}, \\ \lim\limits_{x\downarrow-\infty} V_1(t,x,\xi) = 0, & \bar{T} < t \leqslant T, \quad 0 \leqslant \xi < \bar{T} \end{cases} \tag{1.3}$$

and

$$
\begin{cases}
\dfrac{\partial V_2}{\partial t} = \dfrac{\partial V_2}{\partial \xi} + \mathcal{A}V_2, & (t,x,\xi) \in \widetilde{\Omega}_2,\\
V_2(t,x,\bar{T}-t) = V_{BS}(t,\bar{S}\mathrm{e}^x), & 0 \leqslant t \leqslant \bar{T}, \quad \bar{x} < x < \infty,\\
V_2(t,\bar{x},\xi) = V_1(t,\bar{x},\xi), & \bar{T} < t \leqslant T, \quad 0 \leqslant \xi < \bar{T},\\
\lim\limits_{x\downarrow\bar{x}} \dfrac{\partial V_2(t,x,\xi)}{\partial x} = \lim\limits_{x\uparrow\bar{x}} \dfrac{\partial V_1(t,x,\xi)}{\partial x}, & \bar{T} < t \leqslant T, \quad 0 \leqslant \xi < \bar{T},\\
V_2(t,x,\bar{T}) = 0, & \bar{T} < t \leqslant T, \quad \bar{x} < x < \infty,\\
\lim\limits_{x\uparrow\infty} V_2(t,x,\xi) = 0, & \bar{T} < t \leqslant T, \quad 0 \leqslant \xi < \bar{T}.
\end{cases}
\tag{1.4}
$$

In the literature, there are three types of valuation methods that are well documented, namely the quasi-analytic approaches, the Monte Carlo simulation and the numerical PDEs methods. Chesney et al. [2] and Schröder et al. [8] proposed a quasi-analytic method that expresses the value of a Parisian option in terms of some four-fold integrals; however the computation of these high-dimensional integrals is very difficult. Haber et al. [4] proposed two 3d-PDEs for the prices of Parisian and Parasian options, and then solve these PDEs using the explicit finite difference methods (FDMs). To ensure the computational stability, it has to require the time step be much smaller than the spatial mesh size, which makes the computation too costly and inefficient. The Monte Carlo simulation (see, e.g., [1]) is not that efficient too, as at least $200,000$ stock price paths need to be simulated and tracing the time accumulated in the "clock" takes vast CPU time. Zhu and Chen [14] presented so-called moving window methods (MWMs) which greatly raise the computational efficiency for valuating Parisian options, but it is still not that fast for pricing Parasian options as the analytical formulas are very different for these two kinds of options. The MWMs are further extended to pricing American-style Parisian options (see [5, 15]).

In this paper, we develop a fast Laplace transform method to solve the coupled 3d-PDEs of Parisian and Parasian options. The Laplace transform method is regarded as an efficient alternative to the commonly used time-stepping method for solving PDEs as it has exponential convergence rates. Laplace transform methods have been applied to the option pricing in the literature, for example, the valuation of regime-switching option pricing (see [6]), the American option pricing (see [12,13]) and the fractional-order option pricing (see [11]). This paper further studies the applications to the difficult problems of Parisian and Parasian option pricing. The main idea is described as follows. The 3d-PDE system is first re-written into a convenient form and then taking the Laplace transform to the PDE system with respect to the calendar time to maturity gives a coupled system consisting of an ODE and a 2d-PDE. Since there is "clock" time and coupling of them, it is not easy to find the solutions to the ODE and 2d-PDE. This paper chooses a specific contour such that the ODE has an analytical form of the solution and the contour is also used to the fast Laplace inversion. This technique leads to both the fast computation of the transformed system of ODEs and PDEs in the Laplace space and the fast contour integral Laplace inversion. Numerical examples show that the Laplace transform method has exponential convergence rates and is much faster than the implicit FDMs, Monte Carlo simulations and moving window methods.

The rest of this paper is arranged as follows. In Sections 2 and 3 the Laplace transform methods (LTMs) are derived to solve the coupled PDEs for Parisian and Parasian options, respectively. In Section 4, numerical examples are provided to compare the Laplace transform methods with the implicit FDMs, Monte Carlo simulations and moving window methods. Conclusions are given in Section 5.

2 LTMs for pricing Parisian options

Instead of using directional derivative as in [14], we perform a coordinate transform on the coupled 3d-PDE systems (1.1) and (1.2), and then obtain the two 2d-PDE systems which are more convenient for the derivation of the Laplace transform methods. Let

$$
\alpha = \frac{1}{\sigma^2}\left(\frac{\sigma^2}{2} + \delta - r\right), \quad \beta = -\frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2} + \delta - r\right)^2 - r, \quad \eta = \bar{T} - \xi. \tag{2.1}
$$

Here, the new time η can be interpreted as the remainder of barrier time. For $(t,x)\in\Omega_1$, let $t=\tau+\bar{T}$ and

$$\mathrm{e}^{\alpha x}W_1(\tau,x)=V_1(t,x). \tag{2.2}$$

Then

$$\frac{\partial V_1}{\partial t}=\mathrm{e}^{\alpha x}\frac{\partial W_1}{\partial \tau},\quad \mathcal{A}V_1=\mathcal{A}[\mathrm{e}^{\alpha x}W_1(\tau,x)]. \tag{2.3}$$

For $(t,x,\xi)\in\Omega_2$, let $t=\tau+\bar{T}-\xi$ and

$$\mathrm{e}^{\alpha x+\beta\eta}W_2(\tau,x,\eta)=V_2(t,x,\xi). \tag{2.4}$$

Then

$$\begin{cases}\dfrac{\partial V_2}{\partial t}=\mathrm{e}^{\alpha x+\beta\eta}\dfrac{\partial W_2}{\partial \tau},\quad \mathcal{A}V_2=\mathcal{A}[\mathrm{e}^{\alpha x+\beta\eta}W_2(\tau,x,\eta)],\\ \dfrac{\partial V_2}{\partial \xi}=-\beta\mathrm{e}^{\alpha x+\beta\eta}W_2(\tau,x,\eta)+\mathrm{e}^{\alpha x+\beta\eta}\left[\dfrac{\partial W_2}{\partial\tau}-\dfrac{\partial W_2}{\partial\eta}\right].\end{cases} \tag{2.5}$$

Substituting (2.2)–(2.5) into (1.1) and (1.2), we derive the following transformed equations that will be easier to apply the Laplace transform methods:

$$\begin{cases}\dfrac{\partial W_1}{\partial\tau}=\dfrac{1}{2}\sigma^2\dfrac{\partial^2 W_1}{\partial x^2}+\beta W_1, & 0<\tau<\infty,\quad -\infty<x<\bar{x},\\ W_1(0,x)=\mathrm{e}^{-\alpha x}V_{BS}(\bar{T},\bar{S}\mathrm{e}^x), & -\infty<x<\bar{x},\\ W_1(\tau,\bar{x})=\mathrm{e}^{\beta\bar{T}}W_2(\tau,\bar{x},\bar{T}), & 0<\tau<\infty,\\ \lim\limits_{x\uparrow\bar{x}}\dfrac{\partial W_1(\tau,x)}{\partial x}=\mathrm{e}^{\beta\bar{T}}\lim\limits_{x\downarrow\bar{x}}\dfrac{\partial W_2(\tau,x,\bar{T})}{\partial x}, & 0<\tau<\infty,\\ \lim\limits_{x\downarrow-\infty}\mathrm{e}^{\alpha x}W_1(\tau,x)=0, & 0<\tau<\infty\end{cases} \tag{2.6}$$

and

$$\begin{cases}\dfrac{\partial W_2}{\partial\eta}=\dfrac{1}{2}\sigma^2\dfrac{\partial^2 W_2}{\partial x^2}, & 0<\tau<\infty,\quad \bar{x}<x<\infty,\quad 0<\eta\leqslant\bar{T},\\ W_2(0,x,\eta)=\mathrm{e}^{-\alpha x-\beta\eta}V_{BS}(\eta,\bar{S}\mathrm{e}^x), & \bar{x}<x<\infty,\quad 0<\eta\leqslant\bar{T},\\ W_2(\tau,\bar{x},\eta)=\mathrm{e}^{-\beta\eta}W_1(\tau-\bar{T}+\eta,\bar{x}), & 0<\tau<\infty,\quad \bar{x}<x<\infty,\\ \lim\limits_{x\downarrow\bar{x}}\dfrac{\partial W_2(\tau,x,\bar{T})}{\partial x}=\mathrm{e}^{-\beta\bar{T}}\lim\limits_{x\uparrow\bar{x}}\dfrac{\partial W_1(\tau,x)}{\partial x}, & 0<\tau<\infty,\quad \bar{x}<x<\infty,\\ \lim\limits_{x\uparrow+\infty}\mathrm{e}^{\alpha x+\beta\eta}W_2(\tau,x,\eta)=0, & 0<\tau<\infty,\quad 0<\eta\leqslant\bar{T},\\ \mathrm{e}^{\alpha x}W_2(\tau,x,0)=0, & 0<\tau<\infty,\quad \bar{x}<x<\infty.\end{cases} \tag{2.7}$$

It is noted that $\mathrm{e}^{\alpha\bar{x}}W_1(\tau-\bar{T}+\eta,\bar{x})=V_1(\tau-\bar{T}+\eta,\bar{x})=V_{BS}(\tau+\eta,\bar{S})$ is a supplemental definition for $-\bar{T}\leqslant\tau-\bar{T}+\eta<0$. The third equation in (2.7) is derived as

$$\begin{aligned}\mathrm{e}^{\alpha\bar{x}+\beta\eta}W_2(\tau,\bar{x},\eta)&=V_2(\tau+\bar{T}-\xi,\bar{x},\xi)=V_2((\tau-\xi)+\bar{T},\bar{x},0)\\&=\mathrm{e}^{\alpha\bar{x}+\beta\eta}W_2(\tau-\xi,\bar{x},\bar{T})=\mathrm{e}^{\alpha\bar{x}}W_1(\tau-\xi,\bar{x})=\mathrm{e}^{\alpha\bar{x}}W_1(\tau-\bar{T}+\eta,\bar{x}).\end{aligned}$$

To apply LTMs to price Parisian options, we first introduce two transformations, under which the coupled PDE system of the Parisian option can be further simplified into a form that is easier for finding the analytical solutions in Laplace domain. Taking Laplace transforms

$$\widehat{W}_1(z,x)=\int_0^\infty\mathrm{e}^{-z\tau}W_1(\tau,x)d\tau,\quad \widehat{W}_2(z,x,\eta)=\int_0^\infty\mathrm{e}^{-z\tau}W_2(\tau,x,\eta)d\tau$$

to both sides of the PDEs (2.6) and (2.7) (the smooth pasting conditions between W_1 and W_2 will be discussed separately), respectively, we have

$$\begin{cases}(z-\beta)\widehat{W}_1-\dfrac{1}{2}\sigma^2\dfrac{\partial^2\widehat{W}_1}{\partial x^2}=\mathrm{e}^{-\alpha x}V_{BS}(\bar{T},\bar{S}\mathrm{e}^x),\quad -\infty<x<\bar{x},\\ \widehat{W}_1(z,\bar{x})=\mathrm{e}^{\beta\bar{T}}\widehat{W}_2(z,\bar{x},\bar{T})=:\widehat{W}_1^*(z),\\ \lim\limits_{x\downarrow-\infty}\mathrm{e}^{\alpha x}\widehat{W}_1(z,x)=0\end{cases} \tag{2.8}$$

and

$$\begin{cases} \dfrac{\partial \widehat{W}_2}{\partial \eta} - \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 \widehat{W}_2}{\partial x^2} = 0, & \bar{x} < x < \infty, \quad 0 < \eta \leqslant \bar{T}, \\ \widehat{W}_2(z, \bar{x}, \eta) = f(z,\eta)\widehat{W}_1^*(z) + g(z,\eta), & 0 < \eta \leqslant \bar{T}, \\ \lim\limits_{x\uparrow\infty} \mathrm{e}^{\alpha x + \beta\eta}\widehat{W}_2(z,x,\eta) = 0, & 0 < \eta \leqslant \bar{T}, \\ \mathrm{e}^{\alpha x}\widehat{W}_2(z,x,0) = 0, & \bar{x} < x < \infty, \end{cases} \tag{2.9}$$

where the functions f and g are defined by

$$f(z,\eta) = \mathrm{e}^{-\beta\eta - z(\bar{T}-\eta)}, \quad g(z,\eta) = \mathrm{e}^{-\beta\eta - z(\bar{T}-\eta)} \int_\eta^{\bar{T}} \mathrm{e}^{z(\bar{T}-y)} V_{BS}(y, \bar{S}) dy.$$

The second equation in (2.9) is derived as

$$\begin{aligned} \widehat{W}_2(z,\bar{x},\eta) &= \mathrm{e}^{-\beta\eta} \int_0^\infty W_1(\tau - \bar{T} + \eta, \bar{x}) \mathrm{e}^{-z\tau} d\tau = \mathrm{e}^{-\beta\eta - z(\bar{T}-\eta)} \int_{-\bar{T}+\eta}^\infty W_1(s,\bar{x}) \mathrm{e}^{-zs} ds \\ &= \mathrm{e}^{-\beta\eta - z(\bar{T}-\eta)} \int_0^\infty W_1(s,\bar{x}) \mathrm{e}^{-zs} ds + \mathrm{e}^{-\beta\eta - z(\bar{T}-\eta)} \int_{-\bar{T}+\eta}^0 W_1(s,\bar{x}) \mathrm{e}^{-zs} ds \\ &= f(z,\eta)\widehat{W}_1(z,\bar{x}) + \mathrm{e}^{-\beta\eta - z(\bar{T}-\eta)} \int_{-\bar{T}+\eta}^0 V_{BS}(s+\bar{T}, \bar{S}) \mathrm{e}^{-zs} ds \\ &= f(z,\eta)\widehat{W}_1(z,\bar{x}) + g(z,\eta) = f(z,\eta)\widehat{W}_1^*(z) + g(z,\eta). \end{aligned}$$

In addition the smooth pasting at barrier $\bar{x}$ is set as

$$\lim_{x\downarrow\bar{x}} \frac{\partial \widehat{W}_2(z,x,\bar{T})}{\partial x} = \mathrm{e}^{-\beta\bar{T}} \lim_{x\uparrow\bar{x}} \frac{\partial \widehat{W}_1(z,x)}{\partial x}. \tag{2.10}$$

From the coupled pricing system (2.8)–(2.10), it can be observed that once $\widehat{W}_1^*(z)$ is found, the PDEs (2.8) and (2.9) can be solved separately. It seems quite natural to treat the determination of $\widehat{W}_1^*(z)$ as a key step for solving the PDE system (2.8)–(2.10).

In the following Lemma 1, we give the analytical solution of $\widehat{W}_1(z,x)$ using constant variation methods. But it is not an easy task to deal with the zero-boundary $\lim_{x\downarrow-\infty}\mathrm{e}^{\alpha x}\widehat{W}_1(z,x) = 0$ for the ODE (2.8) because of the presence of the exponential function $\mathrm{e}^{\alpha x}$. To remove this difficulty, we introduce a new technique. Firstly, we replace the above zero-boundary condition by $\lim_{x\downarrow-\infty}\widehat{W}_1(z,x) = 0$. Then an alternative ODE (2.15) is established, which can be solved more easily and we verify that this analytical solution also solves the original ODE (2.8) with the restriction on the parabola contour defined by (2.17).

In the subsequent analysis, we will need the following two integral functions:

$$\phi(z,x,\eta) = -\frac{2}{\sigma^2} \int_{-\infty}^x \mathrm{e}^{-2\gamma(z)y} \left(\int_{-\infty}^y \mathrm{e}^{(\gamma(z)-\alpha)\zeta} V_{BS}(\eta, \bar{S}\mathrm{e}^\zeta) d\zeta \right) dy, \tag{2.11}$$

$$\psi(z,x,\eta) = \frac{\partial}{\partial x}\phi(z,x,\eta) = -\frac{2}{\sigma^2}\mathrm{e}^{-2\gamma(z)x} \int_{-\infty}^x \mathrm{e}^{(\gamma(z)-\alpha)\zeta} V_{BS}(\eta, \bar{S}\mathrm{e}^\zeta) d\zeta, \tag{2.12}$$

where $\gamma(z) = \frac{1}{\sigma}\sqrt{2(z-\beta)}$ with $\mathrm{Re}(\sqrt{2(z-\beta)}) > 0$ and z falls on the parabola contour Γ defined by (2.17). We do not worry about the integrability of $\phi(z,x,\eta)$ and $\psi(z,x,\eta)$, since

$$\lim_{x\to-\infty} V_{BS}(\eta, \bar{S}\mathrm{e}^x) = O(N(x)) \quad \text{as } x \to -\infty,$$

and the accumulation normal distribution function $N(x)$ is convergent to zero with much higher order than $\mathrm{e}^{2\gamma(z)x}$ as $x \to -\infty$.

Lemma 1. *Let* $\gamma(z) = \frac{1}{\sigma}\sqrt{2(z-\beta)}$ *with* $\mathrm{Re}(\sqrt{2(z-\beta)}) > 0$ *and*

$$C(z,x,\bar{T}) = \widehat{W}_1^*(z) + \phi(z,x,\bar{T}) - \phi(z,\bar{x},\bar{T}). \tag{2.13}$$

Then

$$\widehat{W}_1(z,x) = C(z,x,\bar{T})\mathrm{e}^{\gamma(z)x} \tag{2.14}$$

solves the following auxiliary ODE:

$$\begin{cases} (z-\beta)\widehat{W}_1 - \dfrac{1}{2}\sigma^2\dfrac{\partial^2 \widehat{W}_1}{\partial x^2} = \mathrm{e}^{-\alpha x}V_{BS}(\bar{T},\bar{S}\mathrm{e}^x), \\ \widehat{W}_1(z,\bar{x}) = \widehat{W}_1^*(z), \\ \lim\limits_{x\downarrow -\infty}\widehat{W}_1(z,x) = 0. \end{cases} \tag{2.15}$$

Moreover,

$$\left.\frac{\partial \widehat{W}_1(z,x)}{\partial x}\right|_{x=\bar{x}} = \psi(z,\bar{x},\bar{T}) + \gamma(z)\widehat{W}_1^*(z). \tag{2.16}$$

In addition, let $\mu > \frac{1}{2}\alpha^2\sigma^2$ *and* z *fall on a parabola contour defined by*

$$\Gamma: z(\zeta) = \beta + \mu(\mathrm{i}\zeta+1)^2, \quad -\infty < \zeta < \infty. \tag{2.17}$$

Assume $\gamma(z) = \frac{1}{\sigma}\sqrt{2(z-\beta)}$ *with* $\mathrm{Re}(\sqrt{2(z-\beta)}) > 0$. *Then* $\widehat{W}_1(z,x) = C(z,x,\bar{T})\mathrm{e}^{\gamma(z)x}$ *in* (2.14) *satisfies*

$$\lim_{x\to-\infty}\mathrm{e}^{\alpha x}\widehat{W}_1(z(\zeta),x) = 0, \quad \textit{for all } \zeta\in(-\infty,+\infty), \tag{2.18}$$

and thus $\widehat{W}_1(z,x)$ *with* z *on the contour* Γ *in* (2.17) *solves the ODE* (2.8).

Proof. Assuming $\widehat{W}_1(z,x) = \mathrm{e}^{\gamma(z)x}$ is a solution satisfying the homogeneous form corresponding to the ODE in (2.15), we obtain $\gamma(z) = \pm\frac{1}{\sigma}\sqrt{2(z-\beta)}$. Since $\lim_{x\downarrow-\infty}\widehat{W}_1(z,x) = 0$, we select $\gamma(z) = \frac{1}{\sigma}\sqrt{2(z-\beta)}$ with the restriction $\mathrm{Re}(\sqrt{2(z-\beta)}) > 0$. With this choice of z, we see that $\lim_{x\to-\infty}\mathrm{e}^{\gamma(z)x} = 0$.

Now, we find the solution to the non-homogeneous ODE (2.15) using the constant variation method. To this end, assume that (2.15) has a solution of the form

$$\widehat{W}_1(z,x) = C(z,x,\bar{T})\mathrm{e}^{\gamma(z)x}. \tag{2.19}$$

Substituting (2.19) into (2.15) gives

$$\frac{\partial^2 C(z,x,\bar{T})}{\partial x^2} + 2\gamma(z)\frac{\partial C(z,x,\bar{T})}{\partial x} = -\frac{2}{\sigma^2}\mathrm{e}^{-(\gamma(z)+\alpha)x}V_{BS}(\bar{T},\bar{S}\mathrm{e}^x).$$

Solving the above ODE, we get

$$C(z,x,\bar{T}) = -\frac{2}{\sigma^2}\int_{-\infty}^{x}\mathrm{e}^{-2\gamma(z)y}\left(\int_{-\infty}^{y}\mathrm{e}^{(\gamma(z)-\alpha)\zeta}V_{BS}(\bar{T},\bar{S}\mathrm{e}^{\zeta})d\zeta + \tilde{C}_1(z)\right)dy + \tilde{C}(z). \tag{2.20}$$

Noting $\int_{-\infty}^{x}\mathrm{e}^{-2\gamma(z)y}dy$ is not integrable for any real number $-\infty < x \leqslant 0$, we set $\tilde{C}_1(z) \equiv 0$. If $\widehat{W}_1(z,\bar{x}) = C(z,\bar{x},\bar{T})\mathrm{e}^{\gamma(z)\bar{x}} = \widehat{W}_1^*(z)$, from (2.20) we can easily obtain $\tilde{C}(z) = \widehat{W}_1^*(z) - \phi(z,\bar{x})$ and thus $C(z,x,\bar{T}) = \widehat{W}_1^*(z) + \phi(z,x,\bar{T}) - \phi(z,\bar{x},\bar{T})$.

From the expression of $C(z,x,\bar{T})$ in (2.13), we have $\frac{\partial C(z,x,\bar{T})}{\partial x} = \frac{\partial}{\partial x}\phi(z,x,\bar{T}) = \psi(z,x,\bar{T})$. Therefore,

$$\left.\frac{\partial \widehat{W}_1(z,x)}{\partial x}\right|_{x=\bar{x}} = \psi(z,\bar{x},\bar{T})\mathrm{e}^{\gamma(z)\bar{x}} + \gamma(z)C(z,\bar{x},\bar{T})\mathrm{e}^{\gamma(z)\bar{x}} = \psi(z,\bar{x},\bar{T}) + \gamma(z)\widehat{W}_1^*(z),$$

which is just the statement (2.16).

From the definition of $\gamma(z)$, we have $\gamma(z) = \frac{1}{\sigma}\sqrt{2(z(\zeta)-\beta)} = \frac{\sqrt{2\mu}}{\sigma}(\mathrm{i}\zeta+1)$, which implies

$$\mathrm{Re}(\gamma(z)) + \alpha = \frac{\sqrt{2\mu}}{\sigma} + \alpha > \frac{\sqrt{2}}{\sigma}\frac{|\alpha|\sigma}{\sqrt{2}} + \alpha = |\alpha| + \alpha > 0. \tag{2.21}$$

Since $\lim_{x\to-\infty}|C(z,x,\bar{T})| = |\widehat{W}_1^*(z) - \phi(z,\bar{x},\bar{T})| < \infty$ and $|\mathrm{e}^{\mathrm{Im}(\gamma(z))x\mathrm{i}}| = 1$, we have

$$\begin{aligned}\lim_{x\to-\infty}|\mathrm{e}^{\alpha x}\widehat{W}_1(z,x)| &= \lim_{x\to-\infty}|C(z,x,\bar{T})\mathrm{e}^{\alpha x+\gamma(z)x}|\\ &= \lim_{x\to-\infty}|C(z,x,\bar{T})|\lim_{x\to-\infty}\mathrm{e}^{(\alpha+\mathrm{Re}(\gamma(z)))x}\lim_{x\to-\infty}|\mathrm{e}^{\mathrm{Im}(\gamma(z))x\mathrm{i}}|\\ &= |\widehat{W}_1^*(z) - \phi(z,\bar{x},\bar{T})|\lim_{x\to-\infty}\mathrm{e}^{(\alpha+\mathrm{Re}(\gamma(z)))x} = 0,\end{aligned}$$

which is just the statement of (2.18). The last equality is obtained by applying the estimation (2.21). The proof is thus completed. □

Remark 2. From Lemma 1, we take $\mu > \frac{1}{2}\alpha^2\sigma^2$ to ensure $\lim_{x\to-\infty}\mathrm{e}^{\alpha x}\widehat{W}_1(z,x) = 0$. To this purpose, it is sufficient to set

$$L > \frac{2}{\pi}\alpha^2\sigma^2\tau_1\sqrt{8\Lambda+1} = \frac{2\tau_1}{\pi\sigma^2}\left(\frac{\sigma^2}{2}+\delta-r\right)^2\sqrt{8\Lambda+1} \tag{2.22}$$

in (2.40) of the numerical Laplace inversion. In other words, how large L should be taken depends on the parameter σ, δ, r, τ_1 and Λ. In practical computation, the above restriction is not strong because the quantity of $\alpha^2\sigma^2$ is always small. For example, under the choices of parameters $r = 0.05$, $\tau_1 = 1$, $\Lambda = 3$ and $\sigma = 0.3$, $\sigma = 0.5$, $\sigma = 0.8$, it should be satisfied that $L > 0.0028$, 0.2250, 1.1391, respectively.

We now compute the solution $\widehat{W}_2(z,x,\eta)$ to the PDE (2.9). First of all, we need to carefully deal with the conditions $\lim_{x\uparrow\infty}\mathrm{e}^{\alpha x+\beta\eta}\widehat{W}_2(z,x,\eta) = 0$ and $\mathrm{e}^{\alpha x}\widehat{W}_2(z,x,0) = 0$. To this end, we divide the problem into two cases: $\alpha \geqslant 0$ and $\alpha < 0$.

Case 1. $\alpha \geqslant 0$.

For this case, we cannot find the exact solution to the PDE (2.9) on the semi-infinite domain $[\bar{x},\infty)$; instead we find the approximate solution on the truncated domain $[\bar{x},\bar{X}]$ with $\bar{X}$ being taken as a large enough positive number such that $\widehat{W}_2(z,\bar{X},\eta) \approx 0$, i.e., the PDE (2.9) is re-written approximately as

$$\begin{cases}\dfrac{\partial\widehat{W}_2}{\partial\eta} - \dfrac{1}{2}\sigma^2\dfrac{\partial^2\widehat{W}_2}{\partial x^2} = 0, & \bar{x} < x < \bar{X}, \quad 0 < \eta \leqslant \bar{T},\\ \widehat{W}_2(z,\bar{x},\eta) = f(z,\eta)\widehat{W}_1^*(z) + g(z,\eta), & 0 < \eta \leqslant \bar{T},\\ \widehat{W}_2(z,\bar{X},\eta) = 0, & 0 < \eta \leqslant \bar{T},\\ \widehat{W}_2(z,x,0) = 0, & \bar{x} < x < \bar{X}.\end{cases} \tag{2.23}$$

To solve (2.23), we define a new function $\overline{W}_2(z,x,\eta)$ such that it is governed by a corresponding PDE with homogeneous boundary conditions. Let

$$\overline{W}_2(z,x,\eta) = \widehat{W}_2(z,x,\eta) + F(z,x,\eta), \quad F(z,x,\eta) = -\left(1-\frac{x}{\bar{X}}\right)[f(z,\eta)W_1^*(z) + g(z,\eta)].$$

Then we have a new PDE with zero boundary conditions:

$$\begin{cases}\dfrac{\partial\overline{W}_2}{\partial\eta} - \dfrac{1}{2}\sigma^2\dfrac{\partial^2\overline{W}_2}{\partial x^2} = \dfrac{\partial F}{\partial\eta},\\ \overline{W}_2(z,\bar{x},\eta) = 0,\\ \overline{W}_2(z,\bar{X},\eta) = 0,\\ \overline{W}_2(z,x,0) = F(z,x,0).\end{cases} \tag{2.24}$$

The PDE (2.24) can be solved analytically by using the standard variable separation techniques (see [7]) and then the solution $\widehat{W}_2(z,x,\eta)$ is expressed as follows:

$$\widehat{W}_2(z,x,\eta) = G^f(z,x,\eta)\widehat{W}_1^*(z) + G^g(z,x,\eta) \tag{2.25}$$

and

$$\left.\frac{\partial\widehat{W}_2(z,x,\bar{T})}{\partial x}\right|_{x=\bar{x}} = A^f(z,\bar{T})\widehat{W}_1^*(z) + A^g(z,\bar{T}), \tag{2.26}$$

where

$$G^{\varphi}(z,x,\eta) = -\frac{2}{\pi}\varphi(z,0)\rho(x,\eta;0) - \frac{2}{\pi}\int_0^{\eta}\rho(x,\eta-\zeta;0)\frac{\partial\varphi(z,\zeta)}{\partial\zeta}d\zeta + \left(1-\frac{x}{\bar{X}}\right)\varphi(z,\eta), \tag{2.27}$$

$$A^{\varphi}(z,\eta) = -\frac{2}{\bar{X}}\varphi(z,0)\theta(\eta;0) - \frac{2}{\bar{X}}\int_0^{\eta}\theta(\eta-\zeta;0)\frac{\partial\varphi(z,\zeta)}{\partial\zeta}d\zeta - \frac{1}{\bar{X}}\varphi(z,\eta) \tag{2.28}$$

for φ equal to f or g. Here, $\rho(x,\eta;m)$ and $\theta(\eta;m)$ are defined by

$$\rho(x,\eta;m) = \sum_{n=1}^{\infty}\left(\frac{1}{q^m n^{m+1}}\mathrm{e}^{-q^2n^2\eta}\sin\frac{n\pi}{\bar{X}}x\right), \quad m=0,1,2, \tag{2.29}$$

$$\theta(\eta;m) = \sum_{n=1}^{\infty}\left(\frac{1}{q^m n^m}\mathrm{e}^{-q^2n^2\eta}\right), \quad m=0,1,2, \tag{2.30}$$

where $q = \frac{\pi\sigma}{\sqrt{2}\bar{X}}$. (2.27) and (2.28) with φ equal to f or g are expressed by the infinite series (2.29) and (2.30). The convergence of the expression is given by the following lemma.

Lemma 3. *For any complex value z and real number $0 < \eta \leqslant \bar{T}$, $G^f(z,\eta)$, $G^g(z,\eta)$, $A^f(z,\eta)$ and $A^g(z,\eta)$ are convergent for $\bar{x} < x < \bar{X}$.*

Proof. From the notations G^{φ} and A^{φ} (see (2.27) and (2.28)), it is sufficient to prove the convergence of $\rho(x,\eta;m)$, $\theta(\eta;m)$ and their integrals with respect to η. In fact, we have

$$|\rho(x,\eta;m)| \leqslant \sum_{n=1}^{\infty}\left|\frac{1}{q^m n^{m+1}}\mathrm{e}^{-q^2n^2\eta}\sin\frac{n\pi}{\bar{X}}x\right| \leqslant \begin{cases} \displaystyle\sum_{n=1}^{\infty}\mathrm{e}^{-q^2n^2\eta} < \frac{\mathrm{e}^{-q^2\eta}}{(1-\mathrm{e}^{-q^2\eta})} < \infty, & m=0, \\ \displaystyle\frac{1}{q^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6q^2}, & m=2 \end{cases}$$

and

$$\int_0^{\eta}|\rho(x,\eta-\zeta;0)|d\zeta \leqslant \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\eta}\mathrm{e}^{-q^2n^2(\eta-\zeta)}d\zeta \leqslant \frac{\pi^2}{6q^2}$$

for $x \in (\bar{x},\bar{X})$, $\eta \in (0,\bar{T}]$. It is similar to prove the convergence of $\theta(\eta;m)$ defined by (2.30) as

$$|\theta(\eta;m)| \leqslant \begin{cases} \displaystyle\frac{\mathrm{e}^{-q^2\eta}}{(1-\mathrm{e}^{-q^2\eta})}, & m=0, \\ \displaystyle\frac{\pi^2}{6q^2}, & m=2 \end{cases} \quad \text{and} \quad \int_0^{\eta}|\theta(\zeta;0)|d\zeta \leqslant \frac{\pi^2}{6q^2}$$

for $x \in (\bar{x},\bar{X})$, $\eta \in (0,\bar{T}]$. □

Case 2. $\alpha < 0$.

For this case, the PDE (2.9) is equivalent to

$$\begin{cases} \dfrac{\partial\widehat{W}_2}{\partial\eta} - \dfrac{1}{2}\sigma^2\dfrac{\partial^2\widehat{W}_2}{\partial x^2} = 0, & \bar{x} < x < \infty, \quad 0 < \eta \leqslant \bar{T}, \\ \widehat{W}_2(z,\bar{x},\eta) = f(z,\eta)\widehat{W}_1^*(z) + g(z,\eta), & 0 < \eta \leqslant \bar{T}, \\ \widehat{W}_2(z,x,0) = 0, & \bar{x} < x < \infty. \end{cases} \tag{2.31}$$

The solution to the PDE (2.31) can be formulated by Green's formula (see [7]) as

$$\widehat{W}_2(z,x,\eta) = \widetilde{G}^f(z,x,\eta)\widehat{W}_1^*(z) + \widetilde{G}^g(z,x,\eta), \tag{2.32}$$

where

$$\widetilde{G}^{\varphi}(z,x,\eta) = \varphi(z,\eta) - \varphi(z,0)\int_{\bar{x}}^{\infty}\widetilde{G}(x,\eta;\zeta,0)d\zeta - \int_0^{\eta}\left[\frac{\partial\varphi(z,y)}{\partial y}\int_{\bar{x}}^{\infty}\widetilde{G}(x,\eta;\zeta,y)d\zeta\right]dy \tag{2.33}$$

for φ equal to f or g, and

$$\widetilde{G}(x,\eta;\zeta,y)=\Gamma(x,\eta;\zeta,y)-\Gamma(x,\eta;-\zeta,y), \tag{2.34}$$

$$\Gamma(x,\eta;\zeta,y)=\begin{cases}\dfrac{1}{\sigma\sqrt{2\pi(\eta-y)}}\mathrm{e}^{-(x-\zeta)^2/(2\sigma^2(\eta-y))}, & \eta>y,\\ 0, & \eta\leqslant y.\end{cases} \tag{2.35}$$

We are now ready to present the formula for the value functions in the Laplace space in the following theorem.

Theorem 4. *Assume $\alpha\geqslant 0$, $\mu>\frac{1}{2}\alpha^2\sigma^2$ and z falls on a parabola Γ defined by (2.17). Let $\gamma(z)=\frac{1}{\sigma}\sqrt{2(z-\beta)}$ with $\mathrm{Re}(\sqrt{2(z-\beta)})>0$. Assume that $\widehat{W}_1(z,x)$ and $\widehat{W}_2(z,x,\eta)$ solve the coupled PDE systems (2.8) and (2.9), respectively. Then $\widehat{W}_1(z,\bar{x})=\widehat{W}_1^*(z)$ has the analytical formula*

$$\widehat{W}_1^*(z)=\frac{\psi(z,\bar{x},\bar{T})-\mathrm{e}^{\beta\bar{T}}A^g(z,\bar{T})}{\mathrm{e}^{\beta\bar{T}}A^f(z,\bar{T})-\gamma(z)} \tag{2.36}$$

with A^f and A^g defined by (2.28).

Moreover, $\widehat{W}_1(z,x)$ can be expressed as $\widehat{W}_1(z,x)=C(z,x,\bar{T})\mathrm{e}^{\gamma(z)x}$ with $C(z,x,\bar{T})=\widehat{W}^*(z)+\phi(z,x,\bar{T})-\phi(z,\bar{x},\bar{T})$ and $\widehat{W}_2(z,x,\eta)$ can be computed by formula (2.25), i.e., $\widehat{W}_2(z,x,\eta)=G^f(z,x,\eta)\widehat{W}_1^*(z)+G^g(z,x,\eta)$ with G^f and G^g defined by (2.27).

Proof. From the smooth pasting condition $\frac{\partial}{\partial x}\widehat{W}_1(z,\bar{x})=\mathrm{e}^{\beta\bar{T}}\frac{\partial}{\partial x}\widehat{W}_2(z,\bar{x},\bar{T})$ (see (2.10)), and the formulas (2.16) and (2.26), we have $\psi(z,\bar{x},\bar{T})+\gamma(z)\widehat{W}_1^*(z)=\mathrm{e}^{\beta\bar{T}}(A^f(z,\bar{T})\widehat{W}_1^*(z)+A^g(z,\bar{T}))$, which leads to (2.36). The remainder has been illustrated previously. □

Remark 5. For the case where $\alpha<0$, it only needs to take the following substitution:

$$G^f(z,x,\eta)=\widetilde{G}^f(z,x,\eta),\quad G^g(z,x,\eta)=\widetilde{G}^g(z,x,\eta)$$

and

$$A^f(z,\eta)=\frac{\partial\widetilde{G}^f(z,x,\eta)}{\partial x}\bigg|_{x=\bar{x}},\quad A^g(z,\eta)=\frac{\partial\widetilde{G}^g(z,x,\eta)}{\partial x}\bigg|_{x=\bar{x}}$$

in Theorem 4, since $\widehat{W}_2(z,x,\eta)$ can be calculated by Green's formula (2.32). Functions $\widetilde{G}^f(z,x,\eta)$ and $\widetilde{G}^g(z,x,\eta)$ are defined by (2.33).

We now discuss the Laplace inversion. The method for inverting the Laplace transform is based on numerical integration of the Bromwich complex contour integral

$$W(\tau;x,\xi)=\frac{1}{2\pi\mathrm{i}}\int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty}\mathrm{e}^{z\tau}\widehat{W}(z;x,\xi)dz,\quad s>s_0. \tag{2.37}$$

Here, $\mathrm{i}^2=-1$, $\widehat{W}(z;x,\xi)$ is the value of Parisian or Parasian options in the Laplace time domain $z\in\mathbb{C}$, and s_0 is the convergence abscissa. This means that all the singularities of $\widehat{W}(z;x,\xi)$ lie in the open half-plane $\mathrm{Re}(z)<s$. Since the exponential factor is highly oscillatory on the Bromwich line, $z=s+\mathrm{i}y$, and $\widehat{W}(z;x,\eta)$ typically decays slowly as $|y|\to\infty$, the integral (2.37) is not well suited for numerical integration and the Laplace inversion formulas derived from it are not stable.

Talbot [9] suggested the Bromwich line be deformed into a contour that begins and ends in the left half-plane and $\mathrm{Re}(z)\to-\infty$ at each end. Talbot's original contour has a cotangent shape, and is rather complicated to analyze. Two simpler types of contours have been proposed, recently, namely parabolas and hyperbolas (see [9,10]). For pricing Parisian and Parasian options, the parabola is chosen by (2.17) on which the ODE (2.8) has the analytical form of the solution (see Lemma 1). On the contour (2.17) the Bromwich integral (2.37) becomes

$$W(\tau;x,\xi)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{z(\zeta)\tau}\widehat{W}(z(\zeta);x,\xi)z'(\zeta)d\zeta. \tag{2.38}$$

In Sections 2 and 3, we will analyze how to choose μ such that the integrand in (2.38) exponentially decays as $\mathrm{Re}(z) \to -\infty$. Therefore, the integral (2.38) is approximated by the trapezoidal rule with the uniform mesh size h, as $W_{h,L}(\tau; x, \xi) \approx W(\tau; x, \xi)$, where

$$W_{h,L}(\tau; x, \xi) = \frac{h}{2\pi \mathrm{i}} \sum_{k=-L}^{L} \mathrm{e}^{z(\zeta_k)\tau} \widehat{W}(z(\zeta_k); x, \xi) z'(\zeta_k), \quad \zeta_k = kh. \tag{2.39}$$

The quantity

$$\mathrm{DE} \equiv \|W(\tau; x, \xi) - W_{h,L}(\tau; x, \xi)\|_\infty$$

is often referred to as the discretization error, whereas

$$\mathrm{TE} \equiv \|W_{h,L=\infty}(\tau; x, \xi) - W_{h,L}(\tau; x, \xi)\|_\infty$$

is the truncation error. By matching the discretization error (DE) and truncation error (TE), appropriate parameters h and μ should be determined. If we wish to maintain a small absolute error on an interval $\tau_0 < \tau < \tau_1$, the parameters h and μ are selected by

$$h = \frac{1}{L}\sqrt{8\Lambda + 1}, \quad \mu = \frac{\pi}{4\sqrt{8\Lambda + 1}} \frac{L}{\tau_1}, \quad \Lambda = \frac{\tau_1}{\tau_0}. \tag{2.40}$$

Under these choices of parameters, the corresponding convergence rate of the Laplace inversion is

$$E_L = \mathcal{O}(\mathrm{e}^{-(2\pi/\sqrt{8\Lambda+1})L}), \quad L \to \infty. \tag{2.41}$$

The Parisian and Parasian options require L in (2.40) to satisfy a certain condition (see (2.22)).

After getting $\widehat{W}_1(z, x)$ and $\widehat{W}_2(z, x, \eta)$, the original option price can be restored by using the numerical Laplace inversion formula (2.39). Since both $\widehat{W}_1(z, x)$ and $\widehat{W}_2(z, x, \eta)$ are analytical on the parabola Γ and their values are zeros at $x = -\infty$ and $x = \bar{X}$, they should be bounded for any $z \in \Gamma$. A lot of numerical examples show that $|\widehat{W}_1^*(z)|$, $|\widehat{W}_1(z, x)|$ and $|\widehat{W}_2(z, x, \eta)|$ are uniformly bounded with respect to z. Figure 2 plots the trajectories of Parisian option solutions in the Laplace domain, from which we see that the option prices tend to zero as $\mathrm{Re}(z) \to -\infty$. The uniform boundedness of option solutions $\widehat{W}_1(z, x)$ and $\widehat{W}_2(z, x, \eta)$ ensures that the integrand (2.38) decays as $\mathrm{Re}(z) \to -\infty$, so the Laplace inversion formula (2.39) has the exponential convergence rate with respect to the number of Laplace parameters. Therefore, this kind of Laplace inversion is stable and fast.

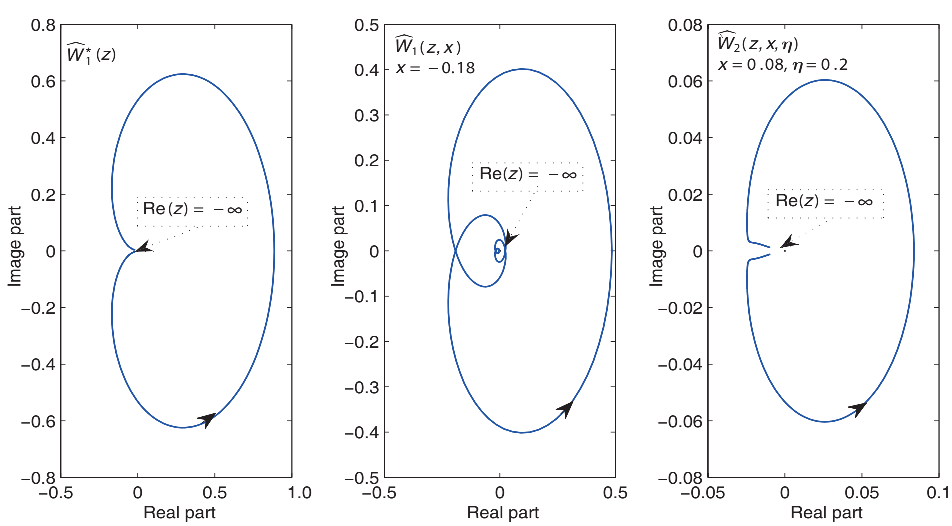


Figure 2 (Color online) Trajectories of Parisian option solutions in the Laplace time domain with parameters $\bar{T} = 0.2$, $\sigma = 0.1$, $r = 0.05$, $K = 10$, $\bar{S} = 12$, $\delta = 0$ and $\bar{X} = 1$

In the end of this section, we give the computational procedure of LTMs for solving Parisian option PDE systems as Algorithm 1.

Algorithm 1 LTMs for solving Parisian options

Step 1. Compute $\widehat{W}_1^*(z)$ by using the formula (2.36).
Step 2. If $x < \bar{x}$, compute $\widehat{W}_1(z,x) = C(z,x,\bar{T})e^{\gamma(z)x}$ for any z falling on the parabola Γ with $C(z,x,\bar{T})$ being defined by (2.13).
Step 3. If $x > \bar{x}$, compute $\widehat{W}_2(z,x,\eta)$ by using the formula (2.25).
Step 4. Apply the Laplace inversion formula (2.39) to restore option values $W_1(\tau,x)$ and $W_2(\tau,x,\eta)$, respectively.
Step 5. Output original option values by transforming $V_1(t,x) = e^{\alpha x}W_1(\tau,x)$ and $V_2(t,x,\xi) = e^{\alpha x+\beta\eta}W_2(\tau,x,\eta)$ with $t = \tau + \bar{T} - \xi$ and $\eta = \bar{T} - \xi$.

3 LTMs for pricing Parasian options

Let $t = \tau + \bar{T} - \xi$, $e^{\alpha x}W_1(\tau,x,\eta) = V_1(t,x,\xi)$ and $e^{\alpha x+\beta\eta}W_2(\tau,x,\eta) = V_2(t,x,\xi)$, where α,β and η are defined by (2.1). Then

$$\begin{cases} \dfrac{\partial V_1}{\partial t} = e^{\alpha x}\dfrac{\partial W_1}{\partial \tau}, \quad \mathcal{A}V_1 = \mathcal{A}[e^{\alpha x}W_1(\tau,x,\eta)], \quad \mathcal{A}V_2 = \mathcal{A}[e^{\alpha x+\beta\eta}W_2(\tau,x,\eta)], \\ \dfrac{\partial V_2}{\partial t} = e^{\alpha x+\beta\eta}\dfrac{\partial W_2}{\partial \tau}, \quad \dfrac{\partial V_2}{\partial \xi} = -\beta e^{\alpha x+\beta\eta}W_2(\tau,x,\eta) + e^{\alpha x+\beta\eta}\left[\dfrac{\partial W_2}{\partial \tau} - \dfrac{\partial W_2}{\partial \eta}\right]. \end{cases} \tag{3.1}$$

Substituting (3.1) into (1.3) and (1.4), we get the following PDE systems:

$$\begin{cases} \dfrac{\partial W_1}{\partial \tau} = \dfrac{1}{2}\sigma^2\dfrac{\partial^2 W_1}{\partial x^2} + \beta W_1, & 0<\tau<\infty, \quad -\infty<x<\bar{x}, \quad 0<\eta\leqslant\bar{T}, \\ W_1(0,x,\eta) = e^{-\alpha x}V_{BS}(\eta,\bar{S}e^x), & -\infty<x<\bar{x}, \quad 0<\eta\leqslant\bar{T}, \\ W_1(\tau,\bar{x},\eta) = e^{\beta\eta}W_2(\tau,\bar{x},\eta), & 0<\tau<\infty, \quad 0<\eta\leqslant\bar{T}, \\ \lim\limits_{x\uparrow\bar{x}}\dfrac{\partial W_1(\tau,x,\eta)}{\partial x} = e^{\beta\eta}\lim\limits_{x\downarrow\bar{x}}\dfrac{\partial W_2(\tau,x,\eta)}{\partial x}, & 0<\tau<\infty, \quad 0<\eta\leqslant\bar{T}, \\ \lim\limits_{x\downarrow-\infty} e^{\alpha x}W_1(\tau,x,\eta) = 0, & 0<\tau<\infty, \quad 0<\eta\leqslant\bar{T}, \\ e^{\alpha x}W_1(\tau,x,0) = 0, & 0<\tau<\infty, \quad -\infty<x<\bar{x} \end{cases} \tag{3.2}$$

and

$$\begin{cases} \dfrac{\partial W_2}{\partial \eta} = \dfrac{1}{2}\sigma^2\dfrac{\partial^2 W_2}{\partial x^2}, & 0<\tau<\infty, \quad \bar{x}<x<\infty, \quad 0<\eta\leqslant\bar{T}, \\ W_2(0,x,\eta) = e^{-\alpha x-\beta\eta}V_{BS}(\eta,\bar{S}e^x), & \bar{x}<x<\infty, \quad 0<\eta\leqslant\bar{T}, \\ W_2(\tau,\bar{x},\eta) = e^{-\beta\eta}W_1(\tau,\bar{x},\eta), & 0<\tau<\infty, \quad 0<\eta\leqslant\bar{T}, \\ \lim\limits_{x\downarrow\bar{x}}\dfrac{\partial W_2(\tau,x,\eta)}{\partial x} = e^{-\beta\eta}\lim\limits_{x\uparrow\bar{x}}\dfrac{\partial W_1(\tau,x,\eta)}{\partial x}, & 0<\tau<\infty, \quad 0<\eta\leqslant\bar{T}, \\ \lim\limits_{x\uparrow+\infty} e^{\alpha x+\beta\eta}W_2(\tau,x,\eta) = 0, & 0<\tau<\infty, \quad 0<\eta\leqslant\bar{T}, \\ e^{\alpha x}W_2(\tau,x,0) = 0, & 0<\tau<\infty, \quad \bar{x}<x<\infty. \end{cases} \tag{3.3}$$

Taking Laplace transforms to both sides of (3.2) and (3.3), respectively, we have an ODE

$$\begin{cases} (z-\beta)\widehat{W}_1 - \dfrac{1}{2}\sigma^2\dfrac{\partial^2\widehat{W}_1}{\partial x^2} = e^{-\alpha x}V_{BS}(\eta,\bar{S}e^x), & -\infty<x<\bar{x}, \quad 0<\eta\leqslant\bar{T}, \\ \widehat{W}_1(z,\bar{x},\eta) = e^{\beta\eta}\widehat{W}_2(z,\bar{x},\eta), & 0<\eta\leqslant\bar{T}, \\ \lim\limits_{x\downarrow-\infty} e^{\alpha x}\widehat{W}_1(z,x,\eta) = 0, & 0<\eta\leqslant\bar{T} \end{cases} \tag{3.4}$$

and a PDE

$$\begin{cases}\dfrac{\partial\widehat{W}_2}{\partial\eta}-\dfrac{1}{2}\sigma^2\dfrac{\partial^2\widehat{W}_2}{\partial x^2}=0, & \bar{x}<x<\infty,\quad 0<\eta\leqslant\bar{T},\\ \widehat{W}_2(z,\bar{x},\eta)=\mathrm{e}^{-\beta\eta}\widehat{W}_1(z,\bar{x},\eta), & 0<\eta\leqslant\bar{T},\\ \lim\limits_{x\to\infty}\mathrm{e}^{\alpha x+\beta\eta}\widehat{W}_2(z,x,\eta)=0, & 0<\eta\leqslant\bar{T},\\ \mathrm{e}^{\alpha x}\widehat{W}_2(z,x,0)=0, & \bar{x}<x<\infty.\end{cases}\tag{3.5}$$

Additionally, the "smooth pasting condition" at barrier $\bar{x}$ is set as

$$\lim_{x\downarrow\bar{x}}\frac{\partial\widehat{W}_2(z,x,\eta)}{\partial x}=\mathrm{e}^{-\beta\eta}\lim_{x\uparrow\bar{x}}\frac{\partial\widehat{W}_1(z,x,\eta)}{\partial x},\quad 0<\eta\leqslant\bar{T}.\tag{3.6}$$

If we regard $\widehat{W}_1(z,\bar{x},\eta)=\widehat{W}_1^*(z,\eta)$ as a known function, then the PDE systems (3.4) and (3.5) can be analytically solved, separately. Lemma 6 gives the formal solution of $\widehat{W}_1(z,x,\eta)$.

Lemma 6. *Let* $\gamma(z)=\frac{1}{\sigma}\sqrt{2(z-\beta)}$ *with* $\mathrm{Re}(\sqrt{2(z-\beta)})>0$ *and* z *fall on a parabola defined by* (2.17). *Assume that* $\mu>\frac{1}{2}\alpha^2\sigma^2$ *and* $\widehat{W}_1(z,\bar{x},\eta)=\widehat{W}_1^*(z,\eta)$ *are given. Then the analytical solution to the ODE* (3.4) *has the expression* $\widehat{W}_1(z,x,\eta)=C(z,x,\eta)\mathrm{e}^{\gamma(z)x}$*, where*

$$C(z,x,\eta)=\widehat{W}_1^*(z,\eta)+\phi(z,x,\eta)-\phi(z,\bar{x},\eta)\tag{3.7}$$

with $\phi(z,x,\eta)$ *defined by* (2.11)*. Furthermore,*

$$\left.\frac{\partial\widehat{W}_1(z,x,\eta)}{\partial x}\right|_{x=\bar{x}}=\psi(z,\bar{x},\eta)+\gamma(z)\widehat{W}_1^*(z,\eta)\tag{3.8}$$

with $\psi(z,x,\eta)$ *given by* (2.12)*.*

Proof. Since the proof is very similar to that of Lemma 1, we omit the details. □

We derive the analytical solution to (3.5) by two cases: $\alpha<0$ and $\alpha\geqslant 0$.

Case 1. $\alpha<0$.

For this case, (3.5) is equivalent to

$$\begin{cases}\dfrac{\partial\widehat{W}_2}{\partial\eta}-\dfrac{1}{2}\sigma^2\dfrac{\partial^2\widehat{W}_2}{\partial x^2}=0, & \bar{x}<x<\infty,\quad 0<\eta\leqslant\bar{T},\\ \widehat{W}_2(z,\bar{x},\eta)=\mathrm{e}^{-\beta\eta}\widehat{W}_1^*(z,\eta), & 0<\eta\leqslant\bar{T},\\ \widehat{W}_2(z,x,0)=0, & \bar{x}<x<\infty.\end{cases}\tag{3.9}$$

The solution to the PDE (3.9) can be formulated by Green's function, i.e.,

$$\widehat{W}_2(z,x,\eta)=\mathrm{e}^{-\beta\eta}\widehat{W}_1^*(z,\eta)-\int_0^\eta\left[\frac{\partial(\mathrm{e}^{-\beta y}\widehat{W}_1^*(z,y))}{\partial y}\int_{\bar{x}}^\infty\widetilde{G}(x,\eta;\zeta,y)d\zeta\right]dy,\tag{3.10}$$

where $\widetilde{G}(x,\eta;\zeta,y)$ is defined by (2.34).

We now calculate $\widehat{W}_1^*$. To this end, we first calculate

$$\int_{\bar{x}}^\infty\left.\frac{\partial\widetilde{G}(x,\eta;\zeta,y)}{\partial x}\right|_{x=\bar{x}}d\zeta=-\frac{1}{\sigma\sqrt{2\pi(\eta-y)}},$$

and then the derivative

$$\left.\frac{\partial\widehat{W}_2(z,x,\eta)}{\partial x}\right|_{x=\bar{x}}=\int_0^\eta\frac{\partial(\mathrm{e}^{-\beta\tau}\widehat{W}_1^*(z,y))}{\partial y}\frac{1}{\sigma\sqrt{2\pi(\eta-y)}}dy.\tag{3.11}$$

Combining (3.8) and (3.11), we get the following integro-differential equation (denoted by IDE-I):

$$\gamma(z)\widehat{W}_1^*(z,\eta)=\int_0^\eta\frac{\partial(\mathrm{e}^{-\beta y}\widehat{W}_1^*(z,y))}{\partial y}\frac{1}{\sigma\sqrt{2\pi(\eta-y)}}dy-\psi(z,\bar{x},\eta).\tag{3.12}$$

The IDE-I can be solved by the FDMs. After numerically computing $\widehat{W}_1^*(z,\eta)$, we can calculate $\widehat{W}_1(z,x,\eta)$ and $\widehat{W}_2(z,x,\eta)$.

Case 2. $\alpha \geqslant 0$.

Similar to (2.23), we find the approximate solution to (3.5) as follows. For a large positive number $\bar{X}$, the solution to (3.5) is approximated by $\widehat{W}_2(z,x,\eta) = \overline{W}_2(z,x,\eta) - \bar{F}(z,x,\eta)$ with $\overline{W}_2(z,x,\eta)$ solving

$$\begin{cases} \dfrac{\partial \overline{W}_2(z,x,\eta)}{\partial \eta} - \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 \overline{W}_2(z,x,\eta)}{\partial x^2} = \dfrac{\partial \bar{F}}{\partial \eta}, \\ \overline{W}_2(z,\bar{x},\eta) = 0, \\ \overline{W}_2(z,\bar{X},\eta) = 0, \\ \overline{W}_2(z,x,0) = \bar{F}(z,x,0) \end{cases} \tag{3.13}$$

and $\bar{F}(z,x,\eta) = -(1-\frac{x}{\bar{X}})\mathrm{e}^{-\beta\eta}\widehat{W}_1^*(z,\eta)$. The solution to (3.13) is given by the standard Fourier expansion method (see [7]) and then

$$\widehat{W}_2(z,x,\eta) = -\frac{2}{\pi}\int_0^\eta \frac{\partial}{\partial \zeta}(\mathrm{e}^{-\beta\zeta}\widehat{W}_1^*(z,\zeta))\rho(x,\eta-\zeta;0)d\zeta + \left(1-\frac{x}{\bar{X}}\right)\mathrm{e}^{-\beta\eta}\widehat{W}_1^*(z,\eta),$$

where $\rho(x,\eta;0)$ is given by (2.29) and

$$\left.\frac{\widehat{W}_2(z,x,\eta)}{\partial x}\right|_{x=\bar{x}} = -\frac{2}{\bar{X}}\int_0^\eta \left[\frac{\partial}{\partial \zeta}(\mathrm{e}^{-\beta\zeta}\widehat{W}_1^*(z,\zeta))\sum_{n=1}^{\infty}\mathrm{e}^{-q^2n^2(\eta-\zeta)}\right]d\zeta - \frac{1}{\bar{X}}\mathrm{e}^{-\beta\eta}\widehat{W}_1^*(z,\eta). \tag{3.14}$$

Using the "smooth pasting condition" (3.6), we get

$$\begin{aligned} \left.\frac{\widehat{W}_2(z,x,\eta)}{\partial x}\right|_{x=\bar{x}} &= \mathrm{e}^{-\beta\eta}\left.\frac{\partial \widehat{W}_1(z,x,\eta)}{\partial x}\right|_{x=\bar{x}} \\ &= \mathrm{e}^{-\beta\eta}(\psi(z,\bar{x},\eta) + \gamma(z)\widehat{W}_1(z,\bar{x},\eta)) \\ &= \mathrm{e}^{-\beta\eta}(\gamma(z)\widehat{W}_1^*(z,\eta) + \psi(z,\bar{x},\eta)). \end{aligned} \tag{3.15}$$

Combining (3.14) with (3.15) yields the 1-dimensional integro-differential equation (denoted by IDE-II):

$$\left(\gamma(z) + \frac{1}{\bar{X}}\right)\mathrm{e}^{-\beta\eta}\widehat{W}_1^*(z,\eta) = -\frac{2}{\bar{X}}\int_0^\eta \left[\frac{\partial}{\partial \zeta}(\mathrm{e}^{-\beta\zeta}\widehat{W}_1^*(z,\zeta))\theta(\eta-\zeta;0)\right]d\zeta - \psi(z,\bar{x},\eta)\mathrm{e}^{-\beta\eta} \tag{3.16}$$

with $\gamma(z) = \frac{1}{\sigma}\sqrt{2(z-\beta)}$, $q = \frac{\pi\sigma}{\sqrt{2}\bar{X}}$ and $\theta(\eta;0) = \sum_{n=1}^{\infty}\mathrm{e}^{-q^2n^2\eta}$.

The solutions to (3.4) and (3.5) are summarized in the following theorem.

Theorem 7. *Assume that $\mu > \frac{1}{2}\alpha^2\sigma^2$ and z falls on a parabola $\Gamma : z(\zeta) = \beta + \mu(\mathrm{i}\zeta+1)^2$, $-\infty < \zeta < \infty$. Let $\gamma(z) = \frac{1}{\sigma}\sqrt{2(z-\beta)}$ with $\mathrm{Re}(\sqrt{2(z-\beta)}) > 0$, and $\widehat{W}_1(z,x,\eta)$ and $\widehat{W}_2(z,x,\eta)$ solve the coupled PDE systems (3.4)–(3.6). Then we have*

(i) *$\widehat{W}_2^*(z,\eta)$ satisfies IDE-I (3.12) for $\alpha < 0$ and IDE-II (3.16) for $\alpha \geqslant 0$, respectively;*

(ii) *$\widehat{W}_1(z,x,\eta)$ can be expressed as $\widehat{W}_1(z,x,\eta) = C(z,x,\eta)\mathrm{e}^{\gamma(z)x}$, where*

$$C(z,x,\eta) = \widehat{W}_1^*(z,\eta) + \phi(z,x,\eta) - \phi(z,\bar{x},\eta) \tag{3.17}$$

with $\phi(z,x,\eta)$ being defined by (2.11);

(iii) *for $\alpha < 0$, $\widehat{W}_2(z,x,\eta)$ is given by Green's formula (3.10) and for $\alpha \geqslant 0$ by the formula*

$$\widehat{W}_2(z,x,\eta) = -\frac{2}{\pi}\int_0^\eta \frac{\partial}{\partial \zeta}(\mathrm{e}^{-\beta\zeta}\widehat{W}_1^*(z,\zeta))\rho(x,\eta-\zeta;0)d\zeta + \left(1-\frac{x}{\bar{X}}\right)\mathrm{e}^{-\beta\eta}\widehat{W}_1^*(z,\eta) \tag{3.18}$$

with $\rho(x,\eta;0) = \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{e}^{-q^2n^2\eta}\sin\frac{n\pi}{\bar{X}}x$ and $q = \frac{\pi\sigma}{\sqrt{2}\bar{X}}$.

Proof. The proof has been given in the previous lines. □

From Theorem 7, we have a general procedure to calculate Parasian options. Firstly, $\widehat{W}_1^*(z,\eta)$ should be computed by solving IDE-I (3.12) or IDE-II (3.16) numerically. Secondly, $\widehat{W}_1(z,x,\eta)$ and $\widehat{W}_2(z,x,\eta)$ can be calculated by (3.17) and (3.18). Thirdly, applying the inverse Laplace transform we get the values of $W_1(\tau,x,\eta)$ and $W_2(\tau,x,\eta)$. Finally, the option prices are obtained by the variable transformations. In the end of this section, we give the computational procedure of LTMs for solving Parasian options in Algorithm 2.

Algorithm 2 LTMs for solving Parasian options

Step 1. Compute $\widehat{W}_1^*(z,\eta)$ by solving IDE-I (3.12) for $\alpha<0$ and IDE-II (3.16) for $\alpha\geqslant 0$.
Step 2. If $x\leqslant\bar{x}$, compute $\widehat{W}_1(z,x,\eta)=C(z,x,\eta)\mathrm{e}^{\gamma(z)x}$ for any z falling on the parabola with $C(z,x,\eta)$ being defined by (3.7).
Step 3. If $x>\bar{x}$, compute $\widehat{W}_2(z,x,\eta)$ using Green's formula (3.10) for $\alpha<0$ and the integral formula (3.18) for $\alpha\geqslant 0$, respectively.
Step 4. Apply the Laplace inversion formula (2.39) to restore option values $W_1(\tau,x,\eta)$ and $W_2(\tau,x,\eta)$, respectively.
Step 5. Output option values by transforming $V_1(t,x,\xi)=\mathrm{e}^{\alpha x}W_1(\tau,x,\eta)$ and $V_2(t,x,\xi)=\mathrm{e}^{\alpha x+\beta\eta}W_2(\tau,x,\eta)$ with $t=\tau+\bar{T}-\xi$ and $\eta=\bar{T}-\xi$.

4 Numerical examples

In this section, we illustrate the accuracy and efficiency of LTMs for pricing Parisian and Parasian options. The results of LTMs are compared with the implicit finite difference methods (IFDMs), Monte Carlo simulations (MCSs) and the moving window methods (MWMs) in [14]. All the numerical experiments are run in MATLAB 8.0 (R2012b) on a PC with the configuration: Intel(R) Core(TM) i5-4200 CPU @ 2.5 GHz and 4.00 GB RAM. This PC configuration is comparable with that of [14].

For the implicit FDMs, the temporal mesh is given by $\tau_m=m\Delta\tau$, $\Delta\tau=T/N_\tau$, $m=0,1,2,\ldots,N_\tau$, and spatial meshes

$$\begin{cases} x_i'=\bar{x}-(\bar{x}+\bar{X})\left(\dfrac{N_{x'}+1-i}{N_{x'}+1}\right)^\nu, & \Delta x'_i=x'_{i+1}-x'_i, \quad i=0,1,\ldots,N_{x'}+1,\\ x_j''=\bar{x}+(\bar{X}-\bar{x})\left(\dfrac{j}{N_{x''}+1}\right)^\nu, & \Delta x''_j=x''_{j+1}-x''_j, \quad j=0,1,\ldots,N_{x''}+1,\\ \xi_\ell=\ell\Delta\xi, \quad \Delta\xi=\bar{T}/N_\xi, \quad \ell=0,1,2,\ldots,N_\xi, \end{cases}$$

where N_τ, $N_{x'}$, $N_{x''}$ and N_ξ are given positive numbers, and $\nu\geqslant 1$ is a parameter to control the mesh distribution. If $\nu=1$, then $\{x_i'\}$ and $\{x_j''\}$ are uniform meshes, whereas $\nu>1$, and there are more grid points near the barrier price $\bar{x}$. In the computation of this example, the mesh sizes are taken as $\Delta\tau=\Delta\xi=0.0005$ and the non-uniform meshes $\{x_i'\}$ and $\{x_j''\}$ are produced with $N_{x'}=N_{x''}=500$ and $\nu=1.5$. The algorithm of Monte Carlo simulations is referred to Bernard and Boyle [1]. For each Monte Carlo simulation, we set $\Delta t=2.5\times 10^{-6}$ and produce 2,000,000 stock price paths. The option values computed by moving window methods are copied directly from the paper [14]. We set the number $L=5$ for the Laplace inversion formula (2.39) and take the mesh size $\Delta\eta=0.0005$ for solving IDE-I (3.12) and IDE-II (3.16) by using the standard FDMs.

The main workload of the LTMs includes the computing of integrals $\phi(z,x,\eta)$ and $\psi(z,x,\eta)$, and the sums (2.29) and (2.30). We use the MATLAB function "quadgk", an adaptive Gauss-Kronrod quadrature, to numerically compute $\phi(z,x,\eta)$ and $\psi(z,x,\eta)$. To efficiently compute the sums (2.29) and (2.30), we define

$$\Pi_{\rho,m,x,\eta}(k)=\sum_{j=(k-1)N_s+1}^{kN_s}\left(\frac{1}{q^m n^{m+1}}\mathrm{e}^{-q^2n^2\eta}\sin\frac{n\pi}{\bar{X}}x\right), \quad k=1,2,\ldots,$$

$$\Pi_{\theta,m,\eta}(k)=\sum_{j=(k-1)N_s+1}^{kN_s}\left(\frac{1}{q^m n^m}\mathrm{e}^{-q^2n^2\eta}\right), \quad k=1,2,\ldots,$$

with $N_s = 400$. The terms $\Pi_{\rho,m,x,\eta}(k)$ and $\Pi_{\theta,m,\eta}(k)$ mean the k-th summation block (each summation block includes N_s terms) of series $\rho(x,\eta;m)$ and $\theta(\eta;m)$, respectively. If a certain integer k satisfies

$$\frac{|\Pi_{\rho,m,x,\eta}(k)|}{|\sum_{j=1}^{k}\Pi_{\rho,m,x,\eta}(j)|} \leqslant 10^{-8} \quad \text{or} \quad \frac{|\Pi_{\theta,m,\eta}(k)|}{|\sum_{j=1}^{k}\Pi_{\rho,m,\eta}(j)|} \leqslant 10^{-8},$$

then we take the approximation

$$\rho(x,\eta;m) \approx \sum_{j=1}^{k}\Pi_{\rho,m,x,\eta}(j) \quad \text{or} \quad \theta(\eta;m) \approx \sum_{j=1}^{k}\Pi_{\theta,m,\eta}(j).$$

For the convenience of comparison, all the values of Parisian and Parasian options listed in this section are the original prices, for example,

$$V_1\left(t, \log\frac{S}{\bar{S}}\right) = V_1(t,x) = U_1(\tau,x) = \mathrm{e}^{\alpha x}W_1(\tau,x),$$
$$V_2\left(t, \log\frac{S}{\bar{S}}, \xi\right) = V_2(t,x,\xi) = U_2(\tau,x,\xi) = \mathrm{e}^{\alpha x+\beta\eta}W_2(\tau,x,\eta)$$

with $t = \tau + \bar{T} - \xi$, $\xi = \bar{T} - \eta$ and $S = \bar{S}\mathrm{e}^x$.

Example 8. In this example, we calculate the Parisian up-and-out call options across the barrier with $\eta = \bar{T}$ (i.e., $\xi = 0$) with parameters: $\sigma = 0.1$, $r = 0.05$, $\delta = 0$, $\bar{T} = 0.2$, $\bar{S} = 12$ and $K = 10$.

Table 1 presents the values of the Parisian up-and-out call options computed by MWMs, MCSs, IFDM and LTMs at $S = \bar{S}$ and $\eta = \bar{T}$. Numbers in the parentheses are the CPU times (seconds) in Table 1. It is shown that the results by LTMs have little difference to those calculated by other methods and the CPU times of the LTMs are much less than that of FDMs, MCSs and MWMs. Moreover, IFDMs, MCSs and MWMs take more CPU time as the time to maturity τ increases, whereas the CPU time remains unchanged for LTMs. Although MWM is an analytical method for the pricing Parisian-type option, it requires much more computational time when $T \gg \bar{T}$. This is because the number of moving time windows grows larger with T increasing. However the CPU time for LTMs is independent of τ and T. So the LTMs are more efficient than the MWMs in this regard.

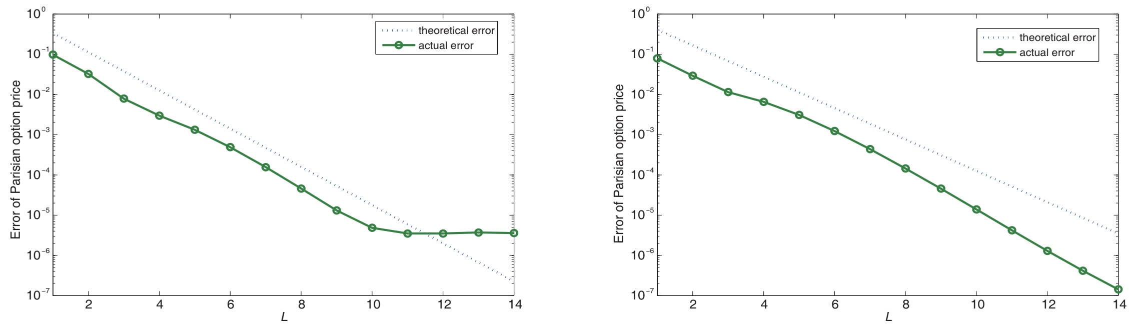
In Figure 3, we plot the errors of the Laplace inversion formula (2.39). Here, the values of Parisian options computed by setting $L = 20$ are regarded as the reference solutions, and the actual errors are measured by L^2 norm. From Figure 3, we see that the actual errors agree well with the theoretical estimation, i.e., the exponential convergence rate (2.41).

Example 9. In this example, we calculate the Parasian up-and-out call options across the barrier with $\eta = \bar{T}$ with the same values of parameters as Example 8.

Table 2 gives the values of the Parasian up-and-out call options at $S = \bar{S}$ and $\eta = \bar{T}$. Table 2 shows that the results of the LTMs agree with those calculated by using IFDMs and MCSs. Also the CPU times listed in Table 2 show that the LTMs spend much less time than the other methods. Also the error plot in Figure 4 exhibits the exponential convergence rate which is consistent with the theoretical result (2.41).

Table 1 The price of the Parisian up-and-out call options

$t = \tau + \bar{T}$	MWM (s)	MCS (s)	IFDM (s)	LTM (s)
0.3	1.3883 (1)	1.3582 (791)	1.3862 (234)	1.3686 (0.7)
0.4	1.0934 (1)	1.0744 (976)	1.1003 (445)	1.0796 (0.6)
0.5	0.8858 (2)	0.8766 (1,209)	0.9005 (682)	0.8825 (0.6)
1.0	0.4296 (3)	0.4212 (1,880)	0.4337 (1,821)	0.4239 (0.6)

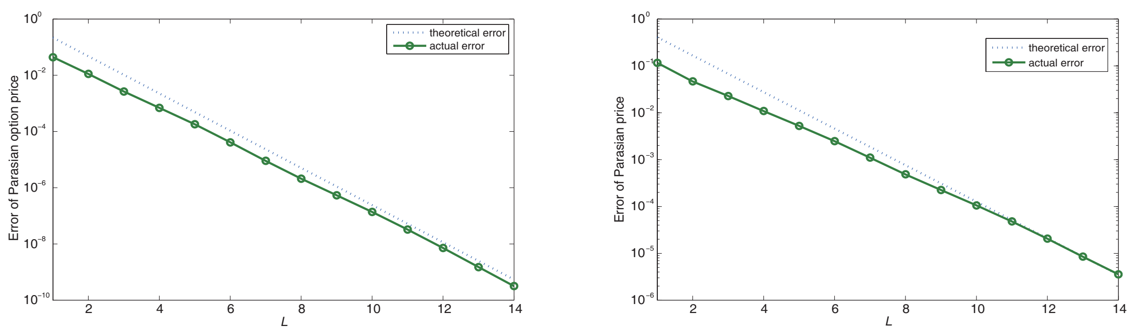


(a) Errors at $\tau = 0.3$ with $\Lambda = \frac{\tau_1}{\tau_0} = \frac{0.4}{0.1} = 4$ (b) Errors at $\tau = 0.4$ with $\Lambda = \frac{\tau_1}{\tau_0} = \frac{0.6}{0.1} = 6$

Figure 3 (Color online) Log-scale plots of the errors for Parisian option values using LTMs. The solid curves represent the actual errors, and the dash-dot lines the theoretical convergence estimate (2.41)

Table 2 The price of the Parasian up-and-out call options

$t = \tau + \bar{T}$	MWM (s)	MCS (s)	IFDM (s)	LTM (s)
0.3	0.9991 (23)	0.9234 (721)	0.9097 (135)	0.9361 (0.9)
0.4	0.7312 (23)	0.6556 (935)	0.6465 (272)	0.6609 (0.9)
0.5	0.5762 (23)	0.4995 (1,202)	0.5055 (408)	0.5085 (0.8)
1.0	0.2462 (23)	0.2198 (2,538)	0.2129 (1,159)	0.2152 (0.8)



(a) Errors at $\tau = 0.2$ with $\Lambda = \frac{\tau_1}{\tau_0} = \frac{0.2}{0.1} = 2$ (b) Errors at $\tau = 0.4$ with $\Lambda = \frac{\tau_1}{\tau_0} = \frac{0.6}{0.1} = 6$

Figure 4 (Color online) Log-scale plots of the errors for Parasian option values using LTMs. The solid curves represent the actual errors, and the dash-dot lines the theoretical convergence estimate (2.41)

5 Concluding remarks

In this paper, the fast Laplace transform methods are derived to solve the complex PDE system arising from Parisian and Parasian option pricing. The key contribution of this paper is that it technically solves the system of ODEs and PDEs derived from the Laplace transform to the original pricing PDE system with respect to the calendar time to maturity. In solving the system of ODEs and PDEs on the Laplace space, the analytical solution to the ODEs is derived on a specific parabola contour which is used in the fast Laplace inversion. Numerical examples show that the Laplace transform methods have exponential convergence rates and are much more efficient than the existing methods in the literature. In the future it will be interesting to extend the Laplace transform methods to the moving window Parisian options which are widely used in the field of convertible bonds in recent years (see [3]).

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