中国科学: 数学　2012 年　第 42 卷　第 11 期: 1079～1093
www.springerlink.com　math.scichina.com
《中国科学》杂志社
SCIENCE CHINA PRESS

Hom-Hopf 代数的对极和 Drinfel'd 偶

王忠伟, 陈园园, 张良云*

南京农业大学理学院, 南京 210095
E-mail: zlyun@njau.edu.cn

收稿日期: 2011-02-28; 接受日期: 2012-04-09; * 通信作者
高等学校博士学科点专项科研基金 (批准号: 20100097110040) 和中央高校基本科研业务费专项资金 (批准号: KYZ201125) 资助项目

摘要　本文证明了带有等价 Hom- 范畴的 Hom- 双代数是 Hom-Hopf 代数, 在基本结构定理成立的前提下, 构造了对极使其成为 Hom-Hopf 代数. 研究了拟三角 Hom-Hopf 代数的对极, 并给出了 Hom-Hopf 代数的 Radford 公式. 最后, 对有限维 Hom-Hopf 代数, 引入了 Drinfel'd 偶的概念, 同时证明了 Drinfel'd 偶是拟三角 Hom-Hopf 代数.

关键词　Hom- 双代数　拟三角 Hom-Hopf 代数　对极　Radford 公式　Drinfel'd 偶

MSC (2010) 主题分类　16W30

0　引言

近年来, Hom- 结构 (Hom-Lie 代数、Hom- 代数、Hom- 余代数、Hom-Hopf 代数、Hom- 模、Hom- 余模和 Hom-Hopf 模) 得到了广泛的研究. 简而言之, Hom- 型结构把之前结构中的恒等映射替换为广义的扭曲映射 α. Hartwig 等人 [1] 引入了 Hom-Lie 代数的概念, 并以此分析了 Witt 和 Virosoro 代数中的某些 q- 分解结构. 同时, Silvestrov [2] 进一步研究了 Hom-Lie 代数, 其思想是用所谓的 Hom-Jacobi 恒等式替换通常的 Jacobi 恒等式, 即,

$$[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0,$$

其中映射 α 为 Lie 代数自同态. 文献 [3] 研究了 Lie 双代数的推广形式: Hom-Lie 双代数, 同时, Chen 等人 [4] 利用 Hom-Lie 代数和 Hom- 余 Lie 代数构造了三角余边缘 Hom-Lie 双代数. Hom- 代数的概念最早是在文献 [5] 中被引入的. 它们不再满足结合性, 其结合律被 Hom- 结合律所替代,

$$\alpha(a)(bc)=(ab)\alpha(c).$$

类似地, Makhlouf 和 Silvestrov [6] 考虑了 Hom- 余代数的 Hom- 余结合律. 进而, Hom- 双代数和 Hom-Hopf 代数的概念也被提出来. Yau [7] 引入了拟三角 Hom-Hopf 代数的概念, 并证明了每个拟三角 Hom-Hopf 代数 (H,α,R) 都是 Hom-Yang-Baxter 方程

$$(R_{13}R_{12})R_{23}=R_{23}(R_{13}R_{12}),\quad R_{13}(R_{12}R_{23})=(R_{23}R_{13})R_{12}$$

英文引用格式: Wang Z W, Chen Y Y, Zhang L Y. The antipode and Drinfel'd double of Hom-Hopf algebras (in Chinese). Sci Sin Math, 2012, 42(11): 1079–1093, doi: 10.1360/012011-138

的解, 其中 $R_{12} = R \otimes 1, R_{23} = 1 \otimes R, R_{13} = (\text{flip} \otimes \text{id})R_{23}$.

Hom- 模, Hom- 余模, Hom-Hopf 模及 Hom- 模代数也在文献 [8,9] 中得到了进一步研究. 同时, 文献 [8] 给出了 Hom-Hopf 模的基本结构定理.

本文的主要目的是利用 Hom- 双代数构造 Hom-Hopf 代数并研究 Hom-Hopf 代数的对极, 同时, 由有限维 Hom-Hopf 代数 (H,α) 构造一个具有拟三角 Hom-Hopf 代数结构的 Drinfel'd 偶

$$D(H) = H^{*\text{cop}} \bowtie H.$$

本文具体安排如下: 第 1 节, 回忆基本概念和结果; 第 2 节, 给出一个 Hom- 双代数成为 Hom-Hopf 代数的充分必要条件, 同时, 由 Hom- 双代数构造 Hom-Hopf 代数; 第 3 节, 研究 Hom-Hopf 代数的对极并给出 Hom-Hopf 代数的 Radford 公式; 第 4 节, 由一个有限维 Hom-Hopf 代数 (H,α) 构造 Drinfel'd 偶: $D(H) = H^{*\text{cop}} \bowtie H$, 并证明 $D(H)$ 是一个拟三角 Hom-Hopf 代数.

1 预备知识

设 k 为域, 所有空间是指域 k 上的线性空间. 对线性空间 M 和 N, 记空间 $M \otimes_k N$ 为 $M \otimes N$.

令 $\mathscr{M}_k$ 为 k- 模范畴. 文献 [8] 定义了一个新的范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$: 对象为二元组 (M,μ), 其中 $M \in \mathscr{M}_k$, $\mu \in \text{Aut}_{\mathscr{M}_k}(M)$ (M 的所有 k- 线性自同构映射构成的集合), 其态射 $f : (M,\mu) \to (N,\nu)$ 满足

$$\nu \circ f = f \circ \mu. \tag{1.1}$$

以后, 称 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 为与 $\mathscr{M}_k$ 相关的 Hom- 范畴. 由文献 [8] 知 $\tilde{\mathcal{H}}(\mathscr{M}_k) = (\tilde{\mathcal{H}}(\mathscr{M}_k), \otimes, (I,I), a, l, r)$ 为 monoidal 范畴: 范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中的对象 (M,μ), (N,ν) 和 (p,π) 的张量积为

$$(M,\mu) \otimes (N,\nu) = (M \otimes N, \mu \otimes \nu), \tag{1.2}$$

结合律和单位约束为

$$a_{M,N,P}((m \otimes n) \otimes p) = \mu(m) \otimes (n \otimes \pi^{-1}(p)),$$
$$l_M(x \otimes m) = r_M(m \otimes x) = x\mu(m).$$

定义 1.1 所谓 Hom- 代数 (见文献 [6,8]) 是指对象 $(H,\alpha) \in \tilde{\mathcal{H}}(\mathscr{M}_k)$, 并且存在 k- 线性映射 $m : H \otimes H \to H$ 和元素 $1_H \in H$, 使得对任意 $h, g, l \in H$,

$$\alpha(hg) = \alpha(h)\alpha(g), \quad \alpha(1_H) = 1_H, \tag{1.3}$$
$$\alpha(h)(gl) = (hg)\alpha(l), \quad h1_H = \alpha(h) = 1_H h, \tag{1.4}$$

这里记 $m(h \otimes g) = hg$.

定义 1.2 所谓 Hom- 余代数 (见文献 [6,8]) 是指对象 $(C,\gamma) \in \tilde{\mathcal{H}}(\mathscr{M}_k)$, 并且存在 k- 线性映射 $\triangle : C \to C \otimes C, \triangle(c) = c_1 \otimes c_2$ 和 $\varepsilon : C \to k$, 使得对任意 $c \in C$,

$$\triangle(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \quad \varepsilon(\gamma(c)) = \varepsilon(c), \tag{1.5}$$
$$\gamma^{-1}(c_1) \otimes \triangle(c_2) = \triangle(c_1) \otimes \gamma^{-1}(c_2), \quad c_1\varepsilon(c_2) = \gamma^{-1}(c) = \varepsilon(c_1)c_2. \tag{1.6}$$

由 (1.6), 我们有

$$c_1 \otimes c_{21} \otimes \gamma(c_{22}) = \gamma(c_{11}) \otimes c_{12} \otimes c_2. \tag{1.6$'$}$$

定义 1.3　设 (C,γ) 为 Hom- 余代数. 若元素 $x \in C$ 满足 $\triangle(x) = x \otimes x$, 则称 x 为 C 中的群象元 (见文献 [10]).

事实上, Hom-(余) 代数为范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中的 (余) 代数.

定义 1.4　所谓 Hom- 双代数 (见文献 [6, 8]) 是指 monoidal 范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中的双代数. 即, (H,α,m,η) 为 Hom- 代数, $(H,\alpha,\triangle,\varepsilon)$ 为 Hom- 余代数使得 $\triangle$ 和 ε 均为 Hom- 代数同态, 也就是说, 对任意 $h, g \in H$,

$$\triangle(hg) = \triangle(h)\triangle(g), \quad \triangle(1_H) = 1_H \otimes 1_H, \tag{1.7}$$

$$\varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad \varepsilon(1_H) = 1_k. \tag{1.8}$$

下面, 回忆文献 [8] 中的 Hom- 模概念.

所谓右 (H,α)-Hom- 模是指由 $(M,\mu) \in \tilde{\mathcal{H}}(\mathscr{M}_k)$ 及范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中的态射 $\psi: M \otimes H \to M, \psi(m \otimes h) = m \cdot h$ 组成, 并且对任意 $h \in H$ 和 $m \in M$ 满足

$$(m \cdot h) \cdot \alpha(g) = \mu(m) \cdot (hg), \tag{1.9}$$

$$m \cdot 1_H = \mu(m). \tag{1.10}$$

而 $\mu \in \tilde{\mathcal{H}}(\mathscr{M}_k)$ 意味着

$$\mu(m \cdot h) = \mu(m) \cdot \alpha(h). \tag{1.11}$$

称范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中的态射 $f: (M,\mu) \to (N,\nu)$ 为右 H- 线性的, 若 f 保持 H- 作用, 即, 满足 $f(m \cdot h) = f(m) \cdot h$. 以下, 用 $\tilde{\mathcal{H}}(\mathscr{M}_k)_H$ 表示右 (H,α)-Hom- 模范畴.

对偶地, 可以定义 Hom- 余模. 设 $C = (C,\gamma)$ 为 Hom- 余代数. 所谓右 (C,γ)-Hom- 余模是指对象 $(M,\mu) \in \tilde{\mathcal{H}}(\mathscr{M}_k)$ 和范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中的 k- 线性映射 $\rho: M \to M \otimes C$, $\rho(m) = m_{(0)} \otimes m_{(1)}$, 使得对任意 $m \in M$ 满足

$$\mu^{-1}(m_{(0)}) \otimes \triangle(m_{(1)}) = m_{(0)(0)} \otimes (m_{(0)(1)} \otimes \gamma^{-1}(m_{(1)})), \tag{1.12}$$

$$m_{(0)}\varepsilon(m_{(1)}) = \mu^{-1}(m). \tag{1.13}$$

而 $\rho \in \tilde{\mathcal{H}}(\mathscr{M}_k)$ 意味着

$$\rho(\mu(m)) = \mu(m_{(0)}) \otimes \gamma(m_{(1)}). \tag{1.14}$$

称范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中的态射 $f: (M,\mu) \to (N,\nu)$ 为右 H- 余线性的, 若 $\rho_N \circ f = (f \otimes \mathrm{id}) \circ \rho_M$. 以下, 用 $\tilde{\mathcal{H}}(\mathscr{M}_k)^C$ 表示右 (C,γ)-Hom- 余模范畴.

定义 1.5　称 Hom- 双代数 (H,α) 为 Hom-Hopf 代数 (见文献 [8]), 若在范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中存在态射 (称之为对极) $S: H \to H$ (也就是, $S \circ \alpha = \alpha \circ S$) 使得

$$S * \mathrm{id} = \eta \circ \varepsilon = \mathrm{id} * S. \tag{1.15}$$

由于 α 是双射且与 S 可交换, 易知 α 的逆映射 α^{-1} 也与 S 可交换, 即 $S \circ \alpha^{-1} = \alpha^{-1} \circ S$.

引理 1.6　设 (H,α) 为 Hom-Hopf 代数, 则由文献 [8] 知, 对任意 $h, g \in H$,

$$S(hg) = S(g)S(h), \quad S(1_H) = 1_H, \tag{1.16}$$

$$\triangle(S(h)) = S(h_2) \otimes S(h_1), \quad \varepsilon \circ S = \varepsilon. \tag{1.17}$$

对极 S 是一个 Hom-(余) 代数反同态.

定义 1.7 称 (M,μ) 为右 (H,α)-Hom-Hopf 模 (见文献 [8]), 若 (M,μ) 既是右 (H,α)-Hom 模, 又是右 (H,α)-Hom 余模, 且对任意 $m \in M$ 和 $h \in H$ 满足如下的相容条件:

$$\rho(m \cdot h) = m_{(0)} \cdot h_1 \otimes m_{(1)} h_2. \tag{1.18}$$

右(H,α)-Hom-Hopf 模之间的态射是一个 k- 线性映射, 同时又是 $\tilde{\mathcal{H}}(\mathscr{M}_k)_H$ 和 $\tilde{\mathcal{H}}(\mathscr{M}_k)^H$ 中的态射.

2 Hom-Hopf 代数的对极构造

本节将证明带有 Hom- 范畴等价的 Hom- 双代数是 Hom-Hopf 代数, 并由 Hom- 双代数构造 Hom-Hopf 代数的对极.

命题 2.1 设 (H,α) 是 Hom- 双代数, 则下列条件等价:

(a) (H,α) 是 Hom-Hopf 代数.

(b) 映射

$$\phi : (H \otimes H, \alpha \otimes \alpha) \to (H \otimes H, \alpha \otimes \alpha), \quad \phi(h \otimes g) = \alpha^{-1}(h) g_1 \otimes \alpha(g_2)$$

是双射.

证明 (a)$\Rightarrow$(b). 设 S 为 H 的对极, 定义

$$\psi : (H \otimes H, \alpha \otimes \alpha) \to (H \otimes H, \alpha \otimes \alpha), \quad \psi(h \otimes g) = \alpha^{-1}(h) S(g_1) \otimes \alpha(g_2),$$

则有

$$\begin{aligned}
\psi \circ \phi(h \otimes g) &= \psi(\alpha^{-1}(h) g_1 \otimes \alpha(g_2)) = (\alpha^{-2}(h)\alpha^{-1}(g_1)) S\alpha(g_{21}) \otimes \alpha^2(g_{22}) \\
&= (\alpha^{-2}(h)\alpha^{-1}(g_1)) \alpha S(g_{21}) \otimes \alpha^2(g_{22}) \overset{(1.4)}{=} \alpha^{-1}(h)(\alpha^{-1}(g_1) S(g_{21})) \otimes \alpha^2(g_{22}) \\
&\overset{(1.6)}{=} \alpha^{-1}(h)(g_{11} S(g_{12})) \otimes \alpha(g_2) \overset{(1.5)}{=} \alpha^{-1}(h)(\varepsilon(g_1) 1_H) \otimes \alpha(g_2) \\
&= \alpha^{-1}(h) 1_H \otimes \alpha(\varepsilon(g_1) g_2) = \alpha(\alpha^{-1}(h)) \otimes \alpha(\alpha^{-1}(g)) = h \otimes g, \\
\phi \circ \psi(h \otimes g) &= \phi(\alpha^{-1}(h) S(g_1) \otimes \alpha(g_2)) = (\alpha^{-2}(h)\alpha^{-1} S(g_1)) \alpha(g_{21}) \otimes \alpha^2(g_{22}) \\
&\overset{(1.4)}{=} \alpha^{-1}(h)(\alpha^{-1} S(g_1) g_{21}) \otimes \alpha^2(g_{22}) = \alpha^{-1}(h)(S(g_{11}) g_{12}) \otimes \alpha(g_2) \\
&\overset{(1.15)}{=} \alpha^{-1}(h) \varepsilon(g_1) 1_H \otimes \alpha(g_2) = h \otimes g,
\end{aligned}$$

故, ψ 为 ϕ 的逆映射.

(b)$\Rightarrow$(a). 考虑 $H \otimes H$ 上的结构,

$\rho : (H \otimes H, \alpha \otimes \alpha) \to (H \otimes H \otimes H, \alpha \otimes \alpha \otimes \alpha), \quad \rho(h \otimes g) = (\alpha^{-1}(h) \otimes g_1) \otimes g_2,$

$\cdot : (H \otimes H \otimes H, \alpha \otimes \alpha \otimes \alpha) \to (H \otimes H, \alpha \otimes \alpha), \quad (h \otimes g) \cdot t = \alpha(h) \otimes gt,$

$\bullet : (H \otimes H \otimes H, \alpha \otimes \alpha \otimes \alpha) \to (H \otimes H, \alpha \otimes \alpha), \quad (h \otimes g) \bullet t = h\alpha(t_1) \otimes g\alpha(t_2).$

通过计算可知, $(H \otimes H, \alpha \otimes \alpha, \cdot, \rho)$ 和 $(H \otimes H, \alpha \otimes \alpha, \bullet, \rho)$ 是右 H-Hom-Hopf 模, 且映射 $\phi : (H \otimes H, \alpha \otimes$

$\alpha, \cdot, \rho) \to (H\otimes H, \alpha\otimes\alpha, \bullet, \rho)$ 是右 H- 线性和右 H- 余线性的. 记 ϕ 的逆映射为 $\psi: H\otimes H \to H\otimes H$, 则 ψ 也是右 H- 线性和右 H- 余线性的. 再由 ϕ 的定义可知: ϕ 与 $\alpha\otimes\alpha$ 是可交换的, 即 $\phi\circ(\alpha\otimes\alpha) = (\alpha\otimes\alpha)\circ\phi$, 从而 ψ 也与 $\alpha\otimes\alpha$ 可交换.

定义

$$S: H \to H, \quad S(h) = i \circ (\mathrm{id}\otimes\varepsilon)\circ\psi(1_H\otimes h),$$

其中 i 是 $H\otimes k$ 到 H 的典范同构映射. 以下证明 S 为 H 的对极.

令

$$\psi(1_H\otimes h) = h^0\otimes h^1,$$

则对任意 $h\in H$, 使用记法

$$S(h) = h^0\varepsilon(h^1).$$

由 ψ 是右 H- 余线性, 得

$$\rho\circ\psi(1_H\otimes h) = (\alpha^{-1}\otimes\triangle)\circ\psi(1_H\otimes h) = (\psi\otimes\mathrm{id})\circ\rho(1_H\otimes h),$$

运用我们的记法, 上式可变为

$$\alpha^{-1}(h^0)\otimes(h^1)_1\otimes(h^1)_2 = (h_1)^0\otimes(h_1)^1\otimes h_2.$$

对上式作用映射$(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\alpha^2)\circ(\alpha^{-1}\otimes\alpha^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\triangle)$, 并根据等式 $\psi\circ(\alpha\otimes\alpha)(1_H\otimes h) = (\alpha\otimes\alpha)\circ\psi(1_H\otimes h)$, 即, $\alpha(h)^0\otimes\alpha(h)^1 = \alpha(h^0)\otimes\alpha(h^1)$, 可得

$$\alpha^{-2}(h^0)\otimes(h^1)_{11}\otimes(h^1)_{12}\otimes\alpha((h^1)_2) = (h_{11})^0\otimes(h_{11})^1\otimes h_{12}\otimes\alpha(h_2).$$

事实上, 一方面,

$$\begin{aligned}
&(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\alpha^2)\circ(\alpha^{-1}\otimes\alpha^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\triangle)(\alpha^{-1}(h^0)\otimes(h^1)_1\otimes(h^1)_2)\\
&\quad= (\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\alpha^2)\circ(\alpha^{-1}\otimes\alpha^{-1}\otimes\mathrm{id}\otimes\mathrm{id})(\alpha^{-1}(h^0)\otimes(h^1)_1\otimes(h^1)_{21}\otimes(h^1)_{22})\\
&\quad= \alpha^{-2}(h^0)\otimes\alpha^{-1}((h^1)_1)\otimes(h^1)_{21}\otimes\alpha^2((h^1)_{22})\\
&\quad\overset{(1.6)}{=} \alpha^{-2}(h^0)\otimes(h^1)_{11}\otimes(h^1)_{12}\otimes\alpha((h^1)_2);
\end{aligned}$$

另一方面,

$$\begin{aligned}
&(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\alpha^2)\circ(\alpha^{-1}\otimes\alpha^{-1}\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\triangle)((h_1)^0\otimes(h_1)^1\otimes h_2)\\
&\quad= (\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\alpha^2)\circ(\alpha^{-1}\otimes\alpha^{-1}\otimes\mathrm{id}\otimes\mathrm{id})((h_1)^0\otimes(h_1)^1\otimes h_{21}\otimes h_{22})\\
&\quad= (\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\alpha^2)(\alpha^{-1}((h_1)^0)\otimes\alpha^{-1}((h_1)^1)\otimes h_{21}\otimes h_{22})\\
&\quad= \alpha^{-1}((h_1)^0)\otimes\alpha^{-1}((h_1)^1)\otimes h_{21}\otimes\alpha^2(h_{22})\\
&\quad= \alpha^{-1}((\alpha(h_{11}))^0)\otimes\alpha^{-1}((\alpha(h_{11}))^1)\otimes h_{12}\otimes\alpha(h_2)\\
&\quad= \alpha^{-1}(\alpha((h_{11})^0))\otimes\alpha^{-1}(\alpha((h_{11})^1))\otimes h_{12}\otimes\alpha(h_2)\\
&\quad= (h_{11})^0\otimes(h_{11})^1\otimes h_{12}\otimes\alpha(h_2).
\end{aligned}$$

由于 ψ 和 ϕ 是互逆的, 所以 $\phi\circ\psi(1_H\otimes h)=1_H\otimes h$, 再运用记法, 我们有

$$\alpha^{-1}(h^0)(h^1)_1\otimes\alpha((h^1)_2)=1_H\otimes h.$$

故由记法 $S(h)=h^0\varepsilon(h^1)$, 得

$$\begin{aligned}S(h_{11})h_{12}\otimes\alpha(h_2)&=(h_{11})^0\varepsilon((h_{11})^1)h_{12}\otimes\alpha(h_2)\\&=\alpha^{-2}(h^0)\varepsilon((h^1)_{11})(h^1)_{12}\otimes\alpha((h^1)_2)\\&=\alpha^{-2}(h^0)\alpha^{-1}((h^1)_1)\otimes\alpha((h^1)_2)\\&=1_H\otimes h.\end{aligned}$$

现在, 将上式作用映射 $\mathrm{id}\otimes\varepsilon$, 得

$$S(h_1)h_2=\varepsilon(h)1_H,\tag{2.1}$$

并且由 $\phi(\alpha S(h_1)\otimes\alpha(h_2))=S(h_1)\alpha(h_{21})\otimes\alpha^2(h_{22})\overset{(1.6)}{=}\alpha S(h_{11})\alpha(h_{12})\otimes\alpha(h_2)\overset{(2.1)}{=}1_H\otimes h$, 得

$$\psi(1_H\otimes h)=\alpha S(h_1)\otimes\alpha(h_2).$$

令

$$\varphi:H\otimes H\to H\otimes H,\quad \varphi(h\otimes g)=\alpha^{-1}(h)S(g_1)\otimes\alpha(g_2),$$

则易证 $\phi\circ\varphi=\mathrm{id}_{H\otimes H}$. 由 ϕ 是双射知 $\varphi=\psi$. 由于 $\psi(1_H\otimes 1_H)=1_H\otimes 1_H$. 并且 ψ 是右 H- 线性的, 所以 $\psi((1\otimes 1)\bullet h)=(1\otimes 1)\cdot h$, 即, $\psi(\alpha^2(h_1)\otimes\alpha^2(h_2))=1\otimes\alpha(h)$. 再由 $\varphi=\psi$ 知

$$1_H\otimes\alpha(h)=\varphi(\alpha^2(h_1)\otimes\alpha^2(h_2))=\alpha(h_1)\alpha^2S(h_{21})\otimes\alpha^3(h_{22}).$$

对上式作用映射 $\alpha^{-1}\otimes\varepsilon$, 可得

$$h_1S(h_2)=\varepsilon(h)1_H.\tag{2.2}$$

由 (2.1) 和 (2.2) 可知: S 为 (H,α) 的对极, 从而 (H,α) 为 Hom-Hopf 代数. □

由命题 2.1 可以得到以下定理, 推广了文献 [11, 定理 1].

定理 2.2 设 (H,α) 是 Hom- 双代数. 假设函子

$$(-)^{\mathrm{co}H}:\tilde{\mathcal{H}}(\mathscr{M}_k)_H^H\to\tilde{\mathcal{H}}(\mathscr{M}_k)$$

是 Hom- 范畴等价的, 则 (H,α) 是 Hom-Hopf 代数.

证明 设 $(M,\mu,\cdot,\rho)$、$(N,\nu,\cdot',\rho')$ 和 $f:(M,\mu)\to(N,\nu)$ 分别是范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)_H^H$ 中的对象和态射. 定义

$$M^{\mathrm{co}H}=\{m\in M\mid\rho(m)=\mu^{-1}(m)\otimes 1_H\},\quad f^{\mathrm{co}H}=f_{|(M^{\mathrm{co}H},\mu_{|_{M^{\mathrm{co}H}}})}.$$

令 $F=(-\otimes H,-\otimes\alpha):\tilde{\mathcal{H}}(\mathscr{M}_k)\to\tilde{\mathcal{H}}(\mathscr{M}_k)_H^H$ 是如下定义的函子: 对于范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中的对象 (N,ν), $(N\otimes H,\nu\otimes\alpha,\cdot,\rho)$ 是范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)_H^H$ 中的对象, 这里 $(n\otimes h)\cdot g=\nu(n)\otimes hg$, $\rho(n\otimes h)=(\nu^{-1}(n)\otimes h_1)\otimes h_2$. 于是诱导函子 F 是余不变函子 $G=(-)^{\mathrm{co}H}:\tilde{\mathcal{H}}(\mathscr{M}_k)_H^H\to\tilde{\mathcal{H}}(\mathscr{M}_k)$ 的右伴随, 参见文 [8, 定理 3.1].

因此, 由假设可知, 函子 F 与 G 是可逆等价的. 从而对任意 $(M,\mu,\cdot,\rho)\in\tilde{\mathcal{H}}(\mathscr{M}_k)_H^H$, 有同构映射

$$\psi_{(M,\mu)}:(M^{\mathrm{co}H}\otimes H,\mu_{|_{M^{\mathrm{co}H}}}\otimes\alpha)\to(M,\mu),\ \psi_{(M,\mu)}(m\otimes h)=m\cdot h.$$

由命题 2.1 的证明知, $(H\otimes H, \alpha\otimes\alpha, \bullet, \rho)$ 是范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)_H^H$ 中的对象. 因为作为右 (H,α)-Hom- 余模有 $(H\otimes H, \alpha\otimes\alpha, \rho)\cong(-\otimes H, -\otimes\alpha)(H,\alpha)$, 所以在范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中, 有 $(H,\alpha)\cong((H\otimes H)^{\mathrm{co}H}, (\alpha\otimes\alpha)_{|(H\otimes H)^{\mathrm{co}H}})$ (同构映射由 $\pi: h\mapsto h\otimes 1_H$ 及伴随给出).

综上, 由 $\psi_{(H\otimes H,\alpha\otimes\alpha)}:((H\otimes H)^{\mathrm{co}H}\otimes H, (\alpha\otimes\alpha)_{|(H\otimes H)^{\mathrm{co}H}}\otimes\alpha)\rightarrow(H\otimes H, \alpha\otimes\alpha)$ 为双射可知, 映射

$$\phi:(H\otimes H, \alpha\otimes\alpha)\rightarrow(H\otimes H, \alpha\otimes\alpha)$$

由形式 $\phi(h\otimes g)=(\alpha^{-1}\otimes\alpha^{-1})\circ\psi_{(H\otimes H,\alpha\otimes\alpha)}\circ(\pi\otimes\mathrm{id}_H)(h\otimes g)=\alpha^{-1}(h)g_1\otimes\alpha(g_2)$ 给出, 也为双射, 从而由命题 2.1 知, (H,α) 是 Hom-Hopf 代数. □

3 拟三角 Hom-Hopf 代数的对极

本节中, 我们主要研究拟三角 Hom-Hopf 代数的对极的平方, 并且给出了 Hom-Hopf 代数的 Radford 公式.

引理 3.1 设 (H,α) 具有对极 S 的 Hom-Hopf 代数, 并对任意 $h\in H$, 记 $\triangle^{\mathrm{cop}}(h)=h_2\otimes h_1$. 定义 H 上的作用: $g\leftarrow h=(S(h_2)\alpha^{-1}(g))\alpha(h_1)$, h, $g\in H$, 则 $(H,\leftarrow,\alpha)$ 为右 (H,α)-Hom- 模.

证明 直接验证. □

注意到

$$\begin{aligned}
S^2(h)\alpha(g) &= (\alpha^{-1}S^2(h)g)1 \overset{(1.6)}{=} (S^2(h_1)g)\varepsilon(h_2)1_H \\
&\overset{(1.15)}{=} (S^2(h_1)g)(\alpha S(h_{21})\alpha(h_{22})) \\
&= (\alpha^{-1}(S^2(h_1)g)\alpha S(h_{21}))\alpha^2(h_{22}) \\
&\overset{(1.3)}{=} ((S^2(\alpha^{-1}(h_1))\alpha^{-1}(g))S(\alpha(h_{21}))\alpha^2(h_{22}) \\
&\overset{(1.6)}{=} ((S^2(h_{11})\alpha^{-1}(g))S(\alpha(h_{12}))\alpha(h_2) \\
&= (g\leftarrow S(h_1))\alpha(h_2),
\end{aligned}$$

故有

$$S^2(h)\alpha(g)=(g\leftarrow S(h_1))\alpha(h_2). \tag{3.1}$$

于是我们有如下结果:

命题 3.2 设 (H,α) 是具有对极 S 的 Hom-Hopf 代数, 并且 $R=R^{(1)}\otimes R^{(2)}\in S(H)\otimes H$ 在 $H\otimes H$ 中存在逆元素, 且对任意 $h\in H$, $\triangle^{\mathrm{cop}}(h)R=R\triangle(h)$, 则元素 $u=S(R^{(2)})R^{(1)}$ 可逆, 且使得对任意 $h\in H$, 有

$$S^2(h)=(\alpha^{-1}(u)\alpha^{-2}(h))u^{-1}.$$

特别地, S 是双射.

证明 首先, 由 $\triangle^{\mathrm{cop}}(h)R=R\triangle(h)$, $h\in H$, 可证

$$R^{-1}\triangle^{cop}(h)=\triangle(h)R^{-1}. \tag{3.2}$$

对任意 $h\in H$, 设元素 $X=X^{(1)}\otimes X^{(2)}\in H\otimes H$ 满足 $\triangle^{\mathrm{cop}}(h)X=X\triangle(h)$, 则等价于

$$h_2X^{(1)}\otimes h_1X^{(2)}=X^{(1)}h_1\otimes X^{(2)}h_2. \tag{3.3}$$

对等式 (3.3) 两边作用映射 $\mathrm{id}\otimes S$, 得 $h_2X^{(1)}\otimes S(X^{(2)})S(h_1)=X^{(1)}h_1\otimes S(h_2)S(X^{(2)})$. 在这种情况下, 有

$$
\begin{aligned}
&(S(X^{(2)})S(h_1))(h_2X^{(1)})=(S(h_2)S(X^{(2)}))(X^{(1)}h_1)\\
&\qquad\Longrightarrow \alpha S(X^{(2)})(S(h_1)\alpha^{-1}(h_2X^{(1)}))=\alpha S(h_2)(S(X^{(2)})\alpha^{-1}(X^{(1)}h_1))\\
&\qquad\Longrightarrow \alpha S(X^{(2)})(\alpha^{-1}(S(h_1)h_2)X^{(1)})=\alpha S(h_2)(\alpha^{-1}(S(X^{(2)})X^{(1)})h_1)\\
&\qquad\Longrightarrow \varepsilon(h)\alpha(S(X^{(2)})X^{(1)})=\alpha S(h_2)(\alpha^{-1}(x)h_1)\quad (\text{记 } x=S(X^{(2)})X^{(1)})\\
&\qquad\Longrightarrow \varepsilon(h)\alpha(x)=(S(h_2)\alpha^{-1}(x))\alpha(h_1).
\end{aligned}
$$

于是, 我们有

$$x\leftarrow h=\alpha(x)\varepsilon(h). \tag{3.4}$$

由式 (3.1) 和 (3.4) 得

$$
\begin{aligned}
S^2(h)\alpha(x)&=(x\leftarrow S(h_1))\alpha(h_2)=\alpha(x)\varepsilon(S(h_1))\alpha(h_2)\\
&=\alpha(x)\alpha(\alpha^{-1}(h))=\alpha(x)h.
\end{aligned}
$$

故

$$S^2(h)x=xh. \tag{3.5}$$

记 $R^{-1}=U^{(1)}\otimes U^{(2)}$. 因 $R^{-1}\triangle^{\mathrm{cop}}(h)=\triangle(h)R^{-1}$, 对任意 $h\in H$, 所以 $U^{(1)}h_2\otimes U^{(2)}h_1=h_1U^{(1)}\otimes h_2U^{(2)}$, 从而 $h_2U^{(2)}\otimes h_1U^{(1)}=U^{(2)}h_1\otimes U^{(1)}h_2$, 故由 (3.5) 得

$$S^2(h)\alpha^{-1}(u)=\alpha^{-1}(u)h,\quad S^2(h)\alpha^{-1}(v)=\alpha^{-1}(v)h. \tag{3.6}$$

由 $U^{(1)}R^{(1)}\otimes U^{(2)}R^{(2)}=1_H\otimes 1_H$, 两边作用映射 $S\otimes\mathrm{id}$ 得 $1_H=(S(R^{(1)})\alpha^{-1}(v))\alpha(R^{(2)})$. 故由 (3.6) 可证 $(S(R^{(1)})S^2(R^{(2)}))v=1_H$, 即,

$$S(u)v=1_H. \tag{3.7}$$

由 (3.7) 知, v 有左逆, 从而 S 是单射. 下证 v 有右逆. 由题假设可知 $u=S(R^{(2)})R^{(1)}\in S(H)$, 从而存在元素 $w\in H$ 使得 $u=S(w)$. 故由等式 (3.5) 和 (3.7) 知 $1_H=S(u)v=S^2(w)v=vw$, 故 v 是可逆的. 由 (3.6) 可证

$$S^2(\alpha^2(h))=(\alpha^{-1}(v)h)v^{-1}.$$

因 $S^2(\alpha^2(h))=(\alpha^{-1}(v)h)v^{-1}$, 故 $S^2(\alpha(v))=(\alpha^{-1}(v)\alpha^{-1}(v))v^{-1}=v\alpha^{-1}(vv^{-1})=\alpha(v)$, 从而 $S^2(v)=v$. 又由 $\alpha^{-1}(v^{-1})(S^2(\alpha^2(h))v)=h$ 知 $h=S((S(v)S\alpha^2(h))\alpha^{-1}(u))$, 即, S 为满射, 从而是双射, 于是 u 是可逆的. 再由 (3.6) 得

$$
\begin{aligned}
(\alpha^{-1}(u)h)u^{-1}&=(S^2(h)\alpha^{-1}(u))u^{-1}\\
&=\alpha(S^2(h))(\alpha^{-1}(u)\alpha^{-1}(u^{-1})\\
&=S^2(\alpha(h))1_H=S^2(\alpha^2(h)),
\end{aligned}
$$

从而

$$S^2(h)=(\alpha^{-1}(u)\alpha^{-2}(h))u^{-1}.$$

命题 3.2 得证. □

定义 3.3 所谓范畴 $\tilde{\mathcal{H}}(\mathscr{M}_k)$ 中的拟三角 Hom-Hopf 代数, 是指四元组 $H=(H,\mu,\eta,\triangle,\varepsilon,\alpha,R)$, 满足 $(H,\mu,\eta,\triangle,\varepsilon,\alpha)$ 为 Hom-Hopf 代数, 且存在元素 $R=R^{(1)}\otimes R^{(2)}\in H\otimes H$ 使得

(QT.1) $\triangle(R^{(1)})\otimes\alpha(R^{(2)})=\alpha(R^{(1)})\otimes\alpha(r^{(1)})\otimes R^{(2)}r^{(2)}$;

(QT.2) $\varepsilon(R^{(1)})R^{(2)}=1_H$;

(QT.3) $\alpha(R^{(1)})\otimes\triangle(R^{(2)})=R^{(1)}r^{(1)}\otimes\alpha(r^{(2)})\otimes\alpha(R^{(2)})$;

(QT.4) $R^{(1)}\varepsilon(R^{(2)})=1_H$;

(QT.5) $\triangle^{\mathrm{cop}}(h)R=R\triangle(h)$.

引理 3.4 设 (H,α) 是具有对极 S 的 Hom-Hopf 代数, 且元素 $R\in H\otimes H$ 满足 (QT.1) 和 (QT.2), 则 R 是可逆的, 其逆元素为 $R^{-1}=S(R^{(1)})\otimes R^{(2)}$.

证明 它的证明是直接的. □

注 3.5 (1) 由命题 3.2 和引理 3.4 可知带有拟三角结构的 Hom-Hopf 代数的对极是双射.

(2) 设 (H,α) 是具有对极 S 的 Hom-Hopf 代数. 若元素 R 可逆且满足 (QT.1), 则 R 也满足 (QT.2). 类似地, 若 R 可逆且满足 (QT.3), 则 R 也满足 (QT.4).

引理 3.6 设 (H,α) 是具有对极 S 的 Hom-Hopf 代数, 元素 $R\in H\otimes H$ 满足 (QT.3) 和 (QT.4), 则 $S(R^{(1)})\otimes R^{(2)}$ 是可逆的, 其逆元素为 $S(R^{(1)})\otimes R^{(2)}$, 故当 (H,α,R) 为拟三角 Hopf 代数时, $R^{(1)}\otimes R^{(2)}=S(R^{(1)})\otimes S(R^{(2)})$.

证明 直接验证. □

设 (H,α,R) 是拟三角 Hom-Hopf 代数, 则由引理 3.6 知 $R\in S(H)\otimes S(H)$. 故由命题 3.2, 有

命题 3.7 设 (H,α,R) 是拟三角 Hom-Hopf 代数, 则 S 是可逆的, 且对任意 $h\in H$, 有

$$S^2(h)=(\alpha^{-1}(u)\alpha^{-2}(h))u^{-1},$$

其中 u 是 H 中的可逆元素, 并满足

$$u=S(R^{(2)})R^{(1)},\quad u^{-1}=R^{(2)}S^2(R^{(1)}),\quad \triangle(u)=Q^{-1}(u\otimes u)=(u\otimes u)Q^{-1},$$

这里 $Q=R_{21}R$,$R_{21}=\tau(R)$.

同理, 若记 $w=S(u)$, 则对任意 $h\in H$,

$$S^2(h)=(\alpha^{-1}(w)\alpha^{-2}(h))w^{-1},$$

其中 w 是 H 中的可逆元素, 并满足

$$w=R^{(1)}S(R^{(2)}),\quad w^{-1}=S^2(R^{(1)})R^{(2)},\quad \triangle(w)=Q^{-1}(w\otimes w)=Q^{-1}(w\otimes w).$$

证明 由命题 3.2 知 u 是可逆元素, 对极 S 是可逆的且对任意 $h\in H$, 有

$$S^2(h)=(\alpha^{-1}(u)\alpha^{-2}(h))u^{-1}.$$

同时, 由引理 3.4 和引理 3.6, 可证

$$(R^{(2)}S^2(R^{(1)}))u=1=u(R^{(2)}S^2(R^{(1)})),$$

故 $u^{-1}=R^{(2)}S^2(R^{(1)})$. 再由引理 3.6 可证

$$\triangle(u)=(u\otimes u)Q^{-1},$$

这里 $Q^{-1}=R^{-1}R_{21}^{-1}$, $R_{21}^{-1}=R^{(2)}\otimes S(R^{(1)})$.

对 v 类似可证. 命题得证. □

做为命题 3.7 的结果, 我们有 Radford 公式:

定理 3.8 设 (H,α,R) 是拟三角 Hom-Hopf 代数, 且 R 是 α- 不变的, 即, $(\alpha\otimes\alpha)(R)=R$, 则对任意 $h\in H$,

$$S^4(h)=(g\alpha^{-2}(h))g^{-1},$$

这里 $g=uw^{-1}$ 是 H 的群象元.

证明 对命题 3.7 中的元素 $u=S(R^{(2)})R^{(1)}$, $w=R^{(1)}S(R^{(2)})$ 和 $Q=R_{21}R$, 若 R 是 α- 不变的, 则 u,w 和 Q 也是 α- 不变的, 从而 g 是 H 的群象元, 这是因为

$$\begin{aligned}\triangle(g)&=\triangle(u)\triangle(w^{-1})=((u\otimes u)Q^{-1})(Q(w^{-1}\otimes w^{-1}))\\&=(((u\otimes u)Q^{-1})Q)(w^{-1}\otimes w^{-1})\\&=((u\otimes u)(Q^{-1}Q))(w^{-1}\otimes w^{-1})\\&=(u\otimes u)(w^{-1}\otimes w^{-1})=g\otimes g.\end{aligned}$$

同时, 由 (QT.2) 和 (QT.4) 易证 $\varepsilon(g)=1$.

再由命题 3.2 知, $S^2(h)=(\alpha^{-1}(u)\alpha^{-2}(h))u$, 故对任意 $h\in H$,

$$\begin{aligned}S^4(h)&=S^2((\alpha^{-1}(u)\alpha^{-2}(h))u^{-1})=S^2((u\alpha^{-2}(h))u^{-1})\ \ (\alpha^{-1}(u)=u)\\&=(uS^2\alpha^{-2}(h))u^{-1}\ \ (S^2(u)=u,S^2(u^{-1})=u^{-1})\\&=(u((v\alpha^{-4}(h))v^{-1}))u^{-1}\ \ (S^2(h)=(\alpha^{-1}(v)\alpha^{-2}(h))\alpha^{-1}(v^{-1}))\\&=(u(v(\alpha^{-4}(h)v^{-1}))u^{-1}=((uv)(\alpha^{-3}(h)v^{-1}))u^{-1}\\&=(((uv)\alpha^{-3}(h))v^{-1})u^{-1}=((uv)\alpha^{-2}(h))(v^{-1}u^{-1})\\&=((uw^{-1})\alpha^{-2}(h))(wu^{-1})\ \ (v=w^{-1})\\&=(g\alpha^{-2}(h))g^{-1}.\end{aligned}$$

定理得证. □

4 Hom-Hopf 代数的 Drinfel'd 偶

本节将定义有限维 Hom-Hopf 代数的 Drinfel'd 偶, 并证明它为拟三角 Hom-Hopf 代数.

设 $(H,\mu,\eta,\triangle,\varepsilon,S,\alpha)$ 为有限维 Hom-Hopf 代数. 由 [7, 例 2.4], $(H^*,\mu^*,\eta^*,\triangle^*,\varepsilon^*,S^*,\alpha^*)$ 也是 Hom-Hopf 代数, 其中

$$\langle\mu^*(\phi),x\otimes y\rangle=\langle\phi,\mu(x,y)\rangle,\quad \langle\triangle^*(\phi\otimes\psi),x\rangle=\langle\phi\otimes\psi,\triangle(x)\rangle,\tag{4.1}$$

$$\eta^*(1_k)=\varepsilon,\quad \varepsilon^*(\phi)=\langle\phi,1_H\rangle,\tag{4.2}$$

$$S^*(\phi)=\phi\circ S,\quad \alpha^*(\phi)=\phi\circ\alpha. \tag{4.3}$$

对任意 $x,y\in H,\ \phi,\psi\in H^*$.

设 (H,α) 为具有双射对极 S 的有限维 Hom-Hopf 代数, 且 α 是对合的, 即 $\alpha^2=\mathrm{id}$. 由于 S 是双射, 从而 S^* 也是双射, 并且对任意 $\phi\in H^*$, 有 $S^{*-1}(\phi)=\phi\circ S^{-1}$. 下面我们将定义 H 的 Drinfel'd 偶.

定理 4.1　所谓 Drinfel'd 偶 $D(H)=H^{*\mathrm{cop}}\bowtie H$ 是定义在向量空间 $H^{*\mathrm{cop}}\otimes H$ 上的 Hom-Hopf 代数, 其乘法为对任意 $f,g\in H^{*\mathrm{cop}}$ 和 $a,b\in H$,

$$(f\bowtie a)(g\bowtie b)=\langle g_1,S^{-1}(a_{22})\rangle\langle g_{22},a_1\rangle fg_{21}\bowtie a_{21}b, \tag{4.4}$$

且单位元为 $1_{H^*}\bowtie 1_H$, $\alpha_{D(H)}=\alpha^*\bowtie\alpha$, 其 Hom- 余代数结构为通常的 Hom- 余代数张量积, 即,

$$\triangle_{D(H)}(f\bowtie a)=(f_2\bowtie a_1)\otimes(f_1\bowtie a_2),\quad \varepsilon_{D(H)}=\varepsilon_{H^*}\otimes\varepsilon, \tag{4.5}$$

它的对极为

$$S_{D(H)}(f\bowtie h)=\langle S^*(f_1),S(h_{22})\rangle\langle f_{22},S(h_1)\rangle S^*(f_{21})\bowtie S(h_{21}). \tag{4.6}$$

证明　首先, 验证乘法的 Hom- 结合性.

$$
\begin{aligned}
&(\alpha^*(f)\bowtie\alpha(a))((g\bowtie b)(l\bowtie c))\\
&\quad=\langle l_1,S^{-1}(b_{22})\rangle\langle l_{22},b_1\rangle(\alpha^*(f)\bowtie\alpha(a))(gl_{21}\bowtie b_{21}c)\\
&\quad=\langle l_1,S^{-1}(b_{22})\rangle\langle l_{22},b_1\rangle\langle g_1l_{211},S^{-1}\alpha(a_{22})\rangle\langle g_{22}l_{2122},\alpha(a_1)\rangle\alpha^*(f)(g_{21}l_{2121})\bowtie\alpha(a_{21})(b_{21}c)\\
&\quad\overset{(1.17)}{=}\langle l_1,S^{-1}(b_{22})\rangle\langle l_{22},b_1\rangle\langle g_1,S^{-1}\alpha(a_{222})\rangle\langle l_{211},S^{-1}\alpha(a_{221})\rangle\langle g_{22},\alpha(a_{11})\rangle\langle l_{2122},\alpha(a_{12})\rangle\\
&\qquad\times\alpha^*(f)(g_{21}l_{2121})\bowtie\alpha(a_{21})(b_{21}c)\\
&\quad\overset{(4.3)}{=}\langle g_1,S^{-1}\alpha(a_{222})\rangle\langle g_{22},\alpha(a_{11})\rangle\langle l_1,S^{-1}(b_{22})\rangle\langle l_{2111},S^{-1}(a_{221})\rangle\langle l_{212},a_{12}\rangle\langle l_{22},b_1\rangle\\
&\qquad\times\alpha^*(f)(g_{21}l_{2112})\bowtie\alpha(a_{21})(b_{21}c)\\
&\quad=\langle g_1,S^{-1}\alpha(a_{222})\rangle\langle g_{22},\alpha(a_{11})\rangle\langle l_1,S^{-1}(b_{22})\rangle\langle\alpha^{*2}(l_{2111}),S^{-1}(a_{221})\rangle\langle l_{212},a_{12}\rangle\langle l_{22},b_1\rangle\\
&\qquad\times\alpha^*(f)(g_{21}l_{2112})\bowtie\alpha(a_{21})(b_{21}c)\quad(\alpha^{*2}=\mathrm{id})\\
&\quad\overset{(1.6)'}{=}\langle g_1,S^{-1}\alpha(a_{222})\rangle\langle g_{22},\alpha(a_{11})\rangle\langle l_1,S^{-1}(b_{22})\rangle\langle l_{211},S^{-1}\alpha(a_{221})\rangle\langle l_{221},a_{12}\rangle\langle l_{222},\alpha(b_1)\rangle\\
&\qquad\times\alpha^*(f)(g_{21}\alpha^*(l_{212}))\bowtie\alpha(a_{21})(b_{21}c)\\
&\quad=\langle g_1,S^{-1}\alpha(a_{222})\rangle\langle g_{22},\alpha(a_{11})\rangle\langle l_1,S^{-1}(b_{22})\rangle\langle\alpha^*(l_{211}),S^{-1}(a_{221})\rangle\langle l_{221},a_{12}\rangle\langle\alpha^*(l_{222}),b_1\rangle\\
&\qquad\times\alpha^*(f)(g_{21}\alpha^*(l_{212}))\bowtie\alpha(a_{21})(b_{21}c)\\
&\quad=\langle g_1,S^{-1}\alpha(a_{222})\rangle\langle g_{22},\alpha(a_{11})\rangle\langle l_1,S^{-1}(b_{22})\rangle\langle l_{21},S^{-1}(a_{221})\rangle\langle l_{2221},\alpha(a_{12})\rangle\langle l_{2222},b_1\rangle\\
&\qquad\times\alpha^*(f)(g_{21}\alpha^*(l_{221}))\bowtie\alpha(a_{21})(b_{21}c)\\
&\quad=\langle g_1,S^{-1}\alpha(a_{222})\rangle\langle g_{22},\alpha(a_{11})\rangle\langle l_1,S^{-1}(b_{22})\rangle\langle l_{21},S^{-1}(a_{221})\rangle\langle\alpha^*(l_{2221}),a_{12}\rangle\langle\alpha^*(l_{2222}),\alpha(b_1)\rangle\\
&\qquad\times\alpha^*(f)(g_{21}\alpha^*(l_{221}))\bowtie\alpha(a_{21})(b_{21}c)\\
&\quad=\langle g_1,S^{-1}\alpha(a_{222})\rangle\langle g_{22},\alpha(a_{11})\rangle\langle l_{11},S^{-1}\alpha(b_{22})\rangle\langle l_{12},S^{-1}(a_{221})\rangle\langle l_{221},a_{12}\rangle\langle l_{222},\alpha(b_1)\rangle\\
&\qquad\times\alpha^*(f)(g_{21}l_{21})\bowtie\alpha(a_{21})(b_{21}c)\\
&\quad=\langle g_1,S^{-1}(a_{2222})\rangle\langle g_{22},a_1\rangle\langle l_{11},S^{-1}\alpha(b_{22})\rangle\langle l_{12},S^{-1}\alpha(a_{2221})\rangle\langle l_{221},a_{21}\rangle\langle l_{222},\alpha(b_1)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\times \alpha^*(f)(g_{21}l_{21}) \bowtie \alpha^2(a_{221})(b_{21}c) \\
&= \langle g_1, S^{-1}(a_{2222})\rangle\langle g_{22}, a_1\rangle\langle l_{11}, S^{-1}\alpha(b_{22})\rangle\langle l_{12}, S^{-1}\alpha(a_{2221})\rangle\langle l_{221}, a_{21}\rangle\langle l_{222}, \alpha(b_1)\rangle \\
&\quad \times \alpha^*(f)(g_{21}l_{21}) \bowtie a_{221}(b_{21}c) \quad (\alpha^2 = \mathrm{id}) \\
&= \langle g_1, S^{-1}\alpha^2(a_{2222})\rangle\langle g_{22}, a_1\rangle\langle l_{11}, S^{-1}\alpha(b_{22})\rangle\langle l_{12}, S^{-1}\alpha(a_{2221})\rangle\langle l_{221}, a_{21}\rangle\langle l_{222}, \alpha(b_1)\rangle \\
&\quad \times \alpha^*(f)(g_{21}l_{21}) \bowtie a_{221}(b_{21}c) \\
&= \langle g_1, S^{-1}\alpha(a_{222})\rangle\langle g_{22}, a_1\rangle\langle l_{11}, S^{-1}\alpha(b_{22})\rangle\langle l_{12}, S^{-1}(a_{221})\rangle\langle l_{221}, \alpha(a_{211})\rangle\langle l_{222}, \alpha(b_1)\rangle \\
&\quad \times \alpha^*(f)(g_{21}l_{21}) \bowtie a_{212}(b_{21}c) \\
&= \langle g_1, S^{-1}(a_{22})\rangle\langle g_{22}, a_1\rangle\langle l_{11}, S^{-1}\alpha(b_{22})\rangle\langle l_{12}, S^{-1}(a_{212})\rangle\langle l_{221}, \alpha^2(a_{2111})\rangle\langle l_{222}, \alpha(b_1)\rangle \\
&\quad \times \alpha^*(f)(g_{21}l_{21}) \bowtie \alpha(a_{2112})(b_{21}c) \\
&= \langle g_1, S^{-1}(a_{22})\rangle\langle g_{22}, a_1\rangle\langle l_{11}, S^{-1}\alpha(b_{22})\rangle\langle l_{12}, S^{-1}(a_{212})\rangle\langle l_{221}, a_{2111}\rangle\langle l_{222}, \alpha(b_1)\rangle \\
&\quad \times \alpha^*(f)(g_{21}l_{21}) \bowtie \alpha(a_{2112})(b_{21}c) \\
&= \langle g_1, S^{-1}(a_{22})\rangle\langle g_{22}, a_1\rangle\langle l_{11}, S^{-1}\alpha(b_{22})\rangle\langle l_{12}, S^{-1}\alpha(a_{2122})\rangle\langle l_{221}, \alpha(a_{211})\rangle\langle l_{222}, \alpha(b_1)\rangle \\
&\quad \times \alpha^*(f)(g_{21}l_{21}) \bowtie \alpha(a_{2121})(b_{21}c) \\
&\overset{(4.1)}{=} \langle g_1, S^{-1}(a_{22})\rangle\langle g_{22}, a_1\rangle\langle \alpha^*(l_1), S^{-1}(a_{2122}b_{22})\rangle\langle \alpha^*(l_{22}), a_{211}b_1\rangle \\
&\quad \times (fg_{21})\alpha^*(l_{21}) \bowtie (a_{2121}b_{21})\alpha(c) \\
&= \langle g_1, S^{-1}(a_{22})\rangle\langle g_{22}, a_1\rangle(fg_{21} \bowtie a_{21}b)(\alpha^*(l) \bowtie \alpha(c)) \\
&= ((f \bowtie a)(g \bowtie b))(\alpha^*(l) \bowtie \alpha(c))).
\end{aligned}
$$

其次, 验证余乘法的 Hom- 余结合性.

$$
\begin{aligned}
&\alpha^{-1}_{D(H)}((f \bowtie a)_1) \otimes \triangle_{D(H)}((f \bowtie a)_2) \\
&\quad = (\alpha^{*-1} \otimes \alpha^{-1})((f \bowtie a)_1) \otimes \triangle_{D(H)}((f \bowtie a)_2) \\
&\quad = (\alpha^* \otimes \alpha)(f_2 \bowtie a_1) \otimes (f_1 \bowtie a_2)_1 \otimes (f_1 \bowtie a_2)_2 \\
&\quad = (\alpha^*(f_2) \bowtie \alpha(a_1)) \otimes (f_{12} \bowtie a_{21}) \otimes (f_{11} \bowtie a_{22}) \\
&\quad \overset{(1.6)}{=} (f_{22} \bowtie a_{11}) \otimes (f_{21} \bowtie a_{12}) \otimes (\alpha^*(f_1) \bowtie \alpha(a_2)) \\
&\quad \overset{(4.5)}{=} (f_2 \bowtie a_1)_1 \otimes (f_2 \bowtie a_1)_2 \otimes (\alpha^* \otimes \alpha)(f_1 \bowtie a_2) \\
&\quad = (f \bowtie a)_{11} \otimes (f \bowtie a)_{12} \otimes (\alpha^* \otimes \alpha)((f \bowtie a)_2) \\
&\quad = \triangle_{D(H)}((f \bowtie a)_1) \otimes \alpha^{-1}_{D(H)}((f \bowtie a)_2).
\end{aligned}
$$

再次, $\triangle_{D(H)}$ 是 Hom- 代数同态, $\varepsilon_{D(H)}$ 是 Hom- 代数同态, 读者可以直接验证.

最后, 验证 $D(H)$ 有对极 $S_{D(H)}$. 由 α 是对合知 $S_{D(H)}$ 与 $\alpha_{D(H)}$ 可交换, 且对任意 $f \in H^*$, $h \in H$,

$$
\begin{aligned}
\alpha_{D(H)} \circ S_{D(H)}(f \bowtie h) &= \alpha_{D(H)}(S^*(f_{21}) \bowtie S(h_{21}))\langle S^*(f_1), S(h_{22})\rangle\langle f_{22}, S(h_1)\rangle \\
&= S^*\alpha^*(f_{21}) \bowtie S\alpha(h_{21}))\langle S^*(f_1), S(h_{22})\rangle\langle f_{22}, S(h_1)\rangle \\
&= S^*\alpha^*(f_{21}) \bowtie S\alpha(h_{21}))\langle S^*\alpha^*(f_1), S\alpha(h_{22})\rangle\langle \alpha^*(f_{22}), S\alpha(h_1)\rangle
\end{aligned}
$$

$$= S_{D(H)}(\alpha^*(f) \bowtie \alpha(h)) = S_{D(H)} \circ \alpha_{D(H)}(f \bowtie h).$$

并且,

$$\begin{aligned}
S_{D(H)}((f \bowtie a)_1)(f \bowtie a)_2 &= S_{D(H)}(f_2 \bowtie a_1)(f_1 \bowtie a_2) \\
&= \langle S^*(f_{21}), S(a_{122})\rangle \langle f_{222}, S(a_{11})\rangle (S^*(f_{221}) \bowtie S(a_{121}))(f_1 \bowtie a_2) \\
&= \langle S^*(f_{21}), S(a_{122})\rangle \langle f_{222}, S(a_{11})\rangle \langle f_{11}, a_{12111}\rangle \langle f_{122}, S(a_{1212})\rangle \\
&\quad \times S^*(f_{221}) f_{121} \bowtie S(a_{12112}) a_2 \\
&\overset{(1.6)}{=} \langle S^*(f_{21}), S\alpha(a_{1222})\rangle \langle f_{222}, S(a_{11})\rangle \langle f_{11}, \alpha(a_{1211})\rangle \langle f_{122}, S(a_{1221})\rangle \\
&\quad \times S^*(f_{221}) f_{121} \bowtie S\alpha(a_{1212}) a_2 \\
&\overset{(4.1)}{=} \langle S^*\alpha^*(f_{21}) f_{122}, S(a_{122})\rangle \langle f_{222}, S(a_{11})\rangle \langle f_{11}, \alpha(a_{1211})\rangle \\
&\quad \times S^*(f_{221}) f_{121} \bowtie S\alpha(a_{1212}) a_2 \\
&\overset{(1.6)}{=} \langle S^*(f_{221}) f_{212}, S(a_{122})\rangle \langle f_{2222}, S\alpha(a_{11})\rangle \langle f_1, a_{1211}\rangle \\
&\quad \times S^*\alpha^*(f_{2221}) f_{211} \bowtie S\alpha(a_{1212}) a_2 \\
&= \langle S^*\alpha^*(f_{2221}) f_{221}, S(a_{122})\rangle \langle f_{22222}, S(a_{11})\rangle \langle f_1, a_{1211}\rangle \\
&\quad \times S^*(f_{22221}) \alpha^*(f_{21}) \bowtie S\alpha(a_{1212}) a_2 \\
&= \langle S^*\alpha^*(f_{2212}) \alpha^*(f_{2211}), S(a_{122})\rangle \langle f_{2222}, S\alpha(a_{11})\rangle \langle f_1, a_{1211}\rangle \\
&\quad \times S^*\alpha^*(f_{2221}) \alpha^*(f_{21}) \bowtie S\alpha(a_{1212}) a_2 \\
&= \langle \underline{S^*\alpha^*(f_{2212}) \alpha^*(f_{2211})}, S(a_{1222})\rangle \langle f_{2222}, S\alpha(a_{11})\rangle \langle f_1, \alpha(a_{121})\rangle \\
&\quad \times S^*\alpha^*(f_{2221}) \alpha^*(f_{21}) \bowtie S\alpha(a_{1221}) a_2 \\
&\overset{(1.15)}{=} \varepsilon_{H^*}(f_{2211}) \varepsilon_H(a_{1222}) \langle f_{222}, S(a_{11})\rangle \langle f_1, \alpha(a_{121})\rangle \\
&\quad \times S^*\alpha^*(f_{2212}) \alpha^*(f_{21}) \bowtie S\alpha(a_{1221}) a_2 \quad (f \in H^{*\mathrm{co}P}) \\
&= \langle f_{222}, S(a_{11})\rangle \langle f_1, \alpha(a_{121})\rangle S^*(f_{221}) \alpha^*(f_{21}) \bowtie S(a_{122}) a_2 \\
&= \langle f_{22}, S\alpha(a_{11})\rangle \langle f_1, \alpha(a_{121})\rangle \underline{S^*(f_{212}) f_{211}} \bowtie S\alpha(a_{122}) a_2 \\
&= \langle f_{22}, S(a_{111})\rangle \langle f_1, \alpha(a_{112})\rangle \varepsilon_{H^*}(f_{21}) 1_{H^*} \bowtie S\alpha(a_{12}) a_2 \\
&= \langle f_2, S\alpha(a_{111})\rangle \langle f_1, \alpha(a_{112})\rangle 1_{H^*} \bowtie S\alpha(a_{12}) a_2 \\
&= \langle f, \alpha(S(a_{111}) a_{112})\rangle 1_{H^*} \bowtie S\alpha(a_{12}) a_2 \\
&= \varepsilon_{H^*}(f) \varepsilon_H(a_{11}) 1_{H^*} \bowtie S\alpha(a_{12}) a_2 = \varepsilon_{H^*}(f) 1_{H^*} \bowtie S(a_1) a_2 \\
&= \varepsilon_{H^*}(f) \varepsilon_H(a) 1_{H^*} \bowtie 1_H.
\end{aligned}$$

类似可证 $((f \bowtie a)_1) S_{D(H)}((f \bowtie a)_2) = \varepsilon_{H^*}(f) \varepsilon_H(a) 1_{H^*} \bowtie 1_H$. 综上所述, Drinfel'd 偶 $D(H)$ 是 Hom-Hopf 代数. □

下面的定理说明了有限维 Hom-Hopf 代数的 Drinfel'd 偶是拟三角的.

定理 4.2 设 (H, α) 为带有双射对极 S 的有限维 Hom-Hopf 代数, 且 $\alpha^2 = \mathrm{id}$. 令 $\{h_i\}$ 为 H 的一组基, $\{h_i^*\}$ 是 H^* 中的相应对偶基, 则 $(D(H), R)$ 是拟三角 Hom-Hopf 代数, 其中

$$R = (\varepsilon_H \bowtie \alpha(h_i)) \otimes (h_i^* \bowtie 1_H).$$

证明 易证 (QT.2) 和 (QT.4) 成立. 为了验证 (QT.1) 和 (QT.3) 成立, 需证

$$h_{i1}^* \otimes h_{i2}^* \otimes h_i = h_i^* \otimes h_j^* \otimes h_i h_j, \tag{4.7}$$

$$h_i^* \otimes h_{i1} \otimes h_{i2} = h_i^* h_j^* \otimes h_i \otimes h_j. \tag{4.8}$$

事实上, 赋值作用等式 (4.7) 和 (4.8) 的两边即可证明.

为了验证 (QT.5) 成立, 一方面,

$$\begin{aligned}
\triangle_{D(H)}^{\mathrm{cop}}(\psi \bowtie x)R &= (\psi_1 \bowtie x_2)(\varepsilon_H \bowtie \alpha(h_i)) \otimes (\psi_2 \bowtie x_1)(h_i^* \bowtie 1_H) \\
&\overset{(4.4)}{=} (\alpha^*(\psi_1) \bowtie x_2\alpha(h_i))) \otimes \langle h_{i1}^*, S^{-1}(x_{122})\rangle \langle h_{i22}^*, x_{11}\rangle (\psi_2 h_{i21}^* \bowtie \alpha(x_{121})) \\
&\overset{(4.7)}{=} \langle h_i^*, S^{-1}(x_{122})\rangle \langle h_k^*, x_{11}\rangle (\alpha^*(\psi_1) \bowtie x_2(\alpha(h_i)(\alpha(h_j)\alpha(h_k)))) \otimes (\psi_2 h_j^* \bowtie \alpha(x_{121})) \\
&= (\alpha^*(\psi_1) \bowtie x_2(S^{-1}\alpha(x_{122})(\alpha(h_i)\alpha(x_{11})))) \otimes (\psi_2 h_i^* \bowtie \alpha(x_{121})) \\
&\overset{(1.4)}{=} (\alpha^*(\psi_1) \bowtie (\alpha(x_2)S^{-1}\alpha(x_{122}))(h_i x_{11})) \otimes (\psi_2 h_i^* \bowtie \alpha(x_{121})) \\
&\overset{(1.6)}{=} (\alpha^*(\psi_1) \bowtie (x_{22}S^{-1}\alpha(x_{212}))(h_i\alpha(x_1))) \otimes (\psi_2 h_i^* \bowtie \alpha(x_{211})) \\
&= (\alpha^*(\psi_1) \bowtie (\alpha(x_{222})S^{-1}\alpha(x_{221}))(h_i\alpha(x_1))) \otimes (\psi_2 h_i^* \bowtie x_{21}) \\
&\overset{(1.15)}{=} (\alpha^*(\psi_1) \bowtie \alpha(h_i)x_1) \otimes (\psi_2 h_i^* \bowtie \alpha(x_2));
\end{aligned}$$

另一方面,

$$\begin{aligned}
R\triangle_{D(H)}(\psi \bowtie x) &= (\varepsilon_H \bowtie \alpha(h_i))(\psi_2 \bowtie x_1) \otimes (h_i^* \bowtie 1_H)(\psi_1 \bowtie x_2) \\
&= \langle \psi_{21}, S^{-1}\alpha(h_{i22})\rangle \langle \psi_{222}, \alpha(h_{i1})\rangle (\alpha^*(\psi_{221}) \bowtie \alpha(h_{i21})x_1) \otimes (h_i^*\psi_1 \bowtie \alpha(x_2)) \\
&\overset{(4.8)}{=} \langle \psi_{21}, S^{-1}\alpha(h_k)\rangle \langle \psi_{222}, \alpha(h_i)\rangle (\alpha^*(\psi_{221}) \bowtie \alpha(h_j)x_1) \otimes ((h_i^*(h_j^* h_k^*))\psi_1 \bowtie \alpha(x_2)) \\
&= (\alpha^*(\psi_{221}) \bowtie \alpha(h_i)x_1) \otimes ((\alpha^*(\psi_{222})(h_i^* S^{*-1}\alpha^*(\psi_{21})))\psi_1 \bowtie \alpha(x_2)) \\
&\overset{(1.4)}{=} (\alpha^*(\psi_{221}) \bowtie \alpha(h_i)x_1) \otimes (\psi_{222}((h_i^* S^{*-1}\alpha^*(\psi_{21}))\alpha^*(\psi_1)) \bowtie \alpha(x_2)) \\
&= (\alpha^*(\psi_{221}) \bowtie \alpha(h_i)x_1) \otimes (\psi_{222}((\alpha^*(h_i^*)(S^{*-1}\alpha^*(\psi_{21})\psi_1))) \bowtie \alpha(x_2)) \\
&= (\psi_{21} \bowtie \alpha(h_i)x_1) \otimes (\alpha^*(\psi_{22})((\alpha^*(h_i^*)(S^{*-1}\alpha^*(\psi_{12})\alpha^*(\psi_{11})))) \bowtie \alpha(x_2)) \\
&\overset{(1.6)}{=} (\alpha^*(\psi_1) \bowtie \alpha(h_i)x_1) \otimes (\psi_2 h_i^* \bowtie \alpha(x_2)).
\end{aligned}$$

定理得证. □

例 4.3 设 G 为有限群, $\alpha: G \to G$ 为群同态, 满足 $\alpha^2 = \mathrm{id}$, 则由文献 [7], 有 Hom-Hopf 代数

$$\begin{aligned}
&kG_\alpha = (kG, \mu_\alpha = \alpha \circ \mu, \triangle_\alpha = \triangle \circ \alpha, S, \alpha), \\
&kG_{\alpha^*} = (k(G), \mu_{\alpha^*} = \alpha^* \circ \triangle, \triangle_{\alpha^*} = \mu^* \circ \alpha^*, S^*, \alpha^*),
\end{aligned}$$

其中 μ, $\triangle$ 和 S 分别是 kG 的乘法、余乘法和对极, $k(G)$ 是由 $kG \to k$ 所有函数构成的 Hom-Hopf 代数, $\triangle^*$ 为 $k(G)$ 的余乘法. Hom- 双代数同构映射 $\alpha^*: k(G) \to k(G)$ 由 $\alpha^*(\phi) = \phi \circ \alpha$ 给出, kG_{α^*} 的乘法和余乘法为:

$$\langle \mu_{\alpha^*}(\phi, \psi), u\rangle = \langle \phi, \alpha(u)\rangle \langle \psi, \alpha(u)\rangle, \quad \langle \triangle_{\alpha^*}(\phi), (u, v)\rangle = \langle \phi, \alpha(uv)\rangle.$$

于是, 有 Drinfel'd 偶 $D(kG_\alpha) = (kG_\alpha)^{*\mathrm{cop}} \bowtie kG_\alpha = kG_{\alpha^*}^{\mathrm{cop}} \bowtie kG_\alpha$, 故由定理 4.2 知它是拟三角 Hom-Hopf 代数, 其具体结构为:

$$
\begin{aligned}
&(\psi_s \bowtie u)(\psi_t \bowtie v) = \delta_{\alpha(t)u^{-1}s^{-1},u^{-1}}\psi_s \bowtie uv,\\
&\triangle_{D(H)}(\psi_s \bowtie u) = \Sigma_{\alpha(s)=ab}\psi_{\alpha(a)} \bowtie \alpha(u) \otimes \psi_{\alpha(b)} \bowtie \alpha(u),\\
&\varepsilon_{D(H)}(\psi_s \bowtie u) = \delta_{s,e}, \quad S_{D(H)}(\psi_s \bowtie u) = \psi_{\alpha(u^{-1})s^{-1}\alpha(u)} \bowtie u^{-1},\\
&R = (\psi_u \bowtie 1_{kG_\alpha}) \otimes (1_{kG_{\alpha^*}} \bowtie u),
\end{aligned}
$$

对任意 $s, t, u, v \in G$, 这里 $\delta_{s,t}$ 表示 Kronecker 符号.

致谢 作者张良云衷心感谢朱永昌教授邀请他到香港科技大学学术访问, 并对审稿人的有益建议深表感谢.

参考文献

1 Hartwig J, Larsson D, Silvestrov S. Deformations of Lie algebras using σ-derivations. J Algebra, 2006, 295: 314–361
2 Silvestrov S. Paradigm of quasi-Lie and quasi-Hom-Lie algebras and quasi-deformations. In: New Technique in Hopf Algebras and Graded Ring Theory, K Vllam Acad Belgie Wet. Brussels: Kunsten (KVAB), 2007: 165–177
3 Yau D. The classical Hom-Yang-Baxter equation and Hom-Lie bialgebras. ArXiv: 0905.1980, 2010
4 Chen Y Y, Wang Y, Zhang L Y. The construction of Hom-Lie bialgebras. J Lie Theory, 2010, 20: 767–783
5 Makhlouf A, Silvestrov S. Hom-Lie Admissible Hom-coalgebras and Hom-Hopf Algebras. In: Silvestrov S, Paal E, Abramov V, et al. eds. Generalized Lie Theory in Mathematics, Physics and Beyond. Berlin-Heidelberg: Springer-Verlag, 2009: 189–206
6 Makhlouf A, Silvestrov S. Hom-algebras and Hom-coalgebras. J Algebra Appl, 2010, 9: 553–589
7 Yau D. Hom-quantum groups I: quasi-triangular Hom-bialgebras. ArXiv: 0912.4128, 2009
8 Caenepeel S, Goyvaerts I. Monoidal Hom-Hopf algebras. Comm Algebra, 2011, 39: 2216–2240
9 Makhlouf A, Silvestrov S. Hom-algebras structures. J Gen Lie Theory Appl, 2008, 2: 51–64
10 Yau D. Hom-bialgebras and comodule Hom-algebras. Inter Elect J Algebra, 2010, 8: 45–64
11 Ion B. The reconstruction of the antipode. Stud Cerc Mat, 1998, 50: 369–372

The antipode and Drinfel'd double of Hom-Hopf algebras

WANG ZhongWei, CHEN YuanYuan & ZHANG LiangYun

Abstract We find that a Hom-bialgebra which admits an equivalence of Hom-categories is a Hom-Hopf algebra, and reconstruct the antipode so that a Hom-biagbra is a Hom-Hopf algebra when the fundamental structure holds. We study the antipode of a quasi-triangular Hom-Hopf algebra and prove the Radford formula for a given Hom-Hopf algebra. In the end, we introduce the concept of the Drinfel'd double for some finite dimensional Hom-Hopf algebra, and show that the Drinfel'd double is a quasi-triangular Hom-Hopf algebra.

Keywords **Hom-Hopf algebra, quasi-triangular Hom-Hopf algebra, antipode, Radford formula, Drinfel'd double**

MSC(2010) **16W30**

doi: 10.1360/012011-138