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Quasi-admissible, raisable nilpotent orbits, and theta representations

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Abstract We study the quasi-admissibility and raisability of some nilpotent orbits of a covering group. In particular, we determine the degree of the cover such that a given split nilpotent orbit is quasi-admissible and non-raisable. The speculated wavefront sets of theta representations are also computed explicitly and are shown to be quasi-admissible and non-raisable. Lastly, we determine the leading coefficients in the Harish-Chandra character expansion of the theta representations of covers of the general linear groups.

Keywords covering groups, nilpotent orbits, wavefront sets, quasi-admissible, raisable, theta representations

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1 Introduction

Let F be a p-adic local field of characteristic 0. Denote by F^{al} the algebraic closure of F. Let $\mathbf{G}$ be a split-connected linear reductive group over F. Consider the group $G := \mathbf{G}(F)$ or its finite degree central covers

$$\mu_n \hookrightarrow \overline{G}^{(n)} \twoheadrightarrow G,$$

where we assume that F contains the full group μ_n of n-th roots of unity. In this paper, we focus exclusively on covering groups that arise from the Brylinski-Deligne framework [8], and we also write $\overline{G} := \overline{G}^{(n)}$ whenever n is understood. If $n = 1$, then the covering groups are just the linear algebraic groups. We consider genuine representations of $\overline{G}$, where μ_n acts via a fixed embedding $\mu_n \hookrightarrow \mathbf{C}^\times$. Denote by $\mathrm{Irr}_{\mathrm{gen}}(\overline{G})$ the set of equivalence classes of irreducible genuine representations of $\overline{G}$. We study the leading wavefront sets of $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$, which are related to the generalized Whittaker models of π. We give a quick recapitulation.

Let $\mathcal{N}$ denote the partially-ordered set of nilpotent orbits in $\mathfrak{g}_F = \mathrm{Lie}(G)$ under the conjugation action of G, where the partial order is given by the closure ordering in the usual topology of $\mathfrak{g}_F$ induced from that of F. Every $(\pi, V_\pi) \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$ defines a character distribution χ_π in a neighborhood of 0 in $\mathfrak{g}_F$.

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Moreover, there exists a compact open subset S_π of 0 such that for every smooth function f with compact support in S_π, one has (see [36,37] for the proof in the linear algebraic case and [50] for an extension to the covering setting)

$$\chi_\pi(f) = \sum_{\mathcal{O}\in\mathcal{N}} c_{\mathcal{O}}(\pi)\cdot\int \hat{f}\mu_{\mathcal{O}}. \tag{1.1}$$

Here, $\mu_{\mathcal{O}}$ is a certain Haar measure on $\mathcal{O}$ properly normalized, and $\hat{f}$ is the Fourier transform of f with respect to the Cartan-Killing form on $\mathfrak{g}_F$ and a non-trivial character

$$\psi : F \to \mathbf{C}^\times;$$

one has $c_{\mathcal{O}}(\pi) := c_{\mathcal{O},\psi}(\pi) \in \mathbf{C}$. Note that implicitly used in (1.1) is an exponent map $\exp : L \to \overline{G}$ defined for a sufficiently small open set $L \subset \mathfrak{g}_F$ containing 0; it is used to "pull-back" the character distribution of π defined on $\overline{G}$ to be on (a small neighborhood of 0 in) $\mathfrak{g}_F$ (see [50, § 4.3]).

Define

$$\mathcal{N}_{\mathrm{tr}}(\pi) = \{\mathcal{O}\in\mathcal{N} : c_{\mathcal{O}}(\pi)\neq 0\}$$

and let

$$\mathcal{N}_{\mathrm{tr}}^{\max}(\pi)\subset\mathcal{N}_{\mathrm{tr}}(\pi)$$

be the subset consisting of all maximal elements in $\mathcal{N}_{\mathrm{tr}}(\pi)$. The set $\mathcal{N}_{\mathrm{tr}}^{\max}(\pi)$ gives the Gelfand-Kirillov dimension

$$d_{\mathrm{GK}}(\pi) := \frac{1}{2}\max\left\{\dim\mathcal{O} : \mathcal{O}\in\mathcal{N}_{\mathrm{tr}}^{\max}(\pi)\right\},$$

which satisfies

$$\dim\pi^K \approx c_\pi\cdot\mathrm{vol}(K)^{-d_{\mathrm{GK}}(\pi)},$$

as certain open compact congruence subgroups $K\subset\overline{G}$ approach to the identity, and here $c_\pi\in\mathbf{C}$ depends only on π. We see [60] for an argument stated for linear G but is actually applicable to the covering $\overline{G}$ using (1.1). It is also known (see [55,67] for the linear algebraic case and [59] for the extension to covering setting) that the set $\mathcal{N}_{\mathrm{tr}}^{\max}(\pi)$ is equal to the set of maximal nilpotent orbits with respect to which the generalized Whittaker models for π are nontrivial.

More precisely, let $(f,h,u)\subset\mathfrak{g}_F$ be an $\mathfrak{sl}_2$-triple. In this case, h is called a neutral element and it gives a filtration

$$\mathfrak{g}_F = \bigoplus_{i\in\mathbf{Z}}\mathfrak{g}^h[i],$$

and $u\in\mathfrak{g}^h[-2]$. Write

$$\mathfrak{n}_{h,u} := \bigoplus_{i\geqslant 2}\mathfrak{g}^h[i]\subset\mathfrak{g}_F,$$

and let

$$N_{h,u} := \exp(\mathfrak{n}_{h,u})\subset G$$

be the unipotent subgroup of G associated with $\mathfrak{n}_{h,u}$. There is a character

$$\psi_u : N_{h,u}\to\mathbf{C}^\times$$

given by

$$\psi_u(n) = \psi(\kappa(u,\log(n))),$$

where $\kappa : \mathfrak{g}_F\times\mathfrak{g}_F\to F$ is the Killing form. Since $\overline{G}$ splits uniquely over every unipotent subgroup $N\subset G$, we view N as a subgroup of $\overline{G}$.

For every $\pi\in\mathrm{Irr}_{\mathrm{gen}}(\overline{G})$ and every pair (h,u) as above, one can use $N_{h,u}$ and ψ_u above to define the degenerate Whittaker model $\pi_{h,u}$ of π. Moreover, for every nilpotent G-orbit $\mathcal{O}\subset\mathfrak{g}_F$, we write

$$\pi_{\mathcal{O}} := \pi_{h,u},$$

which is independent of the choice of any $u \in \mathcal{O}$ and neutral pair (h, u) (see Subsection 2.2 for details). We call $\pi_{\mathcal{O}}$ the generalized Whittaker model of π associated with $\mathcal{O}$. This gives

$$\mathcal{N}_{\mathrm{Wh}}(\pi) = \{\mathcal{O} \subset \mathfrak{g}_F : \pi_{\mathcal{O}} \neq 0\},$$

and we let $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi) \subset \mathcal{N}_{\mathrm{Wh}}(\pi)$ be the subset of maximal elements. Then it was shown in [55, 59, 67] that

- $\mathcal{N}_{\mathrm{tr}}^{\max}(\pi) = \mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ for every $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$, and
- for every $\mathcal{O}$ in the above (equal) sets, one has $c_{\mathcal{O}} = \dim_{\mathbf{C}} \pi_{\mathcal{O}}$.

We call $\mathcal{N}_{\mathrm{tr}}^{\max}(\pi)$ and thus also $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ the wavefront set of π. For $\pi \in \mathrm{Irr}(\mathrm{GL}_r)$, a relation between the full set $\{c_{\mathcal{O}} : \mathcal{O} \in \mathcal{N}_{\mathrm{tr}}(\pi)\}$ and certain degenerate Whittaker models of the Jacquet modules of π is given in the recent work of Gurevich [35].

It is important to understand the set $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$. Indeed, in view of the above relation among $d_{\mathrm{GK}}(\pi)$, $\mathcal{N}_{\mathrm{tr}}^{\max}(\pi)$ and $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$, we see that the set $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ measures the "asymptotic" size of the usually infinite-dimensional π, coined by the Gelfand-Kirillov dimension. Moreover, one can define a global analogue of $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$, which has many deep applications, for example, in the theory of descent and the Gan-Gross-Prasad conjectures and others, see [21, 31, 45] and the references therein. It should be mentioned that the study of $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ for central covers has applications to problems concerning linear algebraic groups as well. A prototype is the theta correspondence relating the representations of SO_{2m} and the double cover of Sp_{2r}, which relies crucially on the fact that $\mathcal{N}_{\mathrm{Wh}}^{\max}(\omega_\psi)$ of the Weil representation ω_ψ consists of the minimal orbits of the ambient symplectic group. Particularly interesting is the usage of theta representations with appropriate $\mathcal{N}_{\mathrm{Wh}}^{\max}$ to understand L-functions for linear algebraic groups (see, for example, [12, 13, 64]).

Nevertheless, determining the set $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ is a difficult problem for general $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$, and this is already the case for linear G. For GL_r, Ginzburg [30] and Jiang [39] gave several speculations for the set $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ in terms of the Moeglin-Waldspurger classification of the spectrum of GL_r (see [52] for some recent progress). For linear group G, the Arthur parametrization of certain unitary representations π is expected to encode information on $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ in terms of their Arthur parameters. For linear G, several conjectures are formulated in [39] regarding $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ using the parameter of π. There has been progress towards these conjectures as in [40, 41, 43, 44, 51], among many others[1)]. We also have the recent work by Ciubotaru, Mason-Brown and Okada toward understanding the stable wavefront sets and their refined versions (see [16, 58]).

For covering groups, the most "fundamental" genuine representation of a covering group $\overline{G}$ is the theta representation $\Theta(\overline{G})$ (see [24, 49]), a prototype of which is called the even Weil representation of the double cover $\overline{\mathrm{Sp}}_{2r}^{(2)}$. It is desirable to have a full description of $\mathcal{N}_{\mathrm{Wh}}^{\max}(\Theta(\overline{G}))$. For $\overline{\mathrm{GL}}_r^{(n)}$, it was proved by Cai [14] and Savin[2)] that for unramified theta representations, one has

$$\mathcal{N}_{\mathrm{Wh}}^{\max}(\Theta(\overline{\mathrm{GL}}_r^{(n)})) = \{(n^a b)\}, \tag{1.2}$$

where $r = an + b$ with $0 \leqslant b < n$. For $n \geqslant r$, this recovers the earlier result of Kazhdan and Patterson [49] on generic theta representations. Formulas analogous to (1.2) were proved for $\overline{\mathrm{SO}}_{2r+1}^{(4)}$ and $\overline{\mathrm{GSpin}}_{2r}^{(2)}$ (see [10, 11, 47]), respectively. The case of $\overline{\mathrm{Sp}}_{2r}^{(n)}$ was also studied extensively by Friedberg and Ginzburg [19–22]. In [27], a uniform but speculative formula of $\mathcal{N}_{\mathrm{Wh}}^{\max}(\Theta(\overline{G}))$ was given for general $\overline{G}$, which was motivated from the works mentioned above.

It has been expected for a long time that over the algebraic closure, the wavefront set is a singleton for general π. On the contrary, Tsai gave a counter-example recently in [66] for certain epipelagic supercuspidal representations of U_7. However, we still expect, as from [16, 58] for linear groups, that for the genuine Iwahori-spherical representations, in particular unramified genuine theta representations, the wavefront set is a singleton over F^{al}.

Although it is difficult to determine precisely $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ for general $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$, some work has been done in the literature to determine whether a candidate $\mathcal{O} \in \mathcal{N}$ could possibly lie in $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ for some

1) Liu B Y, Shahidi F. On Jiang's wavefront sets conjecture for representations in local Arthur packets. Preprint, 2022

2) Savin G. A nice central extension of GL_r. Preprint

π. Pertaining to this are the notions of admissible or quasi-admissible orbits. Equivalently, one has some sufficient condition for an orbit to lie in the complement set

$$\mathcal{N} - \bigcup_{\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})} \mathcal{N}_{\mathrm{Wh}}^{\max}(\pi), \tag{1.3}$$

especially when $\overline{G} = G$ is a linear group.

The notion of admissibility was first proposed by Duflo [18] for real Lie groups. It was then studied both in the p-adic and the real setting in many works later, for example, [56, 57, 65], and we refer the reader to [56, 57, 65] for more extensive references and historical discussion. Utilizing a more general "Whittaker pair", we see that the recent work by Gomez et al. [32,33] enables one to consider degenerate Whittaker models, where the discussion on admissibility and quasi-admissibility of nilpotent orbits was framed generally and already applies to covering groups.

Closely related to admissible (or slightly weaker, quasi-admissible) orbits is the notion of special orbits, which correspond to the special Weyl group representations via the Springer correspondence. It was shown by Moeglin [54] that for classical groups, every admissible orbit has to be special. Thus, the set $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ for classical linear algebraic groups contains necessarily just special orbits. This fact fails to cover groups, as seen already from the three-fold cover of G_2 and many covering groups considered in this paper. In fact, in a different direction, Jiang et al. [42] considered the notion of raisability of nilpotent orbits, which equally applies to covering groups as well. Roughly speaking, if a nilpotent orbit is raisable, then it must lie in the set (1.3).

Our present paper is motivated by the works above. It could be in part considered as an application and explication of [33, 42] in the covering setting, and in part as a sequel to [27]. Below, we give an elaboration of this and also state our main results.

First, by applying and analyzing the techniques from [32,33,42], we determine the orbits $\mathcal{O} \subset \mathfrak{g}_F$ which are $\overline{G}^{(n)}$-quasi-admissible or $\overline{G}^{(n)}$-raisable. As alluded to above, the importance of these two notions lies in the following:

- if $\mathcal{O}$ is $\overline{G}^{(n)}$-raisable or not $\overline{G}^{(n)}$-quasi-admissible, then it does not lie in $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ for any $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G}^{(n)})$.

For technical reasons, we only consider F-split nilpotent orbits. In Section 2, we consider general $\overline{G}$ and give equivalent criteria for quasi-admissibility which are amenable to explicit verification, similarly for raisability (see Propositions 2.4, 2.5 and 2.10).

In Section 3, applying results from Section 2, we analyze groups of each Cartan-type and determine the quasi-admissibility and raisability of each F-split orbit $\mathcal{O}$ of G, whenever possible.

Theorem 1.1. *Let $\overline{G}^{(n)}$ be the n-fold cover of any of* GL_r, SO_{2r+1}, SO_{2r} *and* Sp_{2r}. *Conditions for quasi-admissibility and raisability of an F-split orbit $\mathcal{O} \in \mathcal{N}$ with the associated partition $\mathfrak{p}_{\mathcal{O}} = (p_1^{d_1} \cdots p_i^{d_i} \cdots p_k^{d_k})$ are stipulated in Theorems* 3.1, 3.3 *and* 3.4. *For covering groups of the simply-connected exceptional groups G_2, F_4 and E_r, $6 \leqslant r \leqslant 8$, quasi-admissible and raisable orbits are given in Tables* 1–5, *whenever our method applies.*

For exceptional groups, we check the quasi-admissibility and raisability for all orbits of G_2 and F_4, but only for those orbits which are speculatively the leading wavefront set of theta representations for $E_r, 6 \leqslant r \leqslant 8$. However, the method of the computation clearly applies to any orbit of E_r.

Next, we consider in Section 4 theta representations $\Theta(\nu)$ of $\overline{G}^{(n)}$, where $\nu \in X \otimes \mathbf{R}$ is a certain exceptional character. Naturally, associated with ν is the nilpotent orbit $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu) \subset \mathfrak{g}_{F^{\mathrm{al}}}$, which arises from the Springer correspondence and the j-induction of the sign character ε_ν of $W_\nu \subset W$. This orbit is expected to be $\mathcal{N}_{\mathrm{Wh}}^{\max}(\Theta(\nu)) \otimes F^{\mathrm{al}}$. In Subsection 4.3, we show that the computation of $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu) \subset \mathfrak{g}_{F^{\mathrm{al}}}$ is reduced to Sommers' duality [63] between nilpotent orbits, which generalizes the classical Barbasch-Vogan duality. With an explicit computation, we verify the quasi-admissibility and non-raisability of such orbits.

Theorem 1.2 (Theorem 4.6). *Let $\overline{G}^{(n)}$ be a cover of the classical and exceptional groups considered in Theorem* 1.1.

(i) *The orbit* $\mathcal{O}_{\rm Spr}(j_{W_\nu}^W \varepsilon_\nu) \subset \mathfrak{g}_{F^{\rm al}}$ *for* $\overline{G}^{(n)}$ *of classical groups is explicitly given as in Table* 6; *for* $\overline{G}^{(n)}$ *of exceptional groups, it is given in Tables* 7–11.

(ii) *The* F*-split orbit* $\mathcal{O}_\Theta \subset \mathfrak{g}_F$ *of type* $\mathcal{O}_{\rm Spr}(j_{W_\nu}^W \varepsilon_\nu)$ *is quasi-admissible and not raisable.*

(iii) *If the orbit* $\mathcal{O}_\Theta$ *is the regular orbit of a Levi subgroup of* G*, then it supports the generalized Whittaker model of the theta representation* $\Theta(\nu)$.

Last, in Section 5, we determine $c_{\mathcal{O}}$ for $\mathcal{O}$ in $\mathcal{N}_{\rm Wh}^{\rm max}(\Theta(\overline{\rm GL}_r^{(n)}))$, which was already shown by Savin[2] and Cai [14] to be equal to

$$\mathcal{O}_{\rm Spr}(j_{W_\nu}^W \varepsilon_\nu) = (n^a b).$$

The main result in Section 5 is the following theorem.

Theorem 1.3 (Theorem 5.1)**.** *Assume* $p \nmid n$*. Consider an unramified theta representation* $\Theta(\overline{\rm GL}_r^{(n)})$ *of the Kazhdan-Patterson cover* $\overline{\rm GL}_r^{(n)}$*. Then for the unique orbit* $\mathcal{O} = (n^a b)$ *in* $\mathcal{N}_{\rm Wh}^{\rm max}(\Theta(\overline{\rm GL}_r^{(n)}))$*, one has*

$$c_{\mathcal{O}} = \langle j_{W_\nu}^W(\varepsilon_\nu), \varepsilon_W \otimes \sigma^{\mathscr{X}} \rangle_W.$$

Thus, Conjecture 4.1 *holds for unramified theta representations of* $\overline{\rm GL}_r^{(n)}$.

Here, $\sigma^{\mathscr{X}} : W \to {\rm Perm}(\mathscr{X})$ is the permutation representation of W on $\mathscr{X}$ via the twisted Weyl action (see (4.2)). The proof of Theorem 1.3 relies on the crucial fact that every nilpotent orbit of ${\rm GL}_r$ is of PL-type à la [34], i.e., it is the principal/regular orbit of a Levi subgroup. In the proof of Theorem 1.3, we also use the result of Gomez et al. [33] and some properties of the j-induction. As discussed earlier, the number $c_{\mathcal{O}}$ is equal to the dimension of certain generalized Whittaker models.

For general $\overline{G}^{(n)}$, we expect that whenever the orbit $\mathcal{O}_{\rm Spr}(j_{W_\nu}^W \varepsilon_\nu)$ is of PL-type, then [27, (2.4)] (i.e., (4.4) in Conjecture 4.1 here) can be checked by using similar analysis as in the case of $\overline{\rm GL}_r^{(n)}$. However, it seems to require new ideas to deal with the case of non-PL-type orbits.

2 Quasi-admissible and raisable orbits

2.1 Covering groups

Let F be a p-adic local field of characteristic 0. Let $\mathbf{G}$ be a connected split linear reductive group over F. Denote by $(X, \Phi, \Delta; Y, \Phi^\vee, \Delta^\vee)$ the root datum of $\mathbf{G}$, where X is the character lattice and Y is the cocharacter lattice of a maximal split torus $\mathbf{T} \subset \mathbf{G}$. Here, Δ is a choice of simple roots, and we denote by $Y^{sc} \subset Y$ the coroot lattice and $X^{sc} \subset X$ the root lattice. Denote by W the Weyl group of the coroot system.

Let $Q : Y \to \mathbf{Z}$ be a Weyl-invariant quadratic form, and B_Q be the associated bilinear form. Assume that $F^\times$ contains the full group μ_n of n-th roots of unity. Consider the pair $(D, \eta = \mathbf{1})$, where D is a "bisector" of Q (see [23, § 2.6]). As discussed in [23, § 2.6], the assumption $\eta = \mathbf{1}$ entails a simpler description of the structure of $\overline{G}$. Moreover, if $\mathbf{G}$ is simply-connected semisimple, then we do not lose any generality with such an assumption since every (D, η) is isomorphic to $(D', \mathbf{1})$ for a certain D' (see [23, § 2.6] for a more detailed discussion on this).

In any case, associated with $(D, \mathbf{1})$ one has a covering group $\overline{G}$ of $G := \mathbf{G}(F)$, which is a central extension

$$\mu_n \hookrightarrow \overline{G}^{(n)} \twoheadrightarrow G$$

of G by μ_n. For simplicity, we may also write $\overline{G} := \overline{G}^{(n)}$. For more details, see [8, 23, 68]. Throughout, for every root α, we define

$$n_\alpha := \frac{n}{\gcd(n, Q(\alpha^\vee))}. \tag{2.1}$$

We also write ${\rm Irr}_{\rm gen}(\overline{G})$ for the set of equivalence classes of irreducible genuine representations of $\overline{G}$, where μ_n acts via a fixed embedding $\mu \hookrightarrow \mathbf{C}^\times$.

Definition 2.1. Assume that $\mathbf{G}_{\mathrm{der}}$ is almost simple. Then the number

$$\mathrm{Inv}_{\mathrm{BD}}(\overline{G}) := Q(\alpha^\vee) \in \mathbf{Z},$$

where $\alpha^\vee$ is any short coroot, is called the Brylinski-Deligne invariant associated with $\overline{G}$.

Note that the Brylinski-Deligne invariant does not depend on n. Moreover, it behaves functorially as follows. Let $\zeta : \mathbf{G} \to \mathbf{H}$ be an algebraic group homomorphism. It induces a group homomorphism $\zeta^\natural : Y_G \to Y_H$ on the cocharacter lattices of $\mathbf{G}$ and $\mathbf{H}$. Let $\overline{H}$ be an n-fold cover of H associated with a quadratic form Q_H. The pull-back n-fold cover $\zeta^*(\overline{H})$ of G via ζ is associated with $Q \circ \zeta^\natural$. In particular, we have

$$\mathrm{Inv}_{\mathrm{BD}}(\zeta^*(\overline{H})) = Q \circ \zeta^\natural(\alpha_G^\vee),$$

where $\alpha_G^\vee$ is any short coroot of G.

2.2 Degenerate Whittaker models

For the terminology and notations in this subsection, we follow [32, 33]. In particular, the Lie algebra $\mathfrak{sl}_2$ over F has a standard basis $\{e_+, h_0, e_-\}$. Recall that $\mathfrak{g}_F$ denotes the Lie algebra of G.

Let $u \in \mathfrak{g}_F$ be a nilpotent element. Given a semisimple element $h \in \mathfrak{g}_F$, one has a decomposition

$$\mathfrak{g}_F = \bigoplus_{i \in I} \mathfrak{g}^h[i],$$

where $I \subset F$ and $\mathfrak{g}^h[i]$ denotes the i-th eigenspace of the adjoint action of h on $\mathfrak{g}_F$. The element $h \in \mathfrak{g}_F$ is called $\mathbf{Q}$-semisimple if $I \subset \mathbf{Q}$. The pair (h, u) is called a Whittaker pair if h is $\mathbf{Q}$-semisimple and $u \in \mathfrak{g}^h[-2]$. A Whittaker pair is called a neutral pair if there exists a nilpotent element $f \in \mathfrak{g}_F$ such that (f, h, u) is an $\mathfrak{sl}_2$-triple, in which case h is called a neutral element for u. For any nilpotent u, the Jacobson-Morozov theorem gives a homomorphism $\gamma : \mathfrak{sl}_2 \to \mathfrak{g}_F$ such that $u = \gamma(e_-)$. Conversely, naturally associated with any γ is an $\mathfrak{sl}_2$-triple $\{f, h, u\} \subset \mathfrak{g}_F$, where h is then a neutral element for u.

Let

$$\kappa : \mathfrak{g}_F \times \mathfrak{g}_F \to F$$

be the Killing form. For any Whittaker pair (h, u), one has a symplectic form

$$\omega_u : \mathfrak{g}_F \times \mathfrak{g}_F \to F$$

given by $\omega_u(x, y) := \kappa(u, [x, y])$. For any rational $i \in \mathbf{Q}$, let

$$\mathfrak{g}^h_{\geqslant i} = \bigoplus_{i' \geqslant i} \mathfrak{g}^h[i'] \quad \text{and} \quad \mathfrak{u}_h := \mathfrak{g}^h_{\geqslant 1}.$$

The restriction $\omega_u|_{\mathfrak{u}_h}$ is well-defined and let $\mathfrak{n}_{h,u}$ be the radical of $\omega_u|_{\mathfrak{u}_h}$. Then

$$[\mathfrak{u}_h, \mathfrak{u}_h] \subset \mathfrak{g}^h_{\geqslant 2} \subset \mathfrak{n}_{h,u}.$$

By [32, Lemma 3.2.6], one has

$$\mathfrak{n}_{h,u} = \mathfrak{g}^h_{\geqslant 2} + \mathfrak{g}^h_1 \cap \mathfrak{g}_u,$$

where $\mathfrak{g}_u$ denotes the centralizer of $u \in \mathfrak{g}$. If the Whittaker pair (h, u) is a neutral pair, then $\mathfrak{n}_{h,u} = \mathfrak{g}^h_{\geqslant 2}$.

Let $U_h = \exp(\mathfrak{u}_h)$ and $N_{h,u} = \exp(\mathfrak{n}_{h,u})$ be the corresponding unipotent subgroups of G. Let

$$\psi : F \to \mathbf{C}^\times$$

be a nontrivial character. Define a character of $N_{h,u}$ by

$$\psi_u(n) = \psi(\kappa(u, \log(n))). \tag{2.2}$$

Let $N'_{h,u} = N_{h,u} \cap \mathrm{Ker}(\psi_u)$. Then $U_h/N'_{h,u}$ is a Heisenberg group with center $N_{h,u}/N'_{h,u}$.

Viewing ψ_u as a character of $N_{h,u}/N'_{h,u}$, we denote by σ_{ψ_u} the unique irreducible representation of $U_h/N'_{h,u}$ with the central character ψ_u. The pull-back of σ_{ψ_u} via the quotient $U_h \twoheadrightarrow U_h/N'_{h,u}$ gives an irreducible representation $\sigma^\sharp_{\psi_u}$ of U_h. The group $\overline{G}$ splits uniquely over every unipotent subgroup of G. Thus, we view all the above U_h and $N_{h,u}$ as subgroups of $\overline{G}$.

The degenerate Whittaker model $\pi_{h,u}$ of $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$ associated with any Whittaker pair (h,u) is by definition (see [33, § 2.5])

$$\pi_{h,u} := ((\mathrm{ind}^{\overline{G}}_{\mu_n \times U_h} \epsilon \otimes \sigma^\sharp_{\psi_u}) \otimes \pi)_G,$$

where $\epsilon : \mu_n \hookrightarrow \mathbf{C}^\times$ is the fixed embedding. For a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}_F$, we write $\pi_{\mathcal{O}} := \pi_{h,u}$ for any $u \in \mathcal{O}$ and any choice (h,u) of neutral pair. Define

$$\mathcal{N}_{\mathrm{Wh}}(\pi) = \{\mathcal{O} \subset \mathfrak{g}_F : \pi_{\mathcal{O}} \neq 0\},$$

and let

$$\mathcal{N}^{\max}_{\mathrm{Wh}}(\pi) \subset \mathcal{N}_{\mathrm{Wh}}(\pi)$$

be the subset consisting of maximal elements.

2.3 *F*-split nilpotent orbits

Let $\gamma = (f,h,u) \subset \mathfrak{g}_F$ be an $\mathfrak{sl}_2$-triple. Let $\mathcal{O}_u \subset \mathfrak{g}_F$. Viewing $\gamma \subset \mathfrak{g}_{F^{\mathrm{al}}}$, one has

$$\mathbf{G}_u = \mathbf{G}_\gamma \cdot \mathbf{N}_u,$$

where $\mathbf{G}_\gamma \subset \mathbf{G}$ is the stabilizer subgroup of γ. The group $\mathbf{G}_\gamma$ is a (possibly disconnected) linear algebraic group and is the reductive part of $\mathbf{G}_u$. In general, we have

$$(\mathcal{O}_u \otimes F^{\mathrm{al}}) \cap \mathfrak{g}_F = \bigsqcup_{i \in I} \mathcal{O}_{u_i},$$

where the F-rational orbits $\mathcal{O}_{u_i}$ are classified by the Galois cohomology group $H^1(F, \mathbf{G}_\gamma)$ (see [61, Chapter III] and [57, § 4]). This group is only a pointed set if $\mathbf{G}_\gamma$ is not abelian.

Let $\mathbf{G}_{\gamma,0} \subseteq \mathbf{G}_\gamma$ be the connected component of the identity in $\mathbf{G}_\gamma$. We get the group

$$\pi_0(\mathbf{G}_\gamma) = \mathbf{G}_\gamma / \mathbf{G}_{\gamma,0}$$

of connected components of $\mathbf{G}_\gamma$. This group is one of the symmetric groups $S_i, 1 \leqslant i \leqslant 5$; moreover, for classical groups, it is either trivial or S_2. One has an exact sequence of pointed sets

$$H^1(F, \mathbf{G}_{\gamma,0}) \longrightarrow H^1(F, \mathbf{G}_\gamma) \xrightarrow{\ \iota\ } H^1(F, \pi_0(\mathbf{G}_\gamma)) \longrightarrow \cdots.$$

In particular, if $\mathbf{G}_{\gamma,0}$ is semisimple and simply-connected, then ι is injective.

The orbit $\mathcal{O}_u$ is called an F-split nilpotent orbit if the reductive group $\mathbf{G}_\gamma$ is split over F. In this paper, we only consider F-split orbits since the computation with covers of a non-split $\mathbf{G}_\gamma$ involves further subtlety. Henceforth, we assume that the orbit $\mathcal{O} \subset \mathfrak{g}_F$ is always F-split without explicating this again.

In this paper, we implement the methods of Moeglin [54], Nevins [56, 57], Jiang et al. [42] and Gomez et al. [33] to give some necessary or sufficient condition for an orbit $\mathcal{O}_u$ to possibly lie in the wavefront set of some genuine representation $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$. In fact, for general $\overline{G}$, we determine (if possible) the F-split orbits of $\mathfrak{g}_F$ that cannot be the wavefront set of any genuine representation π. More precisely, we consider necessary conditions for an F-split orbit $\mathcal{O}_u$ to be quasi-admissible in the sense of [33], and also give sufficient conditions for it to be raisable, a notion due to [42]. All these together will enable us to determine the subset of the orbits in $\mathcal{N}$, which never occur in the wavefront set of any $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$.

Some results here also hold in the global setting for genuine automorphic representations.

2.4 $\overline{G}^{(n)}$-quasi-admissible orbits

Recall that by restriction, one has a non-degenerate symplectic form

$$\omega_u : \mathfrak{g}[1] \times \mathfrak{g}[1] \to F.$$

The group $G_\gamma = \mathbf{G}_\gamma(F)$ acts on $\mathfrak{g}[1]$ and preserves the form ω_u. Thus, one has a natural group homomorphism

$$\phi : G_\gamma \to \mathrm{Sp}(\mathfrak{g}[1]).$$

By pull-back of the metaplectic double cover $\mathrm{Mp}(\mathfrak{g}[1])$ of $\mathrm{Sp}(\mathfrak{g}[1])$ via ϕ (i.e., the fiber product of G_γ and $\mathrm{Mp}(\mathfrak{g}[1])$ over $\mathrm{Sp}(\mathfrak{g}[1])$), one has a double cover

$$\mu_2 \hookrightarrow \overline{G}_\gamma^{(2),\phi} \twoheadrightarrow G_\gamma.$$

Here, the metaplectic group $\mathrm{Mp}(\mathfrak{g}[1])$ is uniquely determined by its Brylinski-Deligne invariant

$$\mathrm{Inv}_{\mathrm{BD}}(\mathrm{Mp}(\mathfrak{g}[1])) = 1.$$

The inclusion $G_\gamma \subset G$ gives an inherited covering $\overline{G}_\gamma^{(n)}$. We obtain the fiber product $\overline{G}_\gamma^{(n)} \times_{G_\gamma} \overline{G}_\gamma^{(2),\phi}$ as in the following commutative diagram:

$$\begin{array}{ccccc}
\overline{G}_\gamma^{(n)} \times_{G_\gamma} \overline{G}_\gamma^{(2),\phi} & \longrightarrow & \overline{G}_\gamma^{(2),\phi} & \longrightarrow & \mathrm{Mp}(\mathfrak{g}[1]) \\
\downarrow {\scriptstyle q_\gamma} & & \downarrow & & \downarrow \\
\overline{G}_\gamma^{(n)} & \twoheadrightarrow & G_\gamma & \xrightarrow{\phi} & \mathrm{Sp}(\mathfrak{g}[1]). \\
\downarrow & & \downarrow & & \\
\overline{G}^{(n)} & \twoheadrightarrow & G & &
\end{array}$$

Write $G_{\gamma,\mathrm{der}} \subset G_\gamma$ for the subgroup generated by unipotent elements; it is equal to the derived subgroup of $G_{\gamma,0} := \mathbf{G}_{\gamma,0}(F) \subset G_\gamma$. Thus, we have the inclusions

$$G_{\gamma,\mathrm{der}} \subset G_{\gamma,0} \subset G_\gamma.$$

Let $\mathbf{G}_{\gamma,\mathrm{der}} \subset \mathbf{G}_\gamma$ be the derived subgroup, and $\mathbf{f} : \mathbf{G}_{\gamma,sc} \twoheadrightarrow \mathbf{G}_{\gamma,\mathrm{der}}$ be the simply-connected cover. Setting $G_{\gamma,sc} := \mathbf{G}_{\gamma,sc}(F)$, the map $\mathbf{f}$ induces a map

$$f : G_{\gamma,sc} \twoheadrightarrow G_{\gamma,\mathrm{der}} \hookrightarrow \mathbf{G}_{\gamma,\mathrm{der}}(F). \tag{2.3}$$

In particular, the inclusion in (2.3) may not be an equality in general (see [6, § 6] for a detailed discussion).

Henceforth, we write

$$\overline{G}_\gamma^{(n,2)} := \overline{G}_\gamma^{(n)} \times_{G_\gamma} \overline{G}_\gamma^{(2),\phi}.$$

Similarly, for $\star \in \{0, \mathrm{der}, sc\}$, one has the natural pull-back $\overline{G}_{\gamma,\star}^{(n)}$ and $\overline{G}_{\gamma,\star}^{(2),\phi}$, and we set

$$\overline{G}_{\gamma,\star}^{(n,2)} := \overline{G}_{\gamma,\star}^{(n)} \times_{G_{\gamma,\star}} \overline{G}_{\gamma,\star}^{(2),\phi}.$$

One has by construction the following commutative diagram:

$$\begin{array}{ccc}
\overline{G}_{\gamma,\star}^{(n,2)} & \longrightarrow & \overline{G}_{\gamma,\star}^{(2),\phi} \\
\downarrow & \searrow {\scriptstyle p_{\gamma,\star}} & \downarrow \\
\overline{G}_{\gamma,\star}^{(n)} & \twoheadrightarrow & G_{\gamma,\star},
\end{array} \tag{2.4}$$

where $p_{\gamma,\star}$ is the canonical quotient map.

Definition 2.2 (See [33]). A representation of a group H with $\mu_n \times \mu_2 \subset Z(H)$ is called $(n,2)$-genuine if the central subgroups μ_n and μ_2 both act faithfully. An F-split nilpotent orbit $\mathcal{O} = \mathcal{O}_u \subset \mathfrak{g}$ is called

- $\overline{G}^{(n)}$-admissible if the map $p_{\gamma,0} : \overline{G}^{(n,2)}_{\gamma,0} \twoheadrightarrow G_{\gamma,0}$ in (2.4) splits;
- $\overline{G}^{(n)}$-quasi-admissible if $\overline{G}^{(n,2)}_{\gamma}$ admits a finite-dimensional $(n,2)$-genuine representation.

For simplicity, we may just use admissibility and quasi-admissibility when the underlying covering group is clear from the context. Assume that $p_{\gamma,0}$ splits. Then we have an isomorphism

$$\overline{G}^{(n,2)}_{\gamma,0} \simeq (\mu_n \times \mu_2) \times G_{\gamma,0},$$

and thus $\overline{G}^{(n,2)}_{\gamma,0}$ clearly has a one-dimensional $(n,2)$-genuine character. Since $\overline{G}^{(n,2)}_{\gamma,0}$ is of finite index in $\overline{G}^{(n,2)}_{\gamma}$, induction gives a finite-dimensional $(n,2)$-genuine representation of $\overline{G}^{(n,2)}_{\gamma}$. Hence, we see that

$$\text{admissible} \Rightarrow \text{quasi-admissible}$$

for F-split orbits $\mathcal{O}$.

To understand further the quasi-admissibility, we denote by $(\overline{G}^{(n,2)}_{\gamma})_+ \subset \overline{G}^{(n,2)}_{\gamma}$ the subgroup generated by unipotent elements. For $\star \in \{0, \mathrm{der}\}$, one has a commutative diagram

$$\begin{array}{ccccc} \mu_n \times \mu_2 & \hookrightarrow & \overline{G}^{(n,2)}_{\gamma,\star} & \xrightarrow{p_{\gamma,\star}} & G_{\gamma,\star} \\ \iota^D \uparrow & & \uparrow & & \uparrow \\ \mathrm{Ker}(p^D_{\gamma}) & \hookrightarrow & (\overline{G}^{(n,2)}_{\gamma})_+ & \xrightarrow{p^D_{\gamma}} & G_{\gamma,\mathrm{der}}. \end{array} \tag{2.5}$$

Here, p^D_{γ} is just the restriction of $p_{\gamma,\star}$ to $(\overline{G}^{(n,2)}_{\gamma})_+$; it is surjective since $G_{\gamma,\mathrm{der}}$ is generated by unipotent elements. Note that ι^D may not be surjective in general.

Lemma 2.3. *Let $\mathcal{O} = \mathcal{O}_u$ be an F-split $\overline{G}^{(n)}$-quasi-admissible orbit.*

(i) *If n is odd, then $\mathrm{Ker}(p^D_{\gamma}) = \{1\}$ and thus $p_{\gamma,0}$ splits over $G_{\gamma,\mathrm{der}}$.*

(ii) *If $n = 2m$ is even, then either $\mathrm{Ker}(p^D_{\gamma}) = \{1\}$ or $\mathrm{Ker}(p^D_{\gamma}) = \Delta(\mu_2)$, where $\Delta : \mu_2 \hookrightarrow \mu_n \times \mu_2$ is the diagonal embedding.*

Proof. Let σ be a finite-dimensional $(n,2)$-genuine representation of $\overline{G}^{(n,2)}_{\gamma}$, viewed as a representation of $\overline{G}^{(n,2)}_{\gamma,0}$ by restriction. By [33, Lemma 4.5] (see also the proof of [33, Proposition 6.4]), we have

$$(\overline{G}^{(n,2)}_{\gamma})_+ \subset \mathrm{Ker}(\sigma).$$

In particular, $\mathrm{Ker}(p^D_{\gamma}) \subset \mathrm{Ker}(\sigma)$. Since σ is $(n,2)$-genuine, this immediately gives the result. □

Now we set $n^* = \mathrm{lcm}(n,2)$ and consider the multiplication map

$$\mathfrak{m} : \mu_n \times \mu_2 \to \mu_{n^*}. \tag{2.6}$$

By push-out via $\mathfrak{m}$, we obtain the following:

$$\begin{array}{ccccc} \mu_n \times \mu_2 & \hookrightarrow & \overline{G}^{(n,2)}_{\gamma,\mathrm{der}} & \xrightarrow{p_{\gamma,\mathrm{der}}} & G_{\gamma,\mathrm{der}} \\ \downarrow \mathfrak{m} & & \downarrow & & \| \\ \mu_{n^*} & \hookrightarrow & \mathfrak{m}_*(\overline{G}^{(n,2)}_{\gamma,\mathrm{der}}) & \xrightarrow{p^{\mathfrak{m}}_{\gamma,\mathrm{der}}} & G_{\gamma,\mathrm{der}}. \end{array} \tag{2.7}$$

Assume that $G_{\gamma} = \prod_{j \in J} G_{\gamma,j}$, where $G_{\gamma,j,\mathrm{der}}$ is almost simple for every j. Then one has

$$\begin{array}{ccccc} \mu_{n^*} & \hookrightarrow & \mathfrak{m}_*(\overline{G}^{(n,2)}_{\gamma,j,0}) & \xrightarrow{p^{\mathfrak{m}}_{\gamma,j,0}} & G_{\gamma,j,0} \\ \| & & \uparrow & & \uparrow \\ \mu_{n^*} & \hookrightarrow & \mathfrak{m}_*(\overline{G}^{(n,2)}_{\gamma,j,\mathrm{der}}) & \xrightarrow{p^{\mathfrak{m}}_{\gamma,j,\mathrm{der}}} & G_{\gamma,j,\mathrm{der}} \end{array} \tag{2.8}$$

for every $j \in J$. In general, the natural set-theoretic map

$$\prod_j \overline{G}^{(n,2)}_{\gamma,j} \to \overline{G}^{(n,2)}_{\gamma}$$

may not be a group homomorphism, i.e., block commutativity may fail. However, for the derived subgroup, we have the natural group isomorphism

$$\left(\prod_j \overline{G}^{(n,2)}_{\gamma,j,\mathrm{der}}\right)/K \simeq \overline{G}^{(n,2)}_{\gamma,\mathrm{der}},$$

where $K = \{(\zeta_j, \xi_j) \in (\mu_n \times \mu_2)^{|J|} : \prod_j (\zeta_j, \xi_j) = (1,1)\}$. Indeed, if Z_j denotes the cocharacter lattice of $G_{\gamma,j,\mathrm{der}}$ and Z that for $G_{\gamma,\mathrm{der}}$, then any Weyl-invariant quadratic form Q on Z decomposes as $Q = \bigoplus_j Q|_{Z_j}$. This shows that $\overline{G}^{(n)}_{\gamma,j,\mathrm{der}}$ commutes with each other in $\overline{G}^{(n)}_{\gamma,\mathrm{der}}$, and similarly for $\overline{G}^{(2),\phi}_{\gamma,j,\mathrm{der}}$; thus, so does the $\overline{G}^{(n,2)}_{\gamma,j,\mathrm{der}}$.

Proposition 2.4. *Keep notations as above. Then the following are equivalent*:

(i) *The F-split orbit $\mathcal{O}$ is quasi-admissible.*

(ii) *The map $p^{\mathfrak{m}}_{\gamma,\mathrm{der}}$ in* (2.7) *splits.*

(iii) *For every j, the map $p^{\mathfrak{m}}_{\gamma,j,\mathrm{der}}$ in* (2.8) *splits.*

(iv) *For every j, the cover $\mathfrak{m}_*(\overline{G}^{(n,2)}_{\gamma,j,0})$ has a finite-dimensional μ_{n^*}-genuine representation.*

Proof. Let $\mathcal{O}$ be as given. The equivalence between (ii) and (iii) is clear in view of the preceding discussion. We first consider the equivalence between (i) and (ii).

Assuming (i), we have the following two cases as from Lemma 2.3:

• If $\mathrm{Ker}(p^D_\gamma) = \{1\}$, then $p_{\gamma,\mathrm{der}}$ splits over $G_{\gamma,\mathrm{der}}$ and thus $p^{\mathfrak{m}}_{\gamma,\mathrm{der}}$ splits as well.

• If $\mathrm{Ker}(p^D_\gamma) = \Delta(\mu_2)$, then necessarily $2|n$ and $n^* = n$. In this case, by pushing out $(\overline{G}^{(n,2)}_{\gamma})_+$ via ι^D, we obtain an isomorphism of extensions from the bottom two lines as in

$$\begin{array}{ccccc}
\mathrm{Ker}(p^D_\gamma) & \hookrightarrow & (\overline{G}^{(n,2)}_{\gamma})_+ & \xrightarrow{p^D_\gamma} & G_{\gamma,\mathrm{der}} \\
\downarrow{\scriptstyle \iota^D} & & \downarrow & & \| \\
\mu_n \times \mu_2 & \hookrightarrow & \iota^D_*(\overline{G}^{(n,2)}_{\gamma})_+ & \twoheadrightarrow & G_{\gamma,\mathrm{der}} \\
\| & & \simeq\uparrow & & \| \\
\mu_n \times \mu_2 & \hookrightarrow & \overline{G}^{(n,2)}_{\gamma,\mathrm{der}} & \twoheadrightarrow & G_{\gamma,\mathrm{der}}.
\end{array}$$

Since in this case, $\mathrm{Ker}(\mathfrak{m}) = \mathrm{Im}(\iota^D)$, we see that one has a retraction of the short exact sequence

$$\mu_{n^*} \hookrightarrow \mathfrak{m}_*(\overline{G}^{(n,2)}_{\gamma,\mathrm{der}}) \twoheadrightarrow G_{\gamma,\mathrm{der}},$$

which then gives a splitting of $p^{\mathfrak{m}}_{\gamma,\mathrm{der}}$.

Now, assuming that $p^{\mathfrak{m}}_{\gamma,\mathrm{der}}$ splits, we have a homomorphism s such that

$$\begin{array}{ccc}
\mu_n \times \mu_2 & \hookrightarrow & \overline{G}^{(n,2)}_{\gamma,\mathrm{der}}/(\overline{G}^{(n,2)}_{\gamma})_+ \\
\downarrow & \swarrow{\scriptstyle s} & \\
\mu_{n^*} & &
\end{array}$$

commutes. This implies in particular that

$$\begin{array}{ccc}
\mu_n \times \mu_2 & \xrightarrow{\iota} & (\mu_n \times \mu_2)/\mathrm{Ker}(p^D_\gamma) \\
\downarrow{\scriptstyle \mathfrak{m}} & \swarrow{\scriptstyle s} & \\
\mu_{n^*} & &
\end{array}$$

commutes. Since $\mathfrak{m}$ is injective on μ_n and also on μ_2, we see that ι is injective on μ_n and on μ_2 as well. Thus, one has a finite-dimensional $(n,2)$-genuine representation of $\overline{G}^{(n,2)}_{\gamma,\mathrm{der}}$ and also of $\overline{G}^{(n,2)}_{\gamma,0}$ (see [33, Proposition 6.4]).

This equivalence between (iii) and (iv) follows an analogous argument for that of (i) and (ii). This completes the proof. □

In view of Proposition 2.4, we assume now that $G_{\gamma,\mathrm{der}}$ is almost simple and gives an explicit condition for the splitting of $p^{\mathfrak{m}}_{\gamma,\mathrm{der}}$. Consider the Brylinski-Deligne invariant (see Definition 2.1)

$$Q_1 := \mathrm{Inv}_{\mathrm{BD}}(\overline{G}^{(n)}_{\gamma,0}), \quad Q_2 := \mathrm{Inv}_{\mathrm{BD}}(\overline{G}^{(2),\phi}_{\gamma,0}) \tag{2.9}$$

associated with $\overline{G}^{(n)}_{\gamma,0}$ and $\overline{G}^{(2),\phi}_{\gamma,0}$, respectively. Then we can view $\overline{G}^{(n)}_{\gamma,0}$ as an n^*-fold cover associated with $(n^*/n)Q_1$, and view $\overline{G}^{(2),\phi}_{\gamma,0}$ as an n^*-fold cover associated with $(n^*/2)Q_2$. Thus, we see that $\mathfrak{m}_*(\overline{G}^{(n,2)}_{\gamma,\mathrm{der}})$ is an n^*-fold cover with the Brylinski-Deligne invariant $(n^*/n)Q_1 + (n^*/2)Q_2$.

Proposition 2.5. *Let $Q_1 \in \mathbf{Z}$ (resp. $Q_2 \in \mathbf{Z}$) be the Brylinski-Deligne invariant of $\overline{G}^{(n)}_{\gamma,0}$ (resp. $\overline{G}^{(2),\phi}_{\gamma,0}$). Assume that n^* is coprime to the size of $\mathrm{Ker}(\mathbf{f} : \mathbf{G}_{\gamma,sc} \twoheadrightarrow \mathbf{G}_{\gamma,\mathrm{der}})$. Then $p^{\mathfrak{m}}_{\gamma,\mathrm{der}}$ splits if and only if n^* divides $(n^*/n)Q_1 + (n^*/2)Q_2$, or equivalently,*

(i) *$n|Q_1$ when $2|Q_2$, and*

(ii) *$n/\gcd(n,Q_1) = 2$ when $2 \nmid Q_2$.*

Proof. Recall that the homomorphism $\mathbf{f} : \mathbf{G}_{\gamma,sc} \twoheadrightarrow \mathbf{G}_{\gamma,\mathrm{der}}$ induces a map

$$f : G_{\gamma,sc} \twoheadrightarrow G_{\gamma,\mathrm{der}} \hookrightarrow \mathbf{G}_{\gamma,\mathrm{der}}(F).$$

By the pull-back via f, the n^*-fold cover $\overline{G}^{(n^*)}_{\gamma,\mathrm{der}}$ of $G_{\gamma,\mathrm{der}}$ gives an n^*-fold cover $\overline{G}^{(n^*)}_{\gamma,sc}$, as depicted in the following diagram:

$$\begin{array}{ccccc} & & \mu_{n^*} & = & \mu_{n^*} \\ & & \downarrow & & \downarrow \\ \mathrm{Ker}(f) & \xhookrightarrow{\iota} & \overline{G}^{(n^*)}_{\gamma,sc} & \twoheadrightarrow & \overline{G}^{(n^*)}_{\gamma,\mathrm{der}} \\ \| & & s \uparrow\downarrow & & \downarrow q \\ \mathrm{Ker}(f) & \xhookrightarrow{\iota} & G_{\gamma,sc} & \xrightarrow{f} & G_{\gamma,\mathrm{der}}. \end{array} \tag{2.10}$$

Thus, if $\overline{G}^{(n^*)}_{\gamma,\mathrm{der}}$ splits, then $\overline{G}^{(n^*)}_{\gamma,sc}$ splits by its definition as pull-back.

On the other hand, let

$$s : G_{\gamma,sc} \hookrightarrow \overline{G}^{(n^*)}_{\gamma,sc}$$

be a splitting, which is actually unique since $G_{\gamma,sc}$ is equal to its derived subgroup. The assumption implies that $\gcd(|\mathrm{Ker}(f)|, n^*) = 1$, and thus any group homomorphism from $\mathrm{Ker}(f)$ to μ_{n^*} is trivial. This shows that the splitting of $\mathrm{Ker}(f)$ into $\overline{G}^{(n^*)}_{\gamma,sc}$ is unique, i.e., the left lower square in (2.10) involving s commutes. This implies that $\overline{G}^{(n^*)}_{\gamma,\mathrm{der}}$ splits over $G_{\gamma,\mathrm{der}}$.

The above shows that with our assumption on n^*, the map $p^{\mathfrak{m}}_{\gamma,\mathrm{der}}$ splits if and only if $\overline{G}^{(n^*)}_{\gamma,sc}$ splits over $G_{\gamma,sc}$. However, since $G_{\gamma,sc}$ is simply-connected, its n^*-fold cover splits if and only if n^* divides the Brylinski-Deligne invariant. This completes the proof. □

Remark 2.6. In general, consider $\mathbf{f} : \mathbf{G}_{sc} \twoheadrightarrow \mathbf{G}$, the simply-connected cover of a semi-simple group $\mathbf{G}$, which gives $f : G_{sc} \twoheadrightarrow G_{\mathrm{der}} \hookrightarrow G$. There exists a non-split extension of G_{der} whose pull-back is split. For example, consider $\mathbf{G} = \mathrm{SO}_3$ and thus $\mathbf{G}_{sc} = \mathrm{SL}_2$. Then one has $G_{\mathrm{der}} = \mathrm{SL}_2(F)/\{\pm 1\}$, where we identify $\{\pm 1\}$ with the center of $\mathrm{SL}_2(F)$. Consider the double cover

$$\overline{G}^{(2)}_{\mathrm{der}} := (\mu_2 \times \mathrm{SL}_2(F))/\mathrm{Im}(\sigma),$$

where the map $\sigma:\{\pm1\}\to\mu_2\times\mathrm{SL}_2(F)$ is given by $\sigma(a)=((\varpi,a)_2,a)$. It is easy to see that the pull-back of $\overline{G}^{(2)}_{\mathrm{der}}$ to $G_{sc}=\mathrm{SL}_2$ is a split extension. However, $\overline{G}^{(2)}_{\mathrm{der}}$ splits over G_{der} if and only if $(\varpi,-1)_2=1$. In fact, $\overline{G}^{(2)}_{\mathrm{der}}$ is a Brylinski-Deligne cover associated with $(D=0,\eta)$, and is the running example for several interesting phenomena discussed in [23].

However, in the notation of Proposition 2.5, we believe that the splitting of $\overline{G}^{(n^*)}_{\gamma,sc}$ is always equivalent to the splitting of $\overline{G}^{(n^*)}_{\gamma,\mathrm{der}}$, without the assumption "$\cdots$ that n^* is coprime to the size of $\cdots$" imposed there, i.e., the covers $\overline{G}^{(n^*)}_{\gamma,\mathrm{der}}$ that arise are not of the type in the preceding paragraph, essentially due to the fact that we consider in this paper covers associated with $(D,\eta=\mathbf{1})$. We are not able to prove this expectation in full generality, though.

The importance of the quasi-admissibility is given as follows.

Theorem 2.7 (See [33, Theorem 1]). *Let $\mathcal{O}$ be an F-split orbit such that $\mathcal{O}\in\mathcal{N}^{\max}_{\mathrm{Wh}}(\pi)$ for some $\pi\in\mathrm{Irr}_{\mathrm{gen}}(\overline{G})$. Then $\mathcal{O}$ is quasi-admissible.*

Proof. The proof of this result relies on [33, Theorem 1.4 and Proposition 6.3]. The statement in [33, Theorem 1.4(i)] needs a revision in the covering setting. However, the idea of the proof of [33, Theorem 1.4] still applies. Indeed, the proof of [33, Theorem 1.4] actually shows that if $\mathcal{O}\in\mathcal{N}^{\max}_{\mathrm{Wh}}(\pi)$, then the non-zero finite-dimensional space $\pi_{N_{\mathcal{O}},\psi_{\mathcal{O}}}$ affords an $(n,2)$-genuine representation of $\overline{G}^{(n,2)}_{\gamma}$ and thus is quasi-admissible. □

2.5 Admissibility versus quasi-admissibility

Again, consider $\mathbf{f}:\mathbf{G}_{sc}\twoheadrightarrow\mathbf{G}$ of a semisimple $\mathbf{G}$ and the arising $f:G_{sc}\to G$, which is not necessarily surjective. It is possible that a cover over G does not split, but its restriction to G_{der} splits and thus also the pull-back to G_{sc}. Note that if $\mathbf{G}=\mathbf{G}_{\gamma,0}$ for some γ, then this gives an example of orbit which is quasi-admissible but not admissible.

As a more concrete example of this, consider $\mathbf{G}=\mathrm{SO}_k$. Let $\overline{\mathrm{SL}}^{(m)}_k$ be the m-fold cover with the Brylinski-Deligne invariant $Q(\alpha^\vee)$, where $\alpha^\vee$ is any coroot of SL_k. Consider the cover $\overline{\mathrm{SO}}^{(m)}_k$ obtained from restricting $\overline{\mathrm{SL}}_k$ via the inclusion $\mathrm{SO}_k\hookrightarrow\mathrm{SL}_k$. Thus,

$$\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SO}}^{(m)}_k)=2Q(\alpha^\vee)\quad\text{for }k\geqslant4\quad\text{and}\quad\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SO}}^{(m)}_3)=4Q(\alpha^\vee).$$

Lemma 2.8. *Keep notations as above. The cover $\overline{\mathrm{SO}}^{(m)}_k, k\geqslant3$ has a finite-dimensional μ_m-genuine representation if and only if $m|\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SO}}^{(m)}_k)$.*

Proof. We first show the "only if" part. If $\overline{\mathrm{SO}}^{(m)}_k$ has a finite-dimensional μ_m-genuine representation, then it gives rise to one for $\overline{G}_{sc}=\overline{\mathrm{Spin}}_k$. It then follows that $\overline{G}_{sc}$ splits over G_{sc}, which implies that m divides $\mathrm{Inv}_{\mathrm{BD}}(\overline{G}_{sc})=\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SO}}^{(m)}_k)$.

Now for the "if" part, we first assume $m|2Q(\alpha^\vee)$. Then the desired result essentially follows from the discussion in [26, § 2.4]. Let $\mathbf{Z}e$ be the cocharacter lattice of $\mathbf{G}_m$, and $Q:\mathbf{Z}e\to\mathbf{Z}$ be a quadratic form. It gives the cover

$$\mu_m\hookrightarrow\overline{F^\times}\twoheadrightarrow F^\times \tag{2.11}$$

associated with Q. Now assume $m|2Q(e)$. Then the cover $\overline{F^\times}$ splits over $F^{\times2}$, and thus we obtain a cover

$$\mu_m\hookrightarrow\overline{F^\times/2}\twoheadrightarrow F^\times/2,$$

where $F^\times/2:=F^\times/F^{\times2}$. This extension is abelian, though non-split. Consider the spinor norm

$$\mathcal{N}:\mathrm{SO}_k\to F^\times/2$$

and the pull-back cover

$$\mu_m\hookrightarrow\mathcal{N}^*(\overline{F^\times/2})\twoheadrightarrow\mathrm{SO}_k.$$

If we consider the $\overline{\mathrm{SL}}_k^{(m)}$ with the Brylinski-Deligne invariant $Q(\alpha^\vee) := Q(e)$, then by restriction, it gives rise to $\overline{\mathrm{SO}}_k^{(m)}$, and one has

$$\overline{\mathrm{SO}}_k^{(m)} \simeq \mathcal{N}^*(\overline{F^\times/2})$$

(see [26, Example 2.10]). Thus, there is a one-dimensional μ_m-genuine representation of $\overline{\mathrm{SO}}_k$.

For the "if" part, the only remaining case is when $k = 3$ and $m = 4Q(\alpha^\vee)$. In this case, the extension (2.11) associated with $Q(e) := Q(\alpha^\vee)$ also splits over $F^{\times 2}$. The pull-back cover

$$\mu_m \hookrightarrow \mathcal{N}^*(\overline{F^\times/2}) \twoheadrightarrow \mathrm{SO}_3$$

is equal to the cover $\overline{\mathrm{SO}}_3^{(m)}$ of SO_3 restricted from $\overline{\mathrm{SL}}_3^{(m)}$ with $\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SL}}_3^{(m)}) = Q(\alpha^\vee)$ via $\mathrm{SO}_3 \hookrightarrow \mathrm{SL}_3$. Since in this case, $\overline{F^\times/2}$ clearly has a finite-dimensional μ_m-genuine representation, so does the cover $\overline{\mathrm{SO}}_3^{(m)}$.

Combining all the above, we complete the proof. □

Thus, suppose that $\mathcal{O}_u$ and γ are such that
- $G_{\gamma,0} = \mathrm{SO}_k$ as above,
- $\mathrm{Inv}_{\mathrm{BD}}(G_{\gamma,0}^{(n)}) = 2Q(\alpha^\vee)$ with $n|2Q(\alpha^\vee)$, and
- $4|\mathrm{Inv}_{\mathrm{BD}}(G_{\gamma,0}^{(2),\phi})$.

Such an orbit $\mathcal{O}_u$ is always quasi-admissible. However, it might not be admissible if $n = 2Q(\alpha^\vee)$. A concrete example is the orbit B_3 of the exceptional group F_4 (see Subsection 3.5).

2.6 $\overline{G}^{(n)}$-raisable orbits

Now we discuss the notion of $\overline{G}$-raisability following [42]. It pertains to a "local" version of non-quasi-admissibility relative to a "good" choice of an $\mathrm{SL}_2 \subset G_\gamma$, if it exists.

Let $\mathfrak{g}_\gamma \subset \mathfrak{g}$ be the centralizer of γ in $\mathfrak{g}$. Assume that
- (C0) There is a non-trivial map

$$\tau : \mathfrak{sl}_2 \to \mathfrak{g}_\gamma.$$

In this case, we write

$$\mathfrak{sl}_{2,\tau} = \tau(\mathfrak{sl}_2).$$

If we set

$$u_\tau := \tau(e_-(x))$$

for some $0 \neq x \in F$, then

$$\gamma' = \gamma \oplus \tau : \mathfrak{sl}_2 \to \mathfrak{g}$$

is a Jacobson-Morozov map associated with the nilpotent element $u' = u + u_\tau$. Let $\mathfrak{g}[j,l] \subset \mathfrak{g}$ be the space of vectors with γ-weight j and τ-weight l. We assume the following:
- (C1) The τ-weights l are bounded by 2.
- (C2) As $\mathfrak{sl}_{2,\tau}$-module,

$$\mathfrak{g}[1] = \mathfrak{g}[1]^{\mathfrak{sl}_{2,\tau}} \oplus mV_2.$$

- (C3) $\dim \mathfrak{g}[0,2] = 1 + \dim \mathfrak{g}[2,2]$.

By abuse of notation, we still denote by $\tau : \mathrm{SL}_2 \to G_\gamma$ the map corresponding to τ. Also, there is a natural group homomorphism

$$\phi_m : \mathrm{SL}_2 \to \mathrm{Sp}_{2m}$$

arising from $\phi \circ \tau$ and (C2) above. Recall that

$$\phi : G_\gamma \to \mathrm{Sp}(\mathfrak{g}[1])$$

is the natural group homomorphism. This gives rise to the following diagram:

$$\begin{array}{ccccc} \overline{\mathrm{SL}}_2^{(n),\tau} \times_{\mathrm{SL}_2} \overline{\mathrm{SL}}_2^{(2),\phi_m} & \longrightarrow & \overline{\mathrm{SL}}_2^{(2),\phi_m} & \longrightarrow & \mathrm{Mp}_{2m} \\ \downarrow & \searrow^{p_\tau} & \downarrow & & \downarrow \\ \overline{\mathrm{SL}}_2^{(n),\tau} & \twoheadrightarrow & \mathrm{SL}_2 & \xrightarrow{\phi_m} & \mathrm{Sp}_{2m}. \\ \downarrow & & \downarrow \tau & & \\ \overline{G}^{(n)} & \twoheadrightarrow & G & & \end{array} \tag{2.12}$$

From now, we write

$$\overline{\mathrm{SL}}_{2,\tau}^{(n,2)} := \overline{\mathrm{SL}}_2^{(n),\tau} \times_{\mathrm{SL}_2} \overline{\mathrm{SL}}_2^{(2),\phi_m}.$$

The two covering groups $\overline{\mathrm{SL}}_2^{(n),\tau}$ and $\overline{\mathrm{SL}}_2^{(2),\phi_m}$ both arise from the Brylinski-Deligne framework. Consider $n^* = \mathrm{lcm}(n,2)$ and the push-out of $\overline{\mathrm{SL}}_{2,\tau}^{(n,2)}$ via $\mathfrak{m}$ as in (2.6). This gives

$$\begin{array}{ccccc} \mu_n \times \mu_2 & \hookrightarrow & \overline{\mathrm{SL}}_{2,\tau}^{(n,2)} & \twoheadrightarrow & \mathrm{SL}_2 \\ \downarrow \mathfrak{m} & & \downarrow & & \Vert \\ \mu_{n^*} & \hookrightarrow & \mathfrak{m}_*(\overline{\mathrm{SL}}_{2,\tau}^{(n,2)}) & \xrightarrow{p_\tau^{\mathfrak{m}}} & \mathrm{SL}_2. \end{array} \tag{2.13}$$

Definition 2.9 (See [42]). An F-split nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ is called $\overline{G}^{(n)}$-raisable if there exists $\tau : \mathfrak{sl}_2 \rightarrow \mathfrak{g}_\gamma$ satisfying (C0)–(C3) such that the projection $p_\tau^{\mathfrak{m}}$ in (2.13) does not split.

Again, we denote by

$$(Q_1^\tau, Q_2^\tau) := (\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SL}}_2^{(n),\tau}), \mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SL}}_2^{(2),\phi_m})) \tag{2.14}$$

the pair of Brylinski-Deligne invariants associated with $\overline{\mathrm{SL}}_2^{(n),\tau}$ and $\overline{\mathrm{SL}}_2^{(2),\phi_m}$, respectively. We have an analogue of Proposition 2.5, noting that $Q_2^\tau = m$ in this case.

Proposition 2.10. *Keep notations as above. Then $p_\tau^{\mathfrak{m}}$ splits if and only if*

$$n^* | (Q_1^\tau n^*/n + Q_2^\tau n^*/2),$$

or equivalently,

(i) $n|Q_1^\tau$ *when $m \in \mathbf{N}_{\geqslant 0}$ is even, and*

(ii) $n/\gcd(n, Q_1^\tau) = 2$ *when $m \in \mathbf{N}_{\geqslant 0}$ is odd.*

As a first example, we have the following corollary.

Corollary 2.11. *Let $\overline{G}^{(n)}$ be an n-fold cover of an almost simple simply-connected G with $\mathrm{Inv}_{\mathrm{BD}}(\overline{G}^{(n)}) = 1$. Then the zero orbit is always raisable if $n \geqslant 2$.*

Proof. Pick any long root α and let $\tau : \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_{2,\alpha}$ be the identity. In this case, (C1)–(C3) are all satisfied. We have $\mathfrak{g}[1] = 0$. This shows that

$$(Q_1^\tau, Q_2^\tau) = (1, 0).$$

It then follows from Proposition 2.10 that the orbit $\{0\}$ is raisable if $n \geqslant 2$. □

The importance of the raisability is seen from the following result given by Jiang et al. [42].

Theorem 2.12 (See [42]). *Let $\mathcal{O}$ be a $\overline{G}$-raisable orbit. Then it does not lie in the wavefront set $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ of any $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$.*

2.7 A comparison

Consider the following three properties of $\mathcal{O} \subset \mathfrak{g}$:

(P1) $\mathcal{O}$ is $\overline{G}$-raisable;

(P2) $\mathcal{O}$ is not $\overline{G}$-quasi-admissible;

(P3) $\mathcal{O}$ does not lie in $\mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$ for any $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$.

In view of Theorems 2.7 and 2.12, one has the implications

$$(\mathrm{P1}) \Rightarrow (\mathrm{P3}) \Leftarrow (\mathrm{P2}).$$

In general, raisable and non-quasi-admissible orbits are not identical. Even in the case $G_{\gamma,0} = \mathrm{SL}_2$, these two properties are not necessarily equivalent, though can be shown to be so in many cases.

3 Explicit analysis for each Cartan-type

For each covering group $\overline{G}^{(n)}$ and each F-split nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$, our goal is to determine n such that $\mathcal{O}$ is quasi-admissible and raisable.

The computation in this section is case by case. For classical groups of type A to type D, we work out the details. For exceptional groups, we illustrate the method by giving full details for G_2 and F_4. For E_6, E_7 and E_8, we only consider the raisability and quasi-admissibility of those orbits, which are conjecturally equal to the wavefront sets of theta representations. Whenever our method applies, we show that these conjectural wavefront orbits of theta representations are not raisable and are also quasi-admissible.

Let

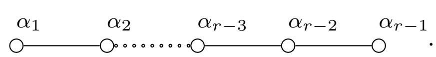
$$\mathfrak{p} = (p_1^{d_1} \cdots p_i^{d_i} \cdots p_k^{d_k})$$

denote a partition of r, where p_i's are distinct with $d_i \in \mathbf{N}_{\geqslant 1}$. For any $p \in \{p_i\}_i$ appearing in $\mathfrak{p}$, we define two functions

$$\mathfrak{A}(\mathfrak{p}, p) = \sum_{\substack{i: p_i > p, \\ p_i - p + 1 \in 2\mathbf{Z}}} d_i, \quad \mathfrak{B}(\mathfrak{p}, p) = \sum_{\substack{i: p_i < p, \\ p_i - p + 1 \in 2\mathbf{Z}}} d_i. \tag{3.1}$$

3.1 Type A

For type A groups, we analyze GL_r instead of general groups for convenience since the parabolic subgroups of the former allow for a simpler description. Consider the Dynkin diagram of simple roots for GL_r:



For the orbit $\mathcal{O} \subset \mathrm{GL}_r$ associated with the partition $\mathfrak{p}_{\mathcal{O}} = (p_1^{d_1} \cdots p_k^{d_k} q_1 \cdots q_m)$ where $d_i \geqslant 2$ and p_i and q_j are distinct, one has

$$G_{\gamma,\mathrm{der}} = \prod_{i=1}^{k} \mathrm{SL}_{d_i}^{\Delta, p_i} \simeq \prod_{i=1}^{k} \mathrm{SL}_{d_i}.$$

Here, $\mathrm{SL}_d^{\Delta,p}$ means the image of the diagonal embedding of SL_d into $\prod_{i=1}^{p} \mathrm{SL}_d$. For each pair (p_i, p_j) with p_i even and p_j odd, we have a map

$$\phi_{p_i,p_j} : \mathrm{SL}_{d_i} \times \mathrm{SL}_{d_j} \to \mathrm{Sp}(\mathfrak{g}_{p_i,p_j}[1])$$

arising from the natural action of $\mathrm{SL}_{d_i} \times \mathrm{SL}_{d_j}$, where

$$\mathfrak{g}_{p_i,p_j}[1] \simeq 2 \min\{p_i, q_j\} \cdot (\mathbf{C}^{d_i} \otimes \mathbf{C}^{d_j})$$

with $\mathbf{C}^{d_i}$ affording the standard representation of GL_{d_i}. We get

$$\mathfrak{g}[1] = \bigoplus_{(p_i,p_j)} \mathfrak{g}_{p_i,p_j}[1]$$

ranging over pairs (p_i, p_j) as above. Then the representation

$$\phi : G_{\gamma,\mathrm{der}} \to \mathrm{Sp}(\mathfrak{g}[1])$$

arises from gluing all the ϕ_{p_i,p_j} together. For each d_i, denote by (Q_{1,d_i}, Q_{2,d_i}) the two Brylinski-Deligne invariants attached to $\mathrm{SL}_{d_i} \subset G_{\gamma,\mathrm{der}}$ as defined in (2.9). We have

$$Q_{1,d_i} = p_i \cdot Q(\alpha^\vee)$$

for any root α of GL_r and $Q_{2,d_i} \in 2\mathbf{Z}$ for every $1 \leqslant i \leqslant k$. By Propositions 2.4 and 2.5, we see that $\mathcal{O}$ is quasi-admissible if and only if $n_\alpha | p_i$ for every i. Recall that n_α is defined in (2.1).

Now we turn to raisability. Here, we choose $1 \leqslant i \leqslant k$ and take

$$\tau_i : \mathrm{SL}_2 \to \mathrm{SL}_{d_i}^{\Delta,p_i} \hookrightarrow G_\gamma$$

to be the embedding corresponding to any simple root of SL_{d_i}. Then this τ_i satisfies (C0)–(C3) as in Subsection 2.6, and regardless of m, one has

$$\overline{\mathrm{SL}}_2^{(2),\phi_m} \simeq \mu_2 \times \mathrm{SL}_2,$$

i.e., $Q_{2,d_i}^{\tau_i} \in 2\mathbf{Z}$. On the other hand, for any Brylinski-Deligne cover of GL_r, the pull-back covering

$$\mu_n \hookrightarrow \overline{\mathrm{SL}}_2^{(n),\tau} \twoheadrightarrow \mathrm{SL}_2$$

has the Brylinski-Deligne invariant $Q_{1,d_i}^{\tau_i} = p_i \cdot Q(\alpha^\vee)$. Hence, by Proposition 2.10, the orbit $\mathcal{O}$ is raisable if $n \nmid p_i \cdot Q(\alpha^\vee)$, i.e., $n_\alpha \nmid p_i$. For the definitions of $Q_{1,d_i}^{\tau_i}$ and $Q_{2,d_i}^{\tau_i}$, see (2.14).

Theorem 3.1. *Keep notations as above and assume $Q(\alpha^\vee) \neq 0$. Then the orbit*

$$\mathcal{O} = (p_1^{d_1} \cdots p_k^{d_k} q_1 \cdots q_m)$$

is

(i) *quasi-admissible if and only if $n_\alpha | p_i$ for every i;*

(ii) *raisable if $n_\alpha \nmid p_i$ for some i.*

In particular, the orbit $\mathcal{O}$ is always $\overline{\mathrm{GL}}_r^{(n)}$-raisable for any $n > |Q(\alpha^\vee)| \cdot \min\{p_i\}$.

Note that every orbit is GL_r-quasi-admissible for the linear GL_r, i.e., when $n = 1$.

Consider the orbit $\mathcal{O}^{r,n} := (n_\alpha^a b)$ of GL_r, where $r = a \cdot n_\alpha + b$ and $0 \leqslant b < n_\alpha$. We believe the following holds.

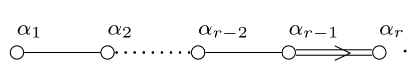
Conjecture 3.2. *Let $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{\mathrm{GL}}_r^{(n)})$. Then one has $\mathcal{O}^{r,n} \leqslant \mathcal{O}$ for every $\mathcal{O} \in \mathcal{N}_{\mathrm{Wh}}^{\max}(\pi)$.*

Here, the orbit $\mathcal{O}^{r,n}$ is equal to the leading wavefront set of the theta representation of $\overline{\mathrm{GL}}_r^{(n)}$ (see (1.2)). If $\mathcal{O}^{r,n}$ is the regular orbit, then Conjecture 3.2 implies that every $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{\mathrm{GL}}_r^{(n)})$ is generic in this case, which is exactly the content of [25, Conjecture 6.9(ii)].

3.2 Types B and D

In this subsection, we consider classical groups of orthogonal type.

For $\mathrm{SO}_{2r+1}, r \geqslant 2$, its Dynkin diagram of simple roots is given as follows:



We consider the natural covering

$$\overline{\mathrm{SO}}_{2r+1} \hookrightarrow \overline{\mathrm{SL}}_{2r+1}$$

obtained from restriction of the covering $\overline{\mathrm{SL}}_{2r+1}$ with the Brylinski-Deligne invariant

$$\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SL}}_{2r+1}) = 1 = Q(\alpha^\vee),$$

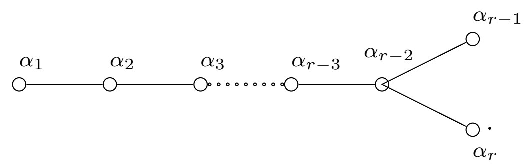
where $\alpha^\vee$ is any coroot of SL_{2r+1}. This gives

$$Q(\alpha_r^\vee) = 4 \cdot Q(\alpha^\vee) = 4 \quad \text{and} \quad Q(\alpha_1^\vee) = 2 \cdot Q(\alpha^\vee) = 2,$$

i.e.,

$$\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SO}}_{2r+1}) = 2.$$

For SO_{2r}, $r \geqslant 3$, we have the Dynkin diagram of simple roots to be



We have the cover $\overline{\mathrm{SO}}_{2r}$ obtained from restricting the above $\overline{\mathrm{SL}}_{2r}$ via $\mathrm{SO}_{2r} \hookrightarrow \mathrm{SL}_{2r}$. In this case,

$$\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{SO}}_{2r}) = 2.$$

Note that we consider the above specific covers only for notational simplicity later. All the argument below applies to covers of $\mathrm{SO}_{2r+1}, \mathrm{SO}_{2r}$ associated with the general quadratic form in the Brylinski-Deligne framework.

Consider a nilpotent orbit $\mathcal{O}$ with the partition

$$\mathfrak{p}_{\mathcal{O}} = (p_1^{d_1} \cdots p_i^{d_i} \cdots p_k^{d_k} q_1^{e_1} \cdots q_j^{e_j} \cdots q_m^{e_m}),$$

where p_i are even, q_j are odd, $d_i \geqslant 1$, and $e_j \geqslant 1$. Thus, d_i are necessarily even. To discuss the quasi-admissibility and raisability of orbit $\mathcal{O}$, we first note that

$$G_{\gamma,0} = \left(\prod_{i=1}^k \mathrm{Sp}_{d_i}^{\Delta,p_i}\right) \times \left(\prod_{j=1}^m \mathrm{SO}_{e_j}^{\Delta,q_j}\right).$$

For each pair (p_i, q_j), we have

$$\mathfrak{g}_{p_i,q_j}[1] \simeq \min\{p_i, q_j\} \cdot (\mathbf{C}^{d_i} \otimes \mathbf{C}^{e_j}),$$

where $\mathbf{C}^{d_i}$ affords the natural action of Sp_{d_i}, and $\mathbf{C}^{e_j}$ affords the natural action of SO_{e_j}; also,

$$\mathfrak{g}[1] = \bigoplus_{(p_i,q_j)} \mathfrak{g}_{p_i,q_j}[1]$$

as G_γ-representations (see [56, § 5]). Since $\mathrm{Inv}_{\mathrm{BD}}(\mathrm{Mp}(\mathfrak{g}[1])) = 1$, we obtain that

- the Brylinski-Deligne invariant for $\overline{\mathrm{Sp}}_{d_i}^{(2),\phi}$ is

$$\sum_j \min\{p_i, q_j\}\, e_j = \left(\sum_{j:p_i>q_j} q_j e_j\right) + p_i\left(\sum_{j:p_i<q_j} e_j\right);$$

- the Brylinski-Deligne invariant for $\overline{\mathrm{SO}}_{e_j}^{(2),\phi}, e_j \geqslant 4$ is

$$\sum_i 2\min\{p_i, q_j\}\, d_i = 2q_j\left(\sum_{i:p_i>q_j} d_i\right) + 2\left(\sum_{i:p_i<q_j} p_i d_i\right).$$

Similarly, we have that

- the Brylinski-Deligne invariant for $\overline{\mathrm{Sp}}_{d_i}^{(n)}$ is p_i;
- the Brylinski-Deligne invariant for $\overline{\mathrm{SO}}_{e_j}^{(n)}, e_j \geqslant 4$ is $2q_j$ (and for $\overline{\mathrm{SO}}_3^{(n)}$, it is $4q_j$).

Note that in general,

$$\overline{G}_{\gamma,0}^{(n,2)} \neq \left(\prod_{i=1}^{k} \overline{\mathrm{Sp}}_{d_i}^{(n,2)}\right) \times_{\mu_n} \left(\prod_{j=1}^{m} \overline{\mathrm{SO}}_{e_j}^{(n,2)}\right).$$

However, $\overline{G}_{\gamma,0}^{(n,2)}$ has the $(n,2)$-genuine representation if and only if each factor has it as well, which follows easily from [33, Proposition 6.4]. Thus, it suffices to consider each factor in $G_{\gamma,0}$. The n^*-fold cover $\mathfrak{m}_*(\overline{\mathrm{Sp}}_{d_i}^{(n,2)})$ has the Brylinski-Deligne invariant

$$(n^* p_i/n) + \left(\sum_j \min\{p_i, q_j\} e_j\right) n^*/2,$$

$\mathfrak{m}_*(\overline{\mathrm{SO}}_{e_j}^{(n,2)})$, $e_j \geqslant 4$ has the Brylinski-Deligne invariant

$$(2n^* q_j/n) + n^* \left(\sum_i \min\{p_i, q_j\} d_i\right),$$

and $\mathfrak{m}_*(\overline{\mathrm{SO}}_3^{(n,2)})$ has the Brylinski-Deligne invariant

$$(4n^* q_j/n) + 2n^* \left(\sum_i \min\{p_i, q_j\} d_i\right).$$

Hence, $\mathfrak{m}_*(\overline{\mathrm{Sp}}_{d_i}^{(n,2)})$, $d_i \geqslant 2$ splits over Sp_{d_i} if and only if the following hold:

- $n|p_i$ and $\mathfrak{B}(\mathfrak{p}_{\mathcal{O}}, p_i) \in 2\mathbf{Z}$ if n is odd;
- $n|p_i$ if n is even and $\mathfrak{B}(\mathfrak{p}_{\mathcal{O}}, p_i) \in 2\mathbf{Z}$;
- $\gcd(n, p_i) = n/2$ if n is even and $\mathfrak{B}(\mathfrak{p}_{\mathcal{O}}, p_i) \notin 2\mathbf{Z}$.

On the other hand, it follows from Lemma 2.8 that $\overline{\mathrm{SO}}_{e_j}^{(n,2)}$, $e_j \geqslant 3$ has a finite-dimensional $(n,2)$-genuine representation in the following cases:

- $n|q_j$ if n is odd;
- $n|(2q_j)$ if n is even and $e_j \geqslant 4$;
- $n|(4q_j)$ if n is even and $e_j = 3$.

Now, regarding raisability, we implement [42, § 8–9] and consider the following two cases:

- (B/D-Sym) Suppose that there exists $d_i \geqslant 2$. Let $\tau : \mathrm{SL}_2 \hookrightarrow \mathrm{Sp}_{d_i}$ be the embedding corresponding to the long simple root of Sp_{d_i}. Then this τ satisfies (C1)–(C3), where $\mathfrak{g}[1] = \mathfrak{g}[1]^{\mathfrak{sl}_{2,\tau}} \oplus mV_2$ with

$$m = p_i \cdot \left(\sum_{q_j : q_j > p_i} e_j\right) + \sum_{q_j : q_j < p_i} q_j \cdot e_j = p_i \cdot \mathfrak{A}(\mathfrak{p}_{\mathcal{O}}, p_i) + \sum_{q_j : q_j < p_i} q_j \cdot e_j.$$

- (B/D-Ort) Suppose that there exist $e_j \geqslant 4$ and thus a Levi subgroup

$$\mathrm{GL}_2 \times \mathrm{SO}_k \subset \mathrm{SO}_{e_j}.$$

Let $\tau : \mathrm{SL}_2 \hookrightarrow \mathrm{SO}_{e_j}$ be corresponding to the root of GL_2. This τ satisfies (C1)–(C3) and $\mathfrak{g}[1] = \mathfrak{g}[1]^{\mathfrak{sl}_{2,\tau}} \oplus mV_2$, where

$$m = q_j \cdot \left(\sum_{p_i : p_i > q_j} d_i\right) + \sum_{p_i : p_i < q_j} p_i \cdot d_i = q_j \cdot \mathfrak{A}(\mathfrak{p}_{\mathcal{O}}, q_j) + \sum_{p_i : p_i < q_j} p_i \cdot d_i.$$

To give a sufficient condition for the orbit $\mathcal{O}$ to be raisable, we proceed with considering the two covering groups $\overline{\mathrm{SL}}_2^{(2),\phi_m}$ and $\overline{\mathrm{SL}}_2^{(n),\tau}$. Again, we have two cases depending on the consideration of symplectic or orthogonal stabilizer.

• (B/D-Sym) We have $Q_2^\tau = m$ and $Q_1^\tau = p_i$ for the Brylinski-Deligne invariants. It follows from Proposition 2.10 that the cover $\mathfrak{m}_*(\overline{\mathrm{SL}}_2^{(n,2)}) \twoheadrightarrow \mathrm{SL}_2$ splits if and only if $\sum_{q_j:q_j<p_i} e_j$ is even and $n|(p_i \cdot Q(\alpha^\vee))$, or $\sum_{q_j:q_j<p_i} e_j$ is odd and $n/\gcd(n, p_i \cdot Q(\alpha^\vee)) = 2$. It follows that the orbit $\mathcal{O}$ is raisable if one of the following holds:

- $n \nmid p_i$ or $\mathfrak{B}(\mathfrak{p}, p_i) \notin 2\mathbf{Z}$ if n is odd;
- $n \nmid p_i$ if n is even and $\mathfrak{B}(\mathfrak{p}, p_i) \in 2\mathbf{Z}$;
- $\gcd(n, p_i) \neq n/2$ if n is even and $\mathfrak{B}(\mathfrak{p}, p_i) \notin 2\mathbf{Z}$.

• (B/D-Ort) In this case, since d_i is always even, we get $m \in 2\mathbf{Z}$. Here, SO_{e_i} has the Brylinski-Deligne invariant equal to $2q_j$. Thus, the covering $\overline{\mathrm{SL}}_2^{(n),\tau}$ has the Brylinski-Deligne invariant $Q_1^\tau = 2q_j$ as well. We see that $\mathfrak{m}_*(\overline{\mathrm{SL}}_2^{(n,2)})$ splits if and only if $n|2q_j$, i.e., the orbit $\mathcal{O}$ is raisable if $n \nmid (2q_j)$.

Theorem 3.3. *Consider an orbit $\mathcal{O}$ with $\mathfrak{p}_\mathcal{O} = (p_1^{d_1} \cdots p_i^{d_i} \cdots p_k^{d_k} q_1^{e_1} \cdots q_j^{e_j} \cdots q_m^{e_m})$ for the orthogonal group $G_r = \mathrm{SO}_{2r}$ or $G_r = \mathrm{SO}_{2r+1}$ with p_i even and q_j odd.*

(i) *It is $\overline{G}_r^{(n)}$-quasi-admissible if and only if for every i and j with $d_i \geqslant 2$ and $e_j \geqslant 3$, the following hold*:

- *$n|p_i$ and $\mathfrak{B}(\mathfrak{p}_\mathcal{O}, p_i) \in 2\mathbf{Z}$ if n is odd*;
- *$n|p_i$ if n is even and $\mathfrak{B}(\mathfrak{p}_\mathcal{O}, p_i) \in 2\mathbf{Z}$*;
- *$\gcd(n, p_i) = n/2$ if n is even and $\mathfrak{B}(\mathfrak{p}_\mathcal{O}, p_i) \notin 2\mathbf{Z}$*;
- *$n|q_j$ if n is odd*;
- *$n|(2q_j)$ if n is even and $e_j \geqslant 4$*;
- *$n|(4q_j)$ if n is even and $e_j = 3$.*

(ii) *It is $\overline{G}_r^{(n)}$-raisable if for some i and j with $d_i \geqslant 2$ and $e_j \geqslant 4$, one of the following holds*:

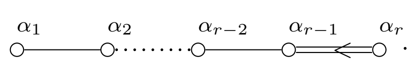
- *$n \nmid p_i$ or $\mathfrak{B}(\mathfrak{p}, p_i) \notin 2\mathbf{Z}$ if n is odd*;
- *$n \nmid p_i$ if n is even and $\mathfrak{B}(\mathfrak{p}, p_i) \in 2\mathbf{Z}$*;
- *$\gcd(n, p_i) \neq n/2$ if n is even and $\mathfrak{B}(\mathfrak{p}, p_i) \notin 2\mathbf{Z}$*;
- *$n \nmid (2q_j)$.*

For $G = \mathrm{SO}_{2r}$ or $G = \mathrm{SO}_{2r+1}$, one can easily see from Theorem 3.3, by applying $n = 1$, that an orbit $\mathcal{O}$ is G-quasi-admissible if and only if it is special (see [17, p. 100]); this also follows from [33, Proposition 1.2] and [56, Theorem 5.8].

We remark that for SO_{2r} with r even, there could be two orbits associated with a very even partition (see [17, p. 70]). However, our argument above equally applies to either case and the results depend only on the partition.

3.3 Type C

Consider the Dynkin diagram for the simple roots of type C_r:



We consider the group $G = \mathrm{Sp}_{2r}$ and a nilpotent orbit

$$\mathcal{O} = (p_1^{d_1} \cdots p_i^{d_i} \cdots p_k^{d_k} q_1^{e_1} \cdots q_j^{e_j} q_m^{e_m}),$$

where p_i is even and q_j is odd. Here, e_j is necessarily even. To discuss the quasi-admissibility, we first note that

$$G_{\gamma,0} = \left(\prod_{i=1}^{k} \mathrm{SO}_{d_i}^{\Delta,p_i} \right) \times \left(\prod_{j=1}^{m} \mathrm{Sp}_{e_j}^{\Delta,q_j} \right).$$

For each pair (p_i, q_j), we have $\mathfrak{g}_{p_i,q_j}[1] \simeq \min\{p_i, q_j\} \cdot (\mathbf{C}^{d_i} \otimes \mathbf{C}^{e_j})$, where $\mathbf{C}^{d_i}$ affords the natural action of SO_{d_i}, and $\mathbf{C}^{e_j}$ affords the natural action of Sp_{e_j}. Also,

$$\mathfrak{g}[1] = \bigoplus_{(p_i,q_j)} \mathfrak{g}_{p_i,q_j}[1],$$

where $G_{\gamma,0}$ acts on each component as given above. Consider the cover $\overline{\mathrm{Sp}}_{2r}^{(n)}$ obtained from the restriction of $\overline{\mathrm{SL}}_{2r}^{(n)}$ via inclusion $\mathrm{Sp}_{2r} \subset \mathrm{SL}_{2r}$, one has

$$\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{Sp}}_{2r}^{(n)}) = Q(\alpha_r^\vee) = Q(\alpha^\vee) = 1,$$

where $\alpha^\vee$ is any coroot of SL_{2r}. We have

- the Brylinski-Deligne invariant for $\overline{\mathrm{Sp}}_{e_j}^{(2),\phi}$ is

$$\sum_i \min\{p_i, q_j\}\, d_i = \bigg(\sum_{i:p_i<q_j} p_i d_i\bigg) + q_j \bigg(\sum_{i:p_i>q_j} d_i\bigg);$$

- the Brylinski-Deligne invariant for $\overline{\mathrm{SO}}_{d_i}^{(2),\phi}$, $d_i \geqslant 4$ is

$$\sum_j 2\min\{p_i, q_j\}\, d_j = 2p_i\bigg(\sum_{j:q_j>p_i} e_j\bigg) + 2\bigg(\sum_{j:p_i>q_j} q_j e_j\bigg).$$

Similarly, we have

- the Brylinski-Deligne invariant for $\overline{\mathrm{Sp}}_{e_j}^{(n)}$ is q_j;
- the Brylinski-Deligne invariant for $\overline{\mathrm{SO}}_{d_i}^{(n)}, d_i \geqslant 4$ is $2p_i$, and it is $4p_i$ for $\overline{\mathrm{SO}}_3^{(n)}$.

Again, for quasi-admissibility, it suffices to consider the splitting of the n^*-fold cover of $\mathfrak{m}_*(\overline{\mathrm{Sp}}_{e_j}^{(n,2)})$ and $\mathfrak{m}_*(\overline{\mathrm{SO}}_{d_i}^{(n,2)})$ over Sp_{e_j} and SO_{d_i}, respectively. The n^*-fold cover $\mathfrak{m}_*(\overline{\mathrm{Sp}}_{e_j}^{(n,2)})$ has the Brylinski-Deligne invariant

$$(n^* q_j/n) + \bigg(\sum_i \min\{p_i, q_j\}\, d_i\bigg) n^*/2,$$

and $\mathfrak{m}_*(\overline{\mathrm{SO}}_{d_i}^{(n,2)})$, $d_i \geqslant 4$ has the Brylinski-Deligne invariant

$$(2n^* p_i/n) + n^*\bigg(\sum_j \min\{p_i, q_j\}\, e_j\bigg).$$

Also, $\mathfrak{m}_*(\overline{\mathrm{SO}}_3^{(n,2)})$ has the Brylinski-Deligne invariant

$$(4n^* p_i/n) + 2n^*\bigg(\sum_j \min\{p_i, q_j\}\, e_j\bigg).$$

Here, $\mathfrak{m}_*(\overline{\mathrm{Sp}}_{e_j}^{(n,2)}), e_j \geqslant 2$ splits over Sp_{e_j} if and only if the following hold:

- $n|q_j$ and $\mathfrak{A}(\mathfrak{p}_{\mathcal{O}}, q_j) \in 2\mathbf{Z}$ if n is odd;
- $n|q_j$ if n is even and $\mathfrak{A}(\mathfrak{p}_{\mathcal{O}}, q_j) \in 2\mathbf{Z}$;
- $\gcd(n, q_j) = n/2$ if n is even and $\mathfrak{A}(\mathfrak{p}_{\mathcal{O}}, q_j) \notin 2\mathbf{Z}$.

On the other hand, $\overline{\mathrm{SO}}_{d_i}^{(n,2)}, d_i \geqslant 3$ has a finite-dimensional $(n, 2)$ representation if the following hold:

- $n|p_i$ if n is odd;
- $n|(2p_i)$ if n is even and $d_i \geqslant 4$;
- $n|(4p_i)$ if n is even and $d_i = 3$.

Now, regarding raisability, we again have the following two cases:

- (C-Sym) Suppose that there exists $e_j \geqslant 2$. Let $\tau : \mathrm{SL}_2 \hookrightarrow \mathrm{Sp}_{e_j}$ be the embedding corresponding to the long simple root of Sp_{e_j}. Then this τ satisfies (C1)–(C3), where $\mathfrak{g}[1] = \mathfrak{g}[1]^{\mathfrak{sl}_{2,\tau}} \oplus mV_2$ with

$$m = q_j \cdot \bigg(\sum_{p_i:p_i>q_j} d_i\bigg) + \sum_{p_i:p_i<q_j} p_i \cdot d_i = q_j \cdot \mathfrak{A}(\mathfrak{p}, q_j) + \sum_{p_i:p_i<q_j} p_i \cdot d_i.$$

We have $Q_2^\tau = m$ and $Q_1^\tau = q_j$. It follows from Proposition 2.10 that $\mathfrak{m}_*(\overline{\mathrm{SL}}_2^{(n,2)}) \twoheadrightarrow \mathrm{SL}_2$ split if and only if $\mathfrak{A}(\mathfrak{p}, q_j)$ is even and $n|q_i$, or $\mathfrak{A}(\mathfrak{p}, q_j)$ is odd and $\gcd(n, q_j) = 2$. This shows that the orbit $\mathcal{O}$ is raisable if one of the following holds:

- $n \nmid q_j$ or $\mathfrak{A}(\mathfrak{p}, q_j) \notin 2\mathbf{Z}$ if n is odd;
- $n \nmid q_j$ if n is even and $\mathfrak{A}(\mathfrak{p}, q_j) \in 2\mathbf{Z}$;
- $\gcd(n, q_j) \neq n/2$ if n is even and $\mathfrak{A}(\mathfrak{p}, q_j) \notin 2\mathbf{Z}$.

• (C-Ort) Suppose that there exist $d_i \geqslant 4$ and thus a Levi subgroup $\mathrm{GL}_2 \times \mathrm{SO}_k \subset \mathrm{SO}_{d_i}$. Let $\tau : \mathrm{SL}_2 \hookrightarrow \mathrm{SO}_{d_i}$ be associated with the root of GL_2. It satisfies (C1)–(C3) and

$$\mathfrak{g}[1] = \mathfrak{g}[1]^{\mathfrak{sl}_{2,\tau}} \oplus mV_2,$$

where

$$m = p_i \cdot \left(\sum_{q_j : q_j > p_i} e_j \right) + \sum_{q_j : q_j < p_i} q_j \cdot e_j.$$

In this case, since e_j is always even, we get $m \in 2\mathbf{Z}$. Here, SO_{d_i} has the Brylinski-Deligne invariant $2p_i$. Thus, the covering $\overline{\mathrm{SL}}_2^{(n),\tau}$ has the Brylinski-Deligne invariant $Q_1^\tau = 2p_i$ as well. We see that $\mathfrak{m}_*(\overline{\mathrm{SL}}_2^{(n,2)})$ splits if and only if $n|(2p_i)$, i.e., the orbit $\mathcal{O}$ is raisable if $n \nmid (2p_i)$.

Theorem 3.4. *Let $\mathcal{O} = (p_1^{d_1} \cdots p_i^{d_i} \cdots p_k^{d_k} q_1^{e_1} \cdots q_j^{e_j} \cdots q_m^{e_m})$ be an F-split symplectic orbit for Sp_{2r} with p_i even and q_j odd. Consider the n-fold cover $\overline{\mathrm{Sp}}_{2r}^{(n)}$ of Sp_{2r} with the Brylinski-Deligne invariant equal to $Q(\alpha_r^\vee) = 1$.*

(i) *The orbit $\mathcal{O}$ is $\overline{\mathrm{Sp}}_{2r}^{(n)}$-quasi-admissible if and only if for every i and j with $d_i \geqslant 3$ and $e_j \geqslant 2$, the following hold*:

- $n|q_j$ *and* $\mathfrak{A}(\mathfrak{p}_{\mathcal{O}}, q_j) \in 2\mathbf{Z}$ *if n is odd*;
- $n|q_j$ *if n is even and* $\mathfrak{A}(\mathfrak{p}_{\mathcal{O}}, q_j) \in 2\mathbf{Z}$;
- $\gcd(n, q_j) = n/2$ *if n is even and* $\mathfrak{A}(\mathfrak{p}_{\mathcal{O}}, q_j) \notin 2\mathbf{Z}$;
- $n|p_i$ *if n is odd*;
- $n|(2p_i)$ *if n is even and $d_i \geqslant 4$*;
- $n|(4p_i)$ *if n is even and $d_i = 3$.*

(ii) *The orbit $\mathcal{O}$ is $\overline{\mathrm{Sp}}_{2r}^{(n)}$-raisable if for some i and j with $e_j \geqslant 2$ and $d_i \geqslant 4$, one of the following holds*:

- $n \nmid q_j$ *or* $\mathfrak{A}(\mathfrak{p}, q_j) \notin 2\mathbf{Z}$ *if n is odd*;
- $n \nmid q_j$ *if n is even and* $\mathfrak{A}(\mathfrak{p}, q_j) \in 2\mathbf{Z}$;
- $\gcd(n, q_j) \neq n/2$ *if n is even and* $\mathfrak{A}(\mathfrak{p}, q_j) \notin 2\mathbf{Z}$;
- $n \nmid (2p_i)$.

Again, by considering $n = 1$ in Theorem 3.4, we see that an orbit $\mathcal{O}$ is Sp_{2r}-quasi-admissible if and only if it is special (see [17, p. 100]), which also follows from [33, Proposition 1.2] and [56, Theorem 5.8].

For a partition $\mathfrak{p}$ and $\sharp \in \{B, C, D\}$, we denote by $\mathfrak{p}_\sharp$ and $\mathfrak{p}^\sharp$ the type $\sharp$-collapse and $\sharp$-expansion of $\mathfrak{p}$, respectively. More precisely, $\mathfrak{p}_\sharp$ is the unique partition of $\sharp$-type that is dominated by $\mathfrak{p}$, and $\mathfrak{p}^\sharp$ is the unique one of $\sharp$-type that dominates $\mathfrak{p}$ (see [17, § 6.3] for more details).

Example 3.5. We consider three special families of orbits of Sp_{2r} and their quasi-admissibility. First, assume that $n \in \mathbf{N}_{\geqslant 1}$ is odd with $2r = an + b$. Consider the orbit

$$\mathcal{O}_C^{2r,n} = (n^a b)_C = \begin{cases} (n^a b), & \text{if } a \text{ is even (and } b \text{ even)}, \\ (n^{a-1}, n-1, b+1), & \text{if } a \text{ is odd.} \end{cases}$$

It is $\overline{\mathrm{Sp}}_{2r}^{(n)}$-quasi-admissible by Theorem 3.4, and is not raisable. Second, if $n = 2k$ with k odd and we write $2r = ka + b$ with $0 \leqslant b < k$, then we have

$$(k+1, \mathcal{O}^{2r-k-1,k})_C = \begin{cases} (k+1, k^a, b), & \text{if } a \text{ is even}, \\ (k+1, k^{a-1}, k-1, b+1), & \text{if } a \text{ is odd.} \end{cases}$$

We see that this orbit is quasi-admissible in this case. If a is even, then the orbit is not raisable since $n/\gcd(n, q_j \cdot Q(\alpha^\vee)) = 2$ in this case. As a last example, consider $n \in 4\mathbf{Z}$, then the orbit is

$$\mathcal{O}_C^{2r,n/2} = \mathcal{O}^{2r,n/2} = ((n/2)^a b),$$

where a and b are both even. This orbit is quasi-admissible. In this case, $p_i = n/2$ and clearly $n|(2p_i \cdot Q(\alpha^\vee))$. Thus, the orbit is not raisable as well.

These orbits are the speculated wavefront sets of the theta representation of $\overline{\mathrm{Sp}}_{2r}^{(n)}$ (see Subsection 4.4 for more details).

In the remaining of this subsection, we consider exceptional groups and discuss in more detail the case of G_2 and F_4. The computation of types $E_i, 6 \leqslant i \leqslant 8$ follows from the same techniques, the details of which however are more involved. Thus, for E_i, we only discuss the quasi-admissibility and raisability of certain nilpotent orbits, which are the speculated stable wavefront sets of the theta representations.

For each orbit written in the Bala-Carter notations, we give information on whether it is special or even, and the structure of $\mathbf{G}_{\gamma,\mathrm{der}}$, which can be obtained from [15,17,38]. By computing the Brylinski-Deligne invariants (Q_1, Q_2) for each of the simple subgroups of $\mathbf{G}_{\gamma,\mathrm{der}}$, we obtain a criterion of quasi-admissibility on n. Similarly, by computing (Q_1^τ, Q_2^τ) for properly chosen τ, we can give the condition on n such that the orbit is $\overline{G}^{(n)}$-raisable. Our computations rely on the extensive results in [38], some of which were already used in [42].

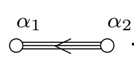
For simplicity, in the remaining of this subsection, we consider an almost-simple simply-connected exceptional group G and its cover $\overline{G}^{(n)}$ with

$$\mathrm{Inv}_{\mathrm{BD}}(\overline{G}^{(n)}) = 1.$$

We use V^k to denote a certain irreducible k-dimensional representation of the underlying group, which is clear from the context.

3.4 Type G_2

Consider the Dynkin diagram of simple roots of G_2:



The results are summarized in Table 1.

The entries for the three orbits $\{0\}, G_2(a_1)$ and G_2 are clear. Thus, we give a brief explanation for the two orbits A_1 and $\tilde{A}_1$.

The minimal orbit A_1 gives $G_{\gamma,\mathrm{der}} = \mathrm{SL}_2$ associated with α_1. One has $\mathfrak{g}[1] \simeq V^4$, which then gives $Q_2 = 4$. We also have $Q_1 = 3$. Thus, the orbit is $\overline{G}_2^{(n)}$-quasi-admissible for $n = 1, 3$. The method of raisability in [42] does not apply.

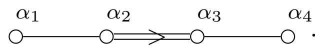
The orbit $\tilde{A}_1$ gives $G_{\gamma,\mathrm{der}} = \mathrm{SL}_2$ associated with the long root α_2, which immediately gives $Q_1 = 1$. On the other hand, $\mathfrak{g}[1] \simeq V^2$, and thus $Q_2 = 1$. This shows that the orbit $\tilde{A}_1$ is quasi-admissible if and only if $n = 2$. The map $\tau = \mathrm{id}$ satisfies (C1)–(C3). If $n \neq 2$, then $\mathfrak{m}_*(\overline{\mathrm{SL}}_{2,\tau}^{(n,2)}) \twoheadrightarrow \mathrm{SL}_2$ does not split and thus $\tilde{A}_1$ is raisable.

Table 1 Nilpotent orbits for G_2

$\mathcal{O}$	special/even?	$\mathbf{G}_{\gamma,\mathrm{der}}$	(Q_1, Q_2)	quasi-admissible, if and only if	raisable if
$\{0\}$	yes/yes	G_2	$(1, 0)$	$n = 1$	$n \geqslant 2$
A_1	no/no	SL_{2,α_1}	$(3, 4)$	$n = 1, 3$	n.a.
$\tilde{A}_1$	no/no	SL_{2,α_2}	$(1, 1)$	$n = 2$	$n \neq 2$
$G_2(a_1)$	yes/yes	1	n.a.	all n	n.a.
G_2	yes/yes	1	n.a.	all n	n.a.

3.5 Type F_4

Consider the Dynkin diagram of simple roots of F_4 as follows:



The results are given in Table 2.

Again, the method for quasi-admissibility and raisability does not apply to distinguished orbits (i.e., the orbits which do not arise from proper Levi subgroups in the Bala-Carter classification of orbits [17, § 8.2]). Thus, it suffices to consider the following orbits:

$$A_1,\ \tilde{A}_1,\ A_1+\tilde{A}_1,\ A_2,\ \tilde{A}_2,\ A_2+\tilde{A}_1,\ B_2,\ \tilde{A}_2+A_1,\ C_3(a_1),\ B_3,\ C_3.$$

We give a case by case discussion.

The orbit A_1 has $G_{\gamma,\mathrm{der}} = \mathrm{Sp}_6$ associated with $\{\alpha_2, \alpha_3, \alpha_4\}$. One has

$$\mathfrak{g}[1] = \bigwedge^3 V_{\mathrm{std}}/V_{\mathrm{std}}$$

as a $G_{\gamma,\mathrm{der}}$-module and is of dimension 14, where V_{std} represents the standard representation of Sp_6. We have $\mathfrak{g}[1] = 4V^1 \oplus 5\cdot V^2$ as an SL_{2,α_2}-module. This gives that $Q_2 = 5$. Thus, it follows from Propositions 2.4 and 2.5 that the orbit A_1 is $\overline{F}_4^{(n)}$-quasi-admissible if and only if $n = 2$. For raisability, we take

$$\mathfrak{sl}_{2,\tau} = \mathfrak{sl}_{2,\alpha_2}$$

to be associated with α_2. Then (C1)–(C3) are satisfied with $m = 5$. This shows that if $2 \nmid n$, then A_1 is $\overline{F}_4^{(n)}$-raisable by Proposition 2.10.

The orbit $\tilde{A}_1$ gives $G_{\gamma,\mathrm{der}} = \mathrm{SL}_4$ associated with $\{\alpha_1, \alpha_2, \alpha_2 + 2\alpha_3\}$. One has $\mathfrak{g}[1] = V_{\mathrm{std}} \oplus V^*_{\mathrm{std}}$, where V_{std} is the standard representation of SL_4. This gives that $Q_2 = 2$. Since $Q_1 = 1$, we see that $\tilde{A}_1$ is $\overline{F}_4^{(n)}$-quasi-admissible if and only if $n = 1$. For raisability, consider $\tau : \mathrm{SL}_2 \hookrightarrow G_{\gamma,\mathrm{der}}$ associated with the simple root α_2 of $G_{\gamma,\mathrm{der}}$, and then (C1)–(C3) are satisfied with $Q_1 = 1$ and $m = Q_2^\tau = 2$. In this case, we see that $\tilde{A}_1$ is raisable if $n \geqslant 2$.

Table 2 Nilpotent orbits for F_4

$\mathcal{O}$	special/even?	$\mathbf{G}_{\gamma,\mathrm{der}}$	(Q_1, Q_2)	quasi-admissible, if and only if	raisable if
$\{0\}$	yes/yes	F_4	$(1,0)$	$n=1$	$n \geqslant 2$
A_1	no/no	Sp_6	$(1,5)$	$n=2$	$n \neq 2$
$\tilde{A}_1$	yes/no	SL_4	$(1,2)$	$n=1$	$n \geqslant 2$
$A_1+\tilde{A}_1$	yes/no	$\mathrm{SL}_{2,\alpha_1}\times\mathrm{SO}_3$	$(1,6),(8,20)$	$n=1$	$n \geqslant 2$
A_2	yes/yes	SL_3	$(2,0)$	$n=1,2$	$n \geqslant 3$
$\tilde{A}_2$	yes/yes	SL_3	$(1,0)$	$n=1$	$n \geqslant 2$
$A_2+\tilde{A}_1$	no/no	SL_2	$(6,11)$	$n=4,12$	n.a.
B_2	no/no	$\mathrm{SL}_{2,\alpha_2}\times\mathrm{SL}_2$	$(1,1),(1,1)$	$n=2$	$n \neq 2$
$\tilde{A}_2+A_1$	no/no	SL_2	$(3,12)$	$n=1,3$	n.a.
$C_3(a_1)$	no/no	SL_{2,α_2}	$(1,3)$	$n=2$	$n \neq 2$
$F_4(a_3)$	yes/yes	1	n.a.	all n	n.a.
B_3	yes/yes	SO_3	$(8,0)$	$n=1,2,4,8$	$n \neq 1,2,4,8$
C_3	yes/no	SL_{2,α_2}	$(1,2)$	$n=1$	$n \geqslant 2$
$F_4(a_2)$	yes/yes	1	n.a.	all n	n.a.
$F_4(a_1)$	yes/yes	1	n.a.	all n	n.a.
F_4	yes/yes	1	n.a.	all n	n.a.

The orbit $A_1+\tilde{A}_1$ gives

$$\mathbf{G}_{\gamma,\mathrm{der}}=\mathrm{SL}_2\times\mathrm{SO}_3,$$

where SL_2 is associated with α_1 and $\mathrm{SO}_3=\mathrm{PGL}_2$ is embedded in SL_3 associated with $\{\alpha_3,\alpha_4\}$. As a $G_{\gamma,\mathrm{der}}$-module, one has

$$\mathfrak{g}[1]=(V^2\boxtimes V^5)\oplus(V^2\boxtimes V^1).$$

Thus, we have $Q_1=1$ for SL_2 and $Q_1(\alpha_3^\vee+\alpha_4^\vee)=8$ for SO_3. Also, $Q_2\in 2\mathbf{Z}$ for both SL_2 and SO_3. Thus, $\mathfrak{m}_*(\overline{\mathrm{SL}}_{2,\tau}^{(n,2)})$ splits over SL_2 if and only if $n=1$; also, $\mathfrak{m}_*(\overline{\mathrm{SO}}_{3,\mathrm{der}}^{(n,2)})$ splits over $\mathrm{SO}_{3,\mathrm{der}}$ if and only if $n=1,2,4,8$. Thus, the orbit $A_1+\tilde{A}_1$ is quasi-admissible if and only if $n=1$. For raisability, consider the embedding

$$\tau=\mathrm{id}\times 1:\mathrm{SL}_2\hookrightarrow G_{\gamma,\mathrm{der}},$$

which satisfies (C0)–(C3) with $Q_2^\tau=m=6$. Since $Q_1^\tau=1$. We see that if $n\geqslant 2$, then $A_1+\tilde{A}_1$ is raisable.

The orbits A_2 and $\tilde{A}_2$ are even, and thus $Q_2=0$ for both of them. For A_2, one has $G_{\gamma,\mathrm{der}}=\mathrm{SL}_3$ associated with α_3 and α_4. For $\tilde{A}_2$, one has $G_{\gamma,\mathrm{der}}=\mathrm{SL}_3$ associated with α_1 and α_2. We get $Q_1=2$ for A_2 and $Q_1=1$ for $\tilde{A}_2$. This gives the criterion for quasi-admissibility. For A_2, let

$$\tau:\mathrm{SL}_2\hookrightarrow G_{\gamma,\mathrm{der}}$$

be the embedding associated with α_3, which gives $Q_1^\tau=2$, and thus we see that if $n\geqslant 3$, then the orbit is raisable. Similarly, if $n\geqslant 2$, then the orbit $\tilde{A}_2$ is raisable.

The orbit $A_2+\tilde{A}_1$ has $G_{\gamma,\mathrm{der}}=\nabla(\mathrm{SL}_2)$, where

$$\nabla=\mathrm{Sym}^2\times\mathrm{id}:\mathrm{SL}_2\hookrightarrow\mathrm{SL}_{3,\alpha_1,\alpha_2}\times\mathrm{SL}_{2,\alpha_4}.$$

In this case,

$$\mathfrak{g}[1]=V^2\oplus V^4$$

as a $G_{\gamma,\mathrm{der}}$-module. We have

$$(Q_1,Q_2)=(6,11).$$

Thus, the orbit is quasi-admissible if $n=4$ or 12. For raisability, the method in [42] does not apply.

The orbit B_2 has

$$G_{\gamma,\mathrm{der}}\simeq\mathrm{SL}_{2,\alpha_2}\times\mathrm{SL}_2\subset\mathrm{Sp}_4,$$

where Sp_4 is associated with α_2 and α_3. Also,

$$\mathfrak{g}[1]=(V^2\boxtimes V^1)\oplus(V^1\boxtimes V^2).$$

For every copy of SL_2 in $G_{\gamma,0}$, one has $Q_1=1$. Also, $Q_2=1$. Thus, the orbit B_2 is quasi-admissible if and only if $n=2$. On the other hand, if we take $\mathrm{SL}_{2,\tau}$ to be associated with α_2, then (C1)–(C3) are satisfied with $m=1$. In this case, the orbit B_2 is raisable if $n\neq 2$.

The orbit $\tilde{A}_2+A_1$ gives

$$G_{\gamma,\mathrm{der}}=\Delta(\mathrm{SL}_2)\subset\mathrm{SL}_{2,\alpha_1}\times\mathrm{SL}_{2,\alpha_3}$$

and

$$\mathfrak{g}[1]=2V^2\oplus V^4.$$

This gives that $(Q_1,Q_2)=(3,12)$. Thus, the orbit is quasi-admissible if and only if $n=1,3$. On the other hand, the method of raisability does not apply.

The orbit $C_3(a_1)$ has $G_{\gamma,\mathrm{der}}\simeq\mathrm{SL}_{2,\alpha_2}$ and $\mathfrak{g}[1]=3V^2$. Thus, $(Q_1,Q_2)=(1,3)$, and the orbit is quasi-admissible if and only if $n=2$. For raisability, taking $\tau=\mathrm{id}$ shows that the orbit is raisable if $n\neq 2$.

The orbit B_3 gives

$$\mathbf{G}_{\gamma,\mathrm{der}}=\mathrm{SO}_3\subset\mathrm{SL}_3$$

associated with α_3 and α_4, and $\mathfrak{g}[1]=0$. Thus, one has

$$(Q_1, Q_2) = (8, 0).$$

This shows that the orbit is quasi-admissible if and only if $n = 1, 2, 4, 8$. Taking $\tau : \mathrm{SL}_2 \to \mathrm{SO}_3 = \mathrm{PGL}_2$ to be the natural map, one sees that the orbit B_3 is raisable if $n \neq 1, 2, 4, 8$.

The orbit C_3 has $G_{\gamma,\mathrm{der}} = \mathrm{SL}_{2,\alpha_2}$ and $\mathfrak{g}[1] = 2V^2$. This gives that $(Q_1, Q_2) = (1, 2)$. Thus, the orbit is quasi-admissible if and only if $n = 1$. Also, taking $\tau = \mathrm{id}$ shows that it is raisable if $n \geqslant 2$.

3.6 Certain orbits for types $E_r, 6 \leqslant r \leqslant 8$

Now we consider several specific non-distinguished orbits for each exceptional group E_r and investigate their $\overline{E}_r^{(n)}$-quasi-admissibility and $\overline{E}_r^{(n)}$-raisability. The consideration of these orbits is motivated from theta representations $\Theta(\overline{E}_r^{(n)})$ since they are expected to be equal to $\mathcal{N}_{\mathrm{Wh}}^{\max}(\Theta(\overline{E}_r^{(n)})) \otimes F^{\mathrm{al}}$ for some n.

More precisely, we consider orbits as follows:

• for E_6, the orbits

$$3A_1,\ 2A_2 + A_1,\ D_4,\ A_4 + A_1,\ D_5;$$

• for E_7, the orbits

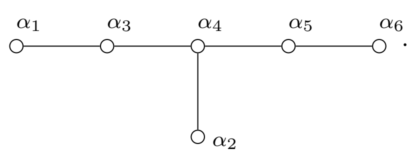
$$4A_1,\ 2A_2 + A_1,\ A_3 + A_2 + A_1,\ A_4 + A_2,\ A_6,\ E_6(a_1);$$

• for E_8, the orbits

$$4A_1,\ 2A_2 + 2A_1,\ 2A_3,\ A_4 + A_3,\ A_6 + A_1,\ A_7.$$

3.6.1 *Type* E_6

Consider the Dynkin diagram for the simple roots of $G = E_6$, following Bourbaki's labeling [7]:



The results are given in Table 3.

The orbit $3A_1$ has

$$G_{\gamma,\mathrm{der}} \simeq \mathrm{SL}_{2,\alpha_2} \times \Delta(\mathrm{SL}_3),$$

where $\Delta : \mathrm{SL}_3 \hookrightarrow \mathrm{SL}_{3,\alpha_1,\alpha_3} \times \mathrm{SL}_{3,\alpha_5,\alpha_6}$ is the diagonal embedding. Also, $\mathfrak{g}[1] = V^2 \boxtimes (V^1 \oplus V_{\mathrm{adj}})$. Thus, for SL_{2,α_2}, we have $(Q_1, Q_2) = (1, 9)$, and for $\Delta(\mathrm{SL}_3)$, one has $(Q_1, Q_2) = (2, 12)$. Hence, $\mathfrak{m}_*(\overline{\mathrm{SL}}_{2,\alpha_2}^{(n,2)})$ splits over SL_{2,α_2} if and only if $n = 2$. On the other hand, the n^*-fold cover of $\Delta(\mathrm{SL}_3)$ splits if and only if $n = 1, 2$. This shows that the orbit is quasi-admissible if and only if $n = 2$. For raisability, τ such that $\tau(\mathrm{SL}_2) = \mathrm{SL}_{2,\alpha_2}$ satisfies (C1)–(C3) with $Q_2^\tau = m = 9$. We see that the orbit is raisable if $n \neq 2$.

Table 3 Some nilpotent orbits for $\overline{E}_6^{(n)}$

$\mathcal{O}$	special/even?	$\mathbf{G}_{\gamma,\mathrm{der}}$	(Q_1, Q_2)	quasi-admissible, if and only if	raisable if
$3A_1$	no/no	$\mathrm{SL}_{2,\alpha_2} \times \mathrm{SL}_3$	$(1,9), (2,12)$	$n = 2$	$n \neq 2$
$2A_2 + A_1$	no/no	SL_2	$(3, 14)$	$n = 1, 3$	n.a.
$D_4(a_1)$	yes/yes	1	n.a.	all n	n.a.
$A_4 + A_1$	yes/no	1	n.a.	all n	n.a.
D_5	yes/yes	1	n.a.	all n	n.a.

The orbit $2A_2 + A_1$ gives $G_{\gamma,\mathrm{der}} = \mathrm{SL}_2$, diagonally embedded into $\mathrm{SL}_{2,\alpha_2} \times \mathrm{SL}_{2,\alpha_3} \times \mathrm{SL}_{2,\alpha_5}$. One has

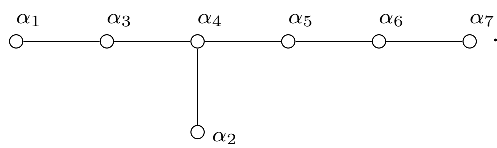
$$\mathfrak{g}[1] = 4V^2 \oplus V^4.$$

This gives $(Q_1, Q_2) = (3, 14)$. Thus, the orbit is quasi-admissible if and only if $n = 1, 3$. For raisability, the method in [42] does not apply.

The orbit $D_4(a_1), A_4 + A_1$ and D_5 all give that $G_{\gamma,\mathrm{der}} = 1$.

3.6.2 *Type E_7*

Consider the Dynkin diagram for the simple roots of E_7:



The results are given in Table 4.

The orbit $4A_1$ has $G_{\gamma,\mathrm{der}} \simeq \mathrm{Sp}_6 \subset \mathrm{SL}_6$, where SL_6 is associated with $\Delta - \{\alpha_2, \alpha_7\}$. Also, as $G_{\gamma,\mathrm{der}}$-module, one has $\mathfrak{g}[1] = 2V^6 \oplus V^{14}$. Let $\mathrm{SL}_{2,\tau} \subset G_{\gamma,\mathrm{der}}$ be associated with the long root. As $\mathrm{SL}_{2,\tau}$-module, we have

$$\mathfrak{g}[1] = 7V^2 \oplus 12V^1$$

with $m = 7$. This shows that

$$(Q_1, Q_2) = (1, 7),$$

and thus the orbit is quasi-admissible if and only if $n = 2$. Moreover, considering the above $\mathrm{SL}_{2,\tau}$, we see that it is raisable if $n \neq 2$.

The orbit $2A_2 + A_1$ has

$$G_{\gamma,\mathrm{der}} = \mathrm{SL}_2^a \times \mathrm{SL}_2^b \subset \mathrm{SL}_{2,\alpha_1} \times \mathrm{SL}_4 \times \mathrm{SL}_{2,\alpha_7}.$$

Here, $\mathrm{SL}_2^a \times \mathrm{SL}_2^b \hookrightarrow \mathrm{SL}_{2,\alpha_1} \times \mathrm{SL}_{2,\alpha_7}$ is the identity, and $\mathrm{SL}_2^a \times \mathrm{SL}_2^b \hookrightarrow \mathrm{SL}_4$ is the tensor embedding, where SL_4 is associated with $\{\alpha_2, \alpha_4, \alpha_5\}$. One has

$$\mathfrak{g}[1] = 2(V_a^2 \boxtimes V_b^3) \oplus (V_a^4 \oplus 2V_a^2) \boxtimes V_b^1.$$

Thus, for SL_2^a, one has $(Q_1, Q_2) = (3, 18)$; for SL_2^b, one has $(Q_1, Q_2) = (3, 16)$. We see that the orbit is quasi-admissible if and only if $n = 1, 3$. The method in [42] does not work for raisability.

The orbit $A_3 + A_2 + A_1$ gives

$$G_{\gamma,\mathrm{der}} \simeq \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_5 \times \mathrm{SL}_3,$$

where the embedding is given by $4\omega_1 \otimes 2\omega_1$. Here, ω_1 is the fundamental weight of SL_2. Also, SL_5 and SL_3 are associated with $\{\alpha_i : 1 \leqslant i \leqslant 4\}$ and $\{\alpha_6, \alpha_7\}$, respectively. One has $(Q_1, Q_2) = (24, 0)$.

Table 4 Some nilpotent orbits for $\overline{E}_7^{(n)}$

$\mathcal{O}$	special/even?	$\mathbf{G}_{\gamma,\mathrm{der}}$	(Q_1, Q_2)	quasi-admissible, if and only if	raisable if
$4A_1$	no/no	Sp_6	$(1, 7)$	$n = 2$	$n \neq 2$
$2A_2 + A_1$	no/no	$\mathrm{SL}_2^a \times \mathrm{SL}_2^b$	$(3, 8), (3, 16)$	$n = 1, 3$	n.a.
$A_3 + A_2 + A_1$	yes/yes	SL_2	$(24, 0)$	$n \vert 24$	n.a.
$A_4 + A_2$	yes/yes	SL_2	$(15, 0)$	$n \vert 15$	n.a.
A_6	yes/yes	SL_2	$(7, 0)$	$n \vert 7$	n.a.
$E_6(a_1)$	yes/yes	1	n.a.	all n	n.a.

Thus, the orbit is quasi-admissible if and only if $n|24$. For $\tau = \mathrm{id} : \mathrm{SL}_2 \to G_{\gamma,\mathrm{der}}$, the condition (C1) is not satisfied, thus the method of [42] does not apply.

The orbit $A_4 + A_2$ gives

$$G_{\gamma,\mathrm{der}} = \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_4 \times \mathrm{SL}_3 \times \mathrm{SL}_{2,\alpha_2},$$

where the embedding is given by $3\omega_1 \otimes 2\omega_1 \otimes \omega_1$. Here, SL_4 and SL_3 are associated with $\{\alpha_5, \alpha_6, \alpha_7\}$ and $\{\alpha_1, \alpha_3\}$, respectively. We have

$$(Q_1, Q_2) = (15, 0),$$

and thus the orbit is quasi-admissible if and only if $n|15$. The conditions for raisability are not satisfied.

The orbit A_6 gives

$$G_{\gamma,\mathrm{der}} = \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_2^a \times \mathrm{SL}_2^b \times \mathrm{SL}_2^c \times \mathrm{SL}_3,$$

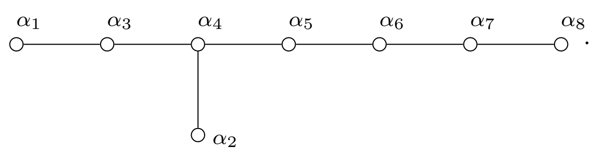
where the embedding is given by identities into $\mathrm{SL}_2^i, i = a, b, c$ and $2\omega_1 : \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_3$. We have $\mathfrak{g}[1] = 0$. We get

$$(Q_1, Q_2) = (7, 0),$$

and thus the orbit is quasi-admissible if and only if $7|n$. Since (C1) for the identity map $\mathrm{SL}_2 \to G_{\gamma,\mathrm{der}}$ is not satisfied, the method of raisability does not apply.

3.6.3 *Type* E_8

Consider the Dynkin diagram for the simple roots of E_8:



The results are given in Table 5.

The orbit $4A_1$ of E_8 gives

$$G_{\gamma,\mathrm{der}} = \mathrm{Sp}_8 \hookrightarrow \mathrm{SL}_8,$$

where SL_8 is associated with $\Delta - \{\alpha_2\}$. Also,

$$\mathfrak{g}[1] = \bigwedge^3 V^8.$$

Now if we take $\mathrm{SL}_{2,\tau}$ to be associated with any long root of $G_{\gamma,\mathrm{der}}$, then (C1)–(C3) are satisfied with $m = 15$. This shows that

$$(Q_1, Q_2) = (1, 15),$$

and thus the orbit is quasi-admissible if and only if $n = 2$. It also shows that the orbit is raisable if $n \neq 2$.

The orbit $2A_3 + 2A_1$ has

$$G_{\gamma,\mathrm{der}} = \mathrm{Sp}_4 \hookrightarrow \mathrm{SL}_4 \times \mathrm{SL}_5$$

Table 5 Some nilpotent orbits for $\overline{E}_8^{(n)}$

$\mathcal{O}$	special/even?	$\mathbf{G}_{\gamma,\mathrm{der}}$	(Q_1, Q_2)	quasi-admissible, if and only if	raisable if
$4A_1$	no/no	Sp_8	$(1, 15)$	$n = 2$	$n \neq 2$
$2A_2 + 2A_1$	no/no	Sp_4	$(3, 34)$	$n = 1, 3$	n.a.
$2A_3$	no/no	Sp_4	$(2, 15)$	$n = 4$	n.a.
$A_4 + A_3$	no/no	SL_2	$(10, 58)$	$n|10$	n.a.
$A_6 + A_1$	yes/no	SL_2	$(7, 14)$	$n = 1, 7$	n.a.
A_7	no/no	SL_2	$(4, 15)$	$n = 8$	n.a.

via the diagonal map, where $\mathrm{Sp}_4 \hookrightarrow \mathrm{SL}_4$ is the canonical inclusion and $\mathrm{SL}_4 = \mathrm{Spin}_5 \twoheadrightarrow \mathrm{SO}_5 \hookrightarrow \mathrm{SL}_5$. Here, SL_4 and SL_5 are associated with $\{\alpha_6, \alpha_7, \alpha_8\}$ and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, respectively. From this, we get $Q_1 = 3$. Taking $\mathrm{SL}_{2,\tau} \subset \mathrm{Sp}_4$ to be associated with the long root of Sp_4, we have

$$\mathfrak{g}[1] = 8V^1 \oplus 8V^2 \oplus 4V^3 \oplus V^4.$$

Thus, $Q_2 = 34$ and this shows that the orbit is quasi-admissible if and only if $n = 1, 3$. The method of raisability does not apply.

For the orbit $2A_3$, consider

$$L = \mathrm{SL}_4^a \times \mathrm{SL}_4^b,$$

which acts on $\mathfrak{g}[1] = V_b^4 \oplus (V_a^4 \boxtimes \bigwedge^2 V_b^4)$. Here, SL_4^a and SL_4^b are associated with $\{\alpha_2, \alpha_3, \alpha_4\}$ and $\{\alpha_6, \alpha_7, \alpha_8\}$, respectively. We have

$$G_{\gamma,\mathrm{der}} = \mathrm{Sp}_4 \hookrightarrow L$$

via the diagonal embedding. Let $\mathrm{SL}_{2,\tau} \subset G_{\gamma,\mathrm{der}}$ be the one associated with the long root. We have that as an $\mathrm{SL}_{2,\tau}$-module,

$$\mathfrak{g}[1] = 8V_1 \oplus 7V_2 \oplus 2V_3.$$

This gives that

$$(Q_1, Q_2) = (2, 15),$$

and thus the orbit is quasi-admissible if and only if $n = 4$. For raisability, the method in [42] does not work.

The orbit $A_4 + A_3$ has

$$G_{\gamma,\mathrm{der}} = \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_3 \times \mathrm{SL}_3,$$

where the embedding is diagonal via $\mathrm{id} : \mathrm{SL}_2 \to \mathrm{SL}_2$ and $\mathrm{Sym}^2 : \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_3$. Here, SL_2 and SL_3 are associated with the connected components of $\Delta - \{\alpha_4, \alpha_7\}$ in the Dynkin diagram. One has

$$\mathfrak{g}[1] = 3V_2 \oplus 3V_4 \oplus V_6,$$

as a $G_{\gamma,\mathrm{der}}$-module. We get

$$(Q_1, Q_2) = (10, 68)$$

and thus the orbit is quasi-admissible if and only if $n|10$. The method of [42] does not apply to raisability.

The orbit $A_6 + A_1$ has

$$G_{\gamma,\mathrm{der}} \simeq \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_3.$$

Here, the three SL_2's and SL_3 are associated with the connected component of $\Delta - \{\alpha_1, \alpha_4, \alpha_6\}$ in the Dynkin diagram. We have

$$\mathfrak{g}[1] = V_4 \oplus 4V_2,$$

and hence

$$(Q_1, Q_2) = (7, 14).$$

This shows that the orbit is quasi-admissible if and only if $n = 1, 7$. On the other hand, the method of [42] for raisability does not apply.

The orbit A_7 gives

$$G_{\gamma,\mathrm{der}} \simeq \mathrm{SL}_2 \hookrightarrow \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2,$$

diagonally embedded in SL_2's associated with α_2, α_3, α_5 and α_8. One has

$$\mathfrak{g}[1] = 5V_2 \oplus V_4$$

as a $G_{\gamma,\mathrm{der}}$-module. This gives that

$$(Q_1, Q_2) = (4, 15).$$

Hence the orbit is quasi-admissible if and only if $n = 8$. The method of [42] does not apply to raisability.

4 Wavefront sets of theta representations

In this section, we consider theta representations $\Theta(\nu)$ of $\overline{G}^{(n)}$, where $\nu \in X \otimes \mathbf{R}$ is a certain exceptional vector. We compute explicitly the F^{al}-orbit $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu) \subset \mathfrak{g}_{F^{\mathrm{al}}}$, which is expected to be the single element in $\mathcal{N}_{\mathrm{Wh}}^{\max}(\Theta(\nu)) \otimes F^{\mathrm{al}}$. We show that a method of determining $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu) \subset \mathfrak{g}_{F^{\mathrm{al}}}$ is given by Sommers' duality [63] between nilpotent orbits, which generalizes the classical Barbasch-Vogan duality. With an explicit computation, we also determine the quasi-admissibility and non-raisability of such orbits.

4.1 Theta representations

We introduce theta representations following the notation and exposition in [27].

Let $\overline{T} \subset \overline{G}$ be the covering torus of $\overline{G}$. We assume that there exists a certain distinguished (finite-dimensional) genuine representation $\pi^\dagger$ of $\overline{T}$ determined by a distinguished genuine character $\chi^\dagger$ of the center $Z(\overline{T})$ of $\overline{T}$ (see [23, § 6–7]). For every $\nu \in X \otimes \mathbf{R}$, there is a map $\delta_\nu : T \to \mathbf{C}^\times$ given by

$$\delta_\nu(y \otimes a) = |a|_F^{\nu(y)}$$

on the generators $y \otimes a \in T$, where $\nu(y)$ is the natural pairing between Y and $X \otimes \mathbf{R}$. For every $\nu \in X \otimes \mathbf{R}$, denote by

$$I(\pi^\dagger, \nu) := \mathrm{Ind}_{\overline{B}}^{\overline{G}}(\pi^\dagger \otimes \delta_\nu)$$

the normalized induced principal series representation of $\overline{G}$.

A vector $\nu \in X \otimes \mathbf{R}$ is called an exceptional character if $\nu(\alpha_{Q,n}^\vee) = 1$ for every $\alpha \in \Delta$. Here, $\alpha_{Q,n}^\vee := n_\alpha \cdot \alpha^\vee$. It follows from the Langlands classification theorem for covers (see [5]) that if $\nu \in X \otimes \mathbf{R}$ is exceptional, then $I(\pi^\dagger, \nu)$ has a unique irreducible quotient $\Theta(\pi^\dagger, \nu)$. We may write $\Theta(\overline{G}, \nu), \Theta(\nu)$ or $\Theta(\overline{G})$ for $\Theta(\pi^\dagger, \nu)$, whenever the emphasis is different; but the dependence on $\pi^\dagger$ and exceptional ν are both understood.

A covering group $\overline{G}$ is called saturated (see [25, Definition 2.1]) if

$$Y^{sc} \cap Y_{Q,n} = Y_{Q,n}^{sc},$$

where the one-sided inclusion $\supset$ always holds. If G is semisimple and simply-connected, then G is saturated if and only if its dual group $\overline{G}^\vee$ is of adjoint type, i.e., $Y_{Q,n} = Y_{Q,n}^{sc}$. We also recall the notion of a persistent cover as follows (see [25, Definition 2.3]). Consider

$$\mathscr{X}_{Q,n}^{sc} := Y/Y_{Q,n}^{sc}, \quad \mathscr{X}_{Q,n} := Y/Y_{Q,n},$$

which are both endowed with the twisted Weyl action

$$w[y] := w(y - \rho^\vee) + \rho^\vee$$

for every w in the Weyl group W. Here, $\rho^\vee$ is the half sum of all positive coroots in $\Phi^\vee$. For every $y \in Y$, let $y^\dagger$ and $y^\ddagger$ denote its image in $\mathscr{X}_{Q,n}^{sc}$ and $\mathscr{X}_{Q,n}$, respectively. An n-fold cover $\overline{G}$ is called persistent if

$$\mathrm{Stab}_W(y^\dagger; \mathscr{X}_{Q,n}^{sc}) = \mathrm{Stab}_W(y^\ddagger; \mathscr{X}_{Q,n})$$

for every $y \in Y$. A saturated cover is always persistent.

4.2 The set $\mathcal{N}_{\mathrm{tr}}^{\max}(\Theta(\pi^\dagger, \nu))$

For every $\nu \in X \otimes \mathbf{R}$, denote by

$$W_\nu = \{w \in W : w(\nu) - \nu \in X^{sc}\} \subset W$$

the integral Weyl subgroup associated with ν. It is a reflection subgroup associated with the root subsystem

$$\Phi_\nu = \{\alpha \in \Phi : \langle \nu, \alpha^\vee \rangle \in \mathbf{Z}\}. \tag{4.1}$$

The MacDonald-Lusztig-Spaltenstein j-induction thus gives an irreducible representation $j_{W_\nu}^W(\varepsilon_{W_\nu})$ of W, where ε_{W_ν} denotes the sign character of W_ν. Write $\varepsilon_\nu = \varepsilon_{W_\nu}$.

Let $\mathfrak{g} \otimes F^{\rm al}$ be the Lie algebra of $\mathbf{G}$ over the algebraically closed field $F^{\rm al}$. Let $x \in \mathfrak{g} \otimes F^{\rm al}$ be a nilpotent element and consider $A_x := \mathbf{G}_{ad,x}/(\mathbf{G}_{ad,x})^o$, the group of connected components of $\mathbf{G}_{ad,x}$. Since A_x depends only on the conjugacy class $\mathcal{O}_x$ of x, for a nilpotent orbit $\mathcal{O} \subset \mathfrak{g} \otimes F^{\rm al}$, we use $A_{\mathcal{O}}$ to denote A_x for any $x \in \mathcal{O}$. Define

$$\mathcal{N}^{\rm en}(\mathbf{G}) = \{(\mathcal{O}, \eta) : \mathcal{O} \in \mathcal{N}(\mathbf{G}) \text{ and } \eta \in {\rm Irr}(A_{\mathcal{O}})\},$$

and the Springer correspondence gives an injective map

$$\mathrm{Spr}^{\mathbf{G}} : \mathrm{Irr}(W) \hookrightarrow \mathcal{N}^{\rm en}(\mathbf{G}),$$

denoted by

$$\mathrm{Spr}^{\mathbf{G}}(\sigma) = (\mathcal{O}^{\mathbf{G}}_{\rm Spr}(\sigma), \eta(\sigma)).$$

If no confusion arises, we write $\mathrm{Spr} = \mathrm{Spr}^{\mathbf{G}}$ and $\mathcal{O}_{\rm Spr} = \mathcal{O}^{\mathbf{G}}_{\rm Spr}$. We call

$$\mathcal{O}^{\mathbf{G}}_{\rm Spr}(\sigma) \subset \mathfrak{g} \otimes F^{\rm al}$$

the nilpotent orbit associated with σ. Here, the normalization is such that $\mathcal{O}^{\mathbf{G}}_{\rm Spr}(\mathbf{1}) = \mathcal{O}_{\rm reg}$, and $\mathcal{O}^{\mathbf{G}}_{\rm Spr}(\varepsilon_W) = \mathcal{O}_0$. Note that for every $\mathcal{O} \in \mathcal{N}(\mathbf{G})$, the pair $(\mathcal{O}, \mathbf{1})$ lies in the image of Spr, i.e., $(\mathcal{O}, \mathbf{1}) = \mathrm{Spr}(\sigma_{\mathcal{O}})$ for a unique $\sigma_{\mathcal{O}} \in \mathrm{Irr}(W)$. This gives us a well-defined injective map

$$\mathrm{Spr}_{\mathbf{1}}^{-1} : \mathcal{N}(\mathbf{G}) \hookrightarrow \mathrm{Irr}(W)$$

given by $\mathrm{Spr}_{\mathbf{1}}^{-1}(\mathcal{O}) := \mathrm{Spr}^{-1}((\mathcal{O}, \mathbf{1}))$. It is clear that $\mathcal{O}_{\rm Spr} \circ \mathrm{Spr}_{\mathbf{1}}^{-1} = \mathrm{id}_{\mathcal{N}(\mathbf{G})}$. However, $\mathrm{Spr}_{\mathbf{1}}^{-1} \circ \mathcal{O}_{\rm Spr}$ may not be the identity map on $\mathrm{Irr}(W)$.

One has the permutation representation

$$\sigma^{\mathscr{X}} : W \to \mathrm{Perm}(\mathscr{X}_{Q,n}) \tag{4.2}$$

given by the twisted Weyl action $w[y] = w(y - \rho^\vee) + \rho^\vee$. We have the following expectation on the stable wavefront set of $\Theta(\pi^\dagger, \nu)$.

Conjecture 4.1 (See [27])**.** *Let $\overline{G}$ be a Brylinski-Deligne n-fold covering of G over a p-adic local field F. Let $\nu \in X \otimes \mathbf{R}$ be exceptional. Consider the Harish-Chandra local character expansion of $\Theta(\pi^\dagger, \nu)$ as in* (1.1)*.*

- *One has*

$$\mathcal{N}^{\max}_{\rm tr}(\Theta(\pi^\dagger, \nu)) \otimes F^{\rm al} = \{\mathcal{O}^{\mathbf{G}}_{\rm Spr}(j_{W_\nu}^W(\varepsilon_\nu))\}. \tag{4.3}$$

- *If furthermore $\overline{G}$ is a persistent cover and $p \nmid n$, then*

$$c_{\mathcal{O}} = \langle j_{W_\nu}^W(\varepsilon_\nu), \varepsilon_W \otimes \sigma^{\mathscr{X}} \rangle_W \tag{4.4}$$

for every orbit $\mathcal{O} \in \mathcal{N}^{\max}_{\rm tr}(\Theta(\pi^\dagger, \nu))$.

In [27], we showed the part of the Conjecture when $\Theta(\pi^\dagger, \nu)$ is generic. Compatibility with existing work in the literature was also verified. This verification depends solely on explicating the right-hand side of (4.3) for several cases of $\overline{G}^{(n)}$ of interest. (During the revision of our paper, the equality (4.3) in Conjecture 4.1 has been verified in the recent work of Karasiewicz et al. [48].)

Remark 4.2. We have revised the statement in Conjecture 4.1 compared with the original form in [27, Conjecture 2.5] since the assumption on "persistence" and the tame condition $p \nmid n$ should concern only (4.4) and not (4.3). Moreover, we have removed a certain process of "saturation" $\tilde{\nu} \in X \otimes \mathbf{R}$ of ν in Conjecture 4.1 above, which appeared [27, Conjecture 2.5]. It seems to us that the saturation may not be necessary. More precisely, if G is almost-simple and simply-connected, then in view of the third-bullet remark in [27, p. 11], a saturation of ν is needed essentially only for $\overline{G} = \overline{\mathrm{Sp}}_{2r}^{(n)}$ with $4|n$ by [27, Lemma 2.3]. (For simplicity, we assume $\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{Sp}}_{2r}^{(n)}) = 1$ here.) For such $\overline{\mathrm{Sp}}_{2r}^{(n)}$, if we write $2r = (n/2)a + b$ with $a \in \mathbf{N}_{\geqslant 0}$ and $0 \leqslant b < n/2$, then

$$\Phi_\nu = (A_a)^{b/2} \times (A_{a-1})^{n/4-b/2}.$$

On the other hand, if we write $r = (n/2)k + s$ with $k \in \mathbf{N}_{\geqslant 0}$ and $0 \leqslant s < n/2$, then

$$\Phi_{\tilde{\nu}} = \begin{cases} (A_{2k-1})^{n/4-1-s} \times (A_{2k})^s \times D_k \times C_k, & \text{if } 0 \leqslant s < n/4, \\ (A_{2k})^{n/2-1-s} \times (A_{2k+1})^{s-n/4} \times D_{k+1} \times C_k, & \text{if } n/4 \leqslant s < n/2. \end{cases}$$

In particular, Φ_ν is not isomorphic to $\Phi_{\tilde{\nu}}$. However, one can check that for such $\overline{\mathrm{Sp}}_{2r}^{(n)}$, we get

$$\mathcal{O}^{\mathbf{G}}_{\mathrm{Spr}}(j^W_{W_\nu}(\varepsilon_\nu)) = \mathcal{O}^{\mathbf{G}}_{\mathrm{Spr}}(j^W_{W_{\tilde{\nu}}}(\varepsilon_{\tilde{\nu}})),$$

which is more explicitly given in Table 6. This explains our venturing to remove the saturation of ν in Conjecture 4.1.

4.3 The method of the computation

Here, we want to compute the orbit $\mathcal{O}^{\mathbf{G}}_{\mathrm{Spr}}(j^W_{W_\nu}(\varepsilon_\nu))$ explicitly for all covers, at least when G is almost simple and simply-connected or is of the classical type. We show that the method of the computation is reduced to the Sommers' duality as in [63].

For any linear algebraic group $\mathbf{G}$, we consider another enhanced set of nilpotent orbits

$$\mathcal{N}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G}) := \{(\mathcal{O}, \mathfrak{c}) : \mathcal{O} \in \mathcal{N}(\mathbf{G}) \text{ and } \mathfrak{c} \in \mathrm{Conj}(A_{\mathcal{O}})\},$$

where $\mathrm{Conj}(A_{\mathcal{O}})$ denotes the set of conjugacy classes of $A_{\mathcal{O}}$. There is a quotient map

$$A_{\mathcal{O}} \twoheadrightarrow \tilde{A}_{\mathcal{O}},$$

where $\tilde{A}_{\mathcal{O}}$ is the Lusztig canonical quotient (see [63, § 5]). This gives a set

$$\tilde{\mathcal{N}}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G}) := \{(\mathcal{O}, \tilde{\mathfrak{c}}) : \mathcal{O} \in \mathcal{N}(\mathbf{G}) \text{ and } \tilde{\mathfrak{c}} \in \mathrm{Conj}(\tilde{A}_{\mathcal{O}})\}$$

together with a natural surjection

$$f : \mathcal{N}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G}) \twoheadrightarrow \tilde{\mathcal{N}}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G})$$

given by $(\mathcal{O}, \mathfrak{c}) \mapsto (\mathcal{O}, \tilde{\mathfrak{c}})$, where $\tilde{\mathfrak{c}}$ is the image of $\mathfrak{c}$.

Recall that there is a natural order-reversing bijection on special orbits $\mathcal{N}^{\mathrm{spe}}(\mathbf{G}) \subset \mathcal{N}(\mathbf{G})$, which can be extended to give the Lusztig-Spaltenstein map

$$d_{\mathrm{LS}} : \mathcal{N}(\mathbf{G}) \twoheadrightarrow \mathcal{N}^{\mathrm{spe}}(\mathbf{G}).$$

Now we have the various extensions of d_{LS} as depicted in the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{N}^{\mathrm{spe}}(\mathbf{G}) & \xrightarrow{\iota} & \mathcal{N}^{\mathrm{spe}}(\mathbf{G}^\vee) & \hookrightarrow & \mathcal{N}(\mathbf{G}^\vee) & \overset{\mathrm{pr}_1}{\twoheadleftarrow} & \tilde{\mathcal{N}}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G}^\vee) \\
& \nwarrow^{d_{\mathrm{LS}}} & \uparrow{\scriptstyle d_{\mathrm{BV}}} & & \uparrow{\scriptstyle d_{\mathrm{Som}}} & \nwarrow^{d_{\mathrm{Som}}} & \uparrow{\scriptstyle d_{\mathrm{Ach}}} \\
& & \mathcal{N}(\mathbf{G}) & \hookrightarrow & \mathcal{N}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G}) & \overset{f}{\twoheadrightarrow} & \tilde{\mathcal{N}}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G}).
\end{array} \tag{4.5}$$

Here, ι is the canonical bijection arising from the identification of the Weyl group W of $\mathbf{G}$ and that of its Langlands dual group $\mathbf{G}^\vee$. Since by definition, there is a bijection between special representations of the Weyl group and the special nilpotent orbits, one has the bijection ι. Moreover, the Barbasch-Vogan duality is given by

$$d_{\mathrm{BV}} := \iota \circ d_{\mathrm{LS}}.$$

The top inclusion in (4.5) is the canonical one, and the bottom inclusion is given by map $\mathcal{O} \mapsto (\mathcal{O}, \mathbf{1})$. On the other hand, the existence of the extended maps

$$d_{\mathrm{Som}} : \mathcal{N}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G}) \to \mathcal{N}(\mathbf{G}^\vee)$$

and d_{Ach} is not trivial and are given in [63] and [1] respectively. It was shown in [63] that the map d_{Som} factors through f and thus can be defined on $\tilde{\mathcal{N}}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G})$ as well.

What pertains to our work is the map d_{Som} and thus we give some elaboration on it. Take any

$$(\mathcal{O}, \mathfrak{c}) \in \mathcal{N}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G}).$$

One picks $e \in \mathcal{O}$ and a semisimple element $s \in Z_{\mathbf{G}}(e)$ such that

$$\phi(s) = \mathfrak{c},$$

where $\phi : Z_{\mathbf{G}}(e) \twoheadrightarrow A_{\mathcal{O}}$ is the canonical quotient map. We take

$$\mathbf{L} := Z_{\mathbf{G}}(s)$$

which is the so-called pseudo-Levi subgroup (see [53, § 6]) of $\mathbf{G}$ with $e \in \mathbf{L}$. Consider the nilpotent orbit $\mathcal{O}^{\mathbf{L}}_e \subset \mathrm{Lie}(\mathbf{L})$. It is known that there exists $\sigma_e \in \mathrm{Irr}(W(\mathbf{L}))$ such that

$$\mathrm{Spr}(\sigma_e) = (d_{\mathrm{LS}}(\mathcal{O}^{\mathbf{L}}_e), \mathbf{1}) \in \mathcal{N}^{\mathrm{en}}(\mathbf{L}).$$

Hence,

$$\sigma_e = \mathrm{Spr}^{-1}_{\mathbf{1}} \circ d_{\mathrm{LS}}(\mathcal{O}^{\mathbf{L}}_e).$$

Now consider

$$j^W_{W(\mathbf{L})}(\sigma_e) \in \mathrm{Irr}(W),$$

where we identify the Weyl groups of $\mathbf{G}$ and $\mathbf{G}^\vee$. The desired orbit arising from d_{Som} is then

$$d_{\mathrm{Som}}((\mathcal{O}, \mathfrak{c})) = \mathcal{O}^{\mathbf{G}^\vee}_{\mathrm{Spr}}(j^W_{W(\mathbf{L})}(\sigma_e)) \in \mathcal{N}(\mathbf{G}^\vee),$$

which is independent of the choices of e and s. In fact, we have

$$\mathrm{Spr}^{\mathbf{G}^\vee}(j^W_{W(\mathbf{L})}(\sigma_e)) = (d_{\mathrm{Som}}(\mathcal{O}, \mathfrak{c}), \mathbf{1}).$$

For $(\mathcal{O}, \mathfrak{c})$ above, one can pick (s, e) such that $\mathcal{O}^{\mathbf{L}}_e$ is a distinguished nilpotent orbit of $\mathbf{L}$. In fact, a generalized Bala-Carter classification for $\mathcal{N}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G})$ was constructed by Sommers [62, p. 548], using such pairs satisfying the minimal "key property". Recall that the Bala-Carter classification gives a bijection

$$f_{\mathrm{BC}} : \{(\mathbf{L}', \mathcal{O}_{\mathbf{L}'})\} \to \mathcal{N}(\mathbf{G}),$$

where $\mathbf{L}' \subset \mathbf{G}$ is a Levi subgroup and $\mathcal{O}_{\mathbf{L}'} \subset \mathrm{Lie}(\mathbf{L})$ a distinguished orbit. By considering more generally a general pseudo-Levi subgroup $\mathbf{L} \subset \mathbf{G}$ and a distinguished orbit $\mathcal{O}_{\mathbf{L}} \subset \mathrm{Lie}(\mathbf{L})$, Sommers showed that there is a natural surjection $\tilde{f}_{\mathrm{BC}} : \{(\mathbf{L}, \mathcal{O}_{\mathbf{L}})\} \twoheadrightarrow \tilde{\mathcal{N}}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G})$ such that the following diagram

$$\begin{array}{ccc} \{(\mathbf{L}', \mathcal{O}_{\mathbf{L}'})\} & \xrightarrow{f_{\mathrm{BC}}} & \mathcal{N}(\mathbf{G}) \\ \downarrow & & \downarrow \\ \{(\mathbf{L}, \mathcal{O}_{\mathbf{L}})\} & \xrightarrow{\tilde{f}_{\mathrm{BC}}} & \tilde{\mathcal{N}}^{\mathrm{en}}_{\mathrm{geo}}(\mathbf{G}) \end{array}$$

commutes, i.e., it extends the classical Bala-Carter classification.

Combining the above, one has a natural map

$$d^{\heartsuit}_{\mathrm{Som}} := d_{\mathrm{Som}} \circ \tilde{f}_{\mathrm{BC}} : \{(\mathbf{L}, \mathcal{O}_{\mathbf{L}})\} \to \mathcal{N}(\mathbf{G}^\vee).$$

The construction of the map d_{Som} immediately gives the following proposition.

Proposition 4.3. *Let $\mathcal{O}$ be a distinguished orbit of a pseudo-Levi $\mathbf{L}$. Then*

$$d^{\heartsuit}_{\rm Som}((\mathbf{L},\mathcal{O}))=\mathcal{O}^{\mathbf{G}^\vee}_{\rm Spr}(j^W_{W(\mathbf{L})}\circ {\rm Spr}^{-1}_{\mathbf{1}}\circ d_{\rm LS}(\mathcal{O})),$$

where $j^W_{W(\mathbf{L})}\circ {\rm Spr}^{-1}_{\mathbf{1}}\circ d_{\rm LS}(\mathcal{O})$ is viewed as the representation of the Weyl group of $\mathbf{G}^\vee$. In particular,

$$d^{\heartsuit}_{\rm Som}(\mathbf{L},\mathcal{O}_{\rm reg})=\mathcal{O}^{\mathbf{G}^\vee}_{\rm Spr}(j^W_{W(\mathbf{L})}(\varepsilon_{W(\mathbf{L})})).$$

For application purposes, we exchange the roles of $\mathbf{G}$ and $\mathbf{G}^\vee$. In particular, retaining the notations in Conjecture 4.1, we have a pseudo-Levi subgroup $\mathbf{L}^\vee_\nu$ whose root system is $\Phi^\vee_\nu$. It thus follows from Proposition 4.3 that

$$\mathcal{O}^{\mathbf{G}}_{\rm Spr}(j^W_{W_\nu}(\varepsilon_\nu))=d^{\heartsuit}_{\rm Som}(\mathbf{L}^\vee_\nu,\mathcal{O}_{\rm reg}). \tag{4.6}$$

4.4 The orbit $\mathcal{O}_{\rm Spr}(j^W_{W_\nu}\varepsilon_\nu)$ for classical groups

For type A_m groups, pseudo-Levi groups are all Levi subgroups, and thus every $\mathbf{L}^\vee_\nu$ corresponds to a partition

$$\mathfrak{p}_\nu=(p_1^{d_1}p_2^{d_2}\cdots p_k^{d_k})$$

of $m+1$. Since $d^{\heartsuit}_{\rm Som}$ extends the Barbasch-Vogan duality, we have

$$\mathcal{O}^{\mathbf{G}}_{\rm Spr}(j^W_{W_\nu}(\varepsilon_\nu))=\mathfrak{p}_\nu^\top,$$

the transpose of the partition $\mathfrak{p}_\nu$. Consider any Brylinski-Deligne $\overline{\rm GL}^{(n)}_r$ of ${\rm GL}_r$ with $n_\alpha=n$. We write $r=an+b$ with $0\leqslant b<n$. Then

$$\mathfrak{p}_\nu=((a+1)^b a^{n-b}),$$

and this gives

$$\mathcal{O}^{\mathbf{G}}_{\rm Spr}(j^W_{W_\nu}(\varepsilon_\nu))=(n^a b).$$

For types B_r, C_r and D_r, we recall the formula for $d_{\rm Som}(\mathcal{O},\mathfrak{c})$ given in [63], which then gives $d^{\heartsuit}_{\rm Som}(\mathbf{L},\mathcal{O}_{\rm reg})$. First, for the pair $(\mathcal{O},\mathfrak{c})\in\mathcal{N}^{\rm en}_{\rm geo}(\mathbf{G})$, one can choose the aforementioned (s,e) with additional properties.

• ${\rm Lie}(\mathbf{L})$ has semisimple rank equal to that of $\mathfrak{g}$, and one has

$${\rm Lie}(\mathbf{L})=\mathfrak{l}_1\oplus\mathfrak{l}_2,$$

where $\mathfrak{l}_2$ is a semisimple Lie algebra of the same type as $\mathfrak{g}$, and $\mathfrak{l}_1$ is a simple Lie algebra containing $-\check{\alpha}\in\check{\Delta}$ in the extended Dynkin diagram with simple roots $\check{\Delta}$.

• one has $e=e_1+e_2, e_i\in\mathfrak{l}_i$, where e_1 is a distinguished element in $\mathfrak{l}_1$.

The Lie algebra $\mathfrak{l}_1$ is of type B, type C or type D if $\mathfrak{g}$ is of type B, type C or type D, respectively. To such a pair (s,e), one can attach a pair of partitions $(\mathfrak{p}_1,\mathfrak{p}_2)$ corresponding to the orbits $\mathcal{O}^{\mathfrak{l}_1}_{e_1}$ and $\mathcal{O}^{\mathfrak{l}_2}_{e_2}$, respectively. In fact, one has

$$\mathfrak{p}_{\mathcal{O}}=\mathfrak{p}_1\cup\mathfrak{p}_2.$$

Theorem 4.4 (See [63, Theorem 12]). *For type B, type C and type D groups, assume that $(\mathfrak{p}_1,\mathfrak{p}_2)$ is associated with $(\mathcal{O},\mathfrak{c})$ as above. Then*

$$d_{\rm Som}(\mathcal{O},\mathfrak{c})=\begin{cases}(\mathfrak{p}_1\cup(\mathfrak{p}_2^-)_C)^\top_C, & \text{if } \mathfrak{g} \text{ is of type } B,\\ (\mathfrak{p}_1\cup(\mathfrak{p}_2^+)_B)^\top_B, & \text{if } \mathfrak{g} \text{ is of type } C,\\ (\mathfrak{p}_1\cup(\mathfrak{p}_{2,D}^\top)^\top)^\top_D, & \text{if } \mathfrak{g} \text{ is of type } D.\end{cases}$$

Here, the operations $\mathfrak{p}^-$ and $\mathfrak{p}^+$ of a partition $\mathfrak{p}$ are given as in [63, *p.* 804].

If $e_1 = 0$ (equivalently $\mathfrak{g}_1 = 0$), then $\mathfrak{p}_1 = \emptyset$; in this case, $d_{\rm Som}$ recovers the Barbasch-Vogan duality. In general, suppose $(\mathcal{O}, \mathfrak{c}) = f_{\rm BC}(\mathbf{L}, \mathcal{O})$ for a distinguished $\mathcal{O}$ of $\mathbf{L}$. We can write $\mathfrak{l} = \mathfrak{g}_1 \oplus \mathfrak{g}_2'$, where $\mathfrak{g}_1$ is a simple Lie algebra (if nonzero) containing $-\check{\alpha} \in \check{\Delta}$, and $\mathfrak{g}_2'$ is a Levi subalgebra of $\mathfrak{g}$. Also, $\mathcal{O} = \mathcal{O}_{e_1} \oplus \mathcal{O}_{e_2}$ with e_1 and e_2 distinguished in $\mathfrak{g}_1$ and $\mathfrak{g}_2'$, respectively. Then $\mathfrak{p}_1 = \mathfrak{p}_{\mathcal{O}_{e_1}^{\mathfrak{g}_1}}$. On the other hand, let $\mathfrak{g}_2 \supset \mathfrak{g}_2'$ be the maximal Levi subalgebra such that its simple roots are disjoint from those of $\mathfrak{g}_1$ in $\check{\Delta}$. Consider the orbit $\mathcal{O}_{e_2}^{\mathfrak{l}_2}$ of e_2 in $\mathfrak{g}_2$, one has $\mathfrak{p}_2 = \mathfrak{p}_{\mathcal{O}_{e_2}^{\mathfrak{g}_2}}$. The above discussion readily applies to the case of our interest when $\mathcal{O}_{\mathbf{L}} = \mathcal{O}_{\rm reg}$ and thus e_1 and e_2 are regular in $\mathfrak{g}_1$ and $\mathfrak{g}_2'$, respectively.

Consider the cover of G of type B_r. We consider both the cases of $G = {\rm Spin}_{2r+1}$ and $G = {\rm SO}_{2r+1}$. For $\overline{G} = \overline{\rm Spin}_{2r+1}^{(n)}$ with the Brylinski-Deligne invariant ${\rm Inv}_{\rm BD}(\overline{\rm Spin}_{2r+1}^{(n)}) = 1$, one has in this case the exceptional character

$$\nu = \begin{cases} \rho/n, & \text{if } n \text{ is odd}, \\ 2\omega_r/n + \left(\displaystyle\sum_{1\leqslant i\leqslant r-1} \omega_i\right)/n = \rho(C_r)/n, & \text{if } n \text{ is even}. \end{cases}$$

Now if we write

$$r = na + b \quad \text{with } 0 \leqslant b < n,$$

then a direct computation gives that

- for odd $n = 2m+1$, we have

$$\Phi_\nu^\vee = \begin{cases} C_a \times (A_{2a-1})^{m-b} \times (A_{2a})^b, & \text{if } 0 \leqslant b \leqslant m, \\ C_{a+1} \times (A_{2a})^{2m-b+1} \times (A_{2a+1})^{b-m-1}, & \text{if } m+1 \leqslant b \leqslant 2m; \end{cases}$$

- for even $n = 2m$, we have

$$\Phi_\nu^\vee = \begin{cases} C_a \times C_a \times (A_{2a-1})^{m-1-b} \times (A_{2a})^b, & \text{if } 0 \leqslant b \leqslant m-1, \\ C_{a+1} \times C_a \times (A_{2a})^{2m-b-1} \times (A_{2a+1})^{b-m}, & \text{if } m \leqslant b \leqslant 2m-1. \end{cases}$$

By applying (4.6) and Theorem 4.4, we see that if $n = 2m+1$ is odd, then

$$\mathcal{O}_{\rm Spr}(j_{W_\nu}^W \varepsilon_\nu) = \mathcal{O}_B^{2r+1,n} = \begin{cases} (n^{2a}, 2b+1), & \text{if } 0 \leqslant b \leqslant m, \\ (n^{2a+1}, 2b+1-n), & \text{if } m+1 \leqslant b \leqslant 2m. \end{cases}$$

On the other hand, if $n = 2m$ is even, then

$$\mathcal{O}_{\rm Spr}(j_{W_\nu}^W \varepsilon_\nu) = \mathcal{O}_B^{2r+1,n} = \begin{cases} (n^{2a}, 2b+1), & \text{if } 0 \leqslant b \leqslant m-1, \\ (n^{2a}, n-1, 2b+1-n, 1), & \text{if } m \leqslant b \leqslant 2m-1. \end{cases}$$

For $\overline{\rm SO}_{2r+1}^{(n)}$, it depends on the parity of $n_{\alpha_1} = n/\gcd(n,2)$. In particular, the exceptional character is of the form

$$\nu = \begin{cases} \rho/n_{\alpha_1}, & \text{if } n_{\alpha_1} \text{ is odd}, \\ 2\omega_r/n_{\alpha_1} + \left(\displaystyle\sum_{1\leqslant i\leqslant r-1} \omega_i\right)/n_{\alpha_r} = \rho(C_r)/n_{\alpha_1}, & \text{if } n_{\alpha_1} \text{ is even}. \end{cases}$$

Thus, the root subsystem associated with an exceptional character of $\overline{\rm SO}_{2r+1}^{(n)}$ equals the root subsystem associated with that of $\overline{\rm Spin}_{2r+1}^{(n_{\alpha_1})}$. This gives us the column for ${\rm SO}_{2r+1}$ in Table 6.

Now consider $\overline{\rm Sp}_{2r}^{(n)}$ with ${\rm Inv}_{\rm BD}(\overline{\rm Sp}_{2r}^{(n)}) = 1$. If n is odd, then a detailed computation of $\mathcal{O}_{\rm Spr}(j_{W_\nu}^W \varepsilon_\nu)$ was already given in [27, §4]. For n even, the root subsystem Φ_ν is given in Remark 4.2 (see also [27, Remark 4.2]). Thus, a similar computation gives the orbit, which we tabulate in the column of ${\rm Sp}_{2r}$ in Table 6.

Table 6 $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$ for classical groups

	$\overline{\mathrm{GL}}_r^{(n)}$	$\overline{\mathrm{SO}}_{2r+1}^{(n)}$	$\overline{\mathrm{Sp}}_{2r}^{(n)}$	$\overline{\mathrm{SO}}_{2r}^{(n)}$
n is odd	$\mathcal{O}^{r,n}$	$\mathcal{O}_B^{2r+1,n}$	$\mathcal{O}_C^{2r,n}$	$\mathcal{O}_D^{2r,n}$
$n=2m$, m is even	$\mathcal{O}^{r,n}$	$\mathcal{O}_B^{2r+1,m}$	$\mathcal{O}_C^{2r,m}$	$(m+1,\mathcal{O}_D^{2r-m-1,m})$
$n=2k$, k is odd	$\mathcal{O}^{r,n}$	$\mathcal{O}_B^{2r+1,k}$	$(k+1,\mathcal{O}_C^{2r-k-1,k})$	$\mathcal{O}_D^{2r,k}$

Consider the group of types D_r, $r \geqslant 2$, where again we analyze both Spin_{2r} and SO_{2r}. Consider Spin_{2r} and its n-fold cover with $\mathrm{Inv}_{\mathrm{BD}}(\overline{\mathrm{Spin}}_{2r}^{(n)})=1$, the exceptional character is $\nu=\rho/n$. We write $r-1=na+b$ with $0\leqslant b<n$. If $n=2m+1$ is odd, then the root subsystem is

$$\Phi_\nu^\vee=\begin{cases} D_{a+1}\times(A_{2a-1})^{m-b}\times(A_{2a})^b, & \text{if } 0\leqslant b\leqslant m,\\ D_{a+1}\times(A_{2a})^{2m-b}\times(A_{2a+1})^{b-m}, & \text{if } m+1\leqslant b\leqslant 2m;\end{cases}$$

also if $n=2m$ is even, then we have

$$\Phi_\nu^\vee=\begin{cases} D_{a+1}\times D_a\times(A_{2a-1})^{m-1-b}\times(A_{2a})^b, & \text{if } 0\leqslant b\leqslant m-1,\\ D_{a+1}\times D_{a+1}\times(A_{2a})^{2m-b-1}\times(A_{2a+1})^{b-m}, & \text{if } m\leqslant b\leqslant 2m-1.\end{cases}$$

By applying Theorem 4.4, we see that if $n=2m+1$ is odd, then

$$\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W\varepsilon_\nu)=\mathcal{O}_D^{2r,n}=\begin{cases}(n^{2a},2b+1,1), & \text{if } 0\leqslant b\leqslant m,\\ (n^{2a+1},2b+2-n), & \text{if } m+1\leqslant b\leqslant 2m.\end{cases}$$

On the other hand, if $n=2m$ is even, then

$$\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W\varepsilon_\nu)=(n+1,\mathcal{O}_D^{2r-n-1,n})=\begin{cases}(n+1,n^{2a-2},n-1,2b+1,1), & \text{if } 0\leqslant b\leqslant m-1,\\ (n+1,n^{2a},2b+1-n), & \text{if } m\leqslant b\leqslant 2m-1.\end{cases}$$

The case of $\overline{\mathrm{SO}}_{2r}^{(n)}$ depends on the parity of $n_\alpha=n/\gcd(n,2)$, where α is any root. In particular, the exceptional character is of the form $\nu=\rho/n_\alpha$. Thus, the root subsystem associated with an exceptional character of $\overline{\mathrm{SO}}_{2r}^{(n)}$ equals the root subsystem associated with that of $\overline{\mathrm{Spin}}_{2r}^{(n_\alpha)}$. This gives us the column for SO_{2r} in Table 6.

Remark 4.5. In a recent work, Bai et al. [2] devised two algorithms to compute the nilpotent orbit associated with the highest weight modules for all classical Lie algebras, which partially extend the recipes and some relevant results given in [3, 4]. One such algorithm in [2] is the so-called "partition algorithm" and is based on Sommers' duality in [63]. In particular, one can check that Table 6 can be recovered by applying [2, § 1.2, Theorem] to an exceptional $\nu\in X\otimes\mathbf{R}$.

4.5 The orbit $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W\varepsilon_\nu)$ for exceptional groups

For exceptional groups, the computation follows from (4.6) and the tables in [63, § 9]. Thus, we have Table 7 for $\overline{G}_2^{(n)}$. For $\overline{F}_4^{(n)}$, we have Table 8. For $\overline{E}_6^{(n)}$, we have Table 9. For $\overline{E}_7^{(n)}$, we have Table 10. For $\overline{E}_8^{(n)}$, we have Table 11.

4.6 Compatibility and remarks

In view of the above tables and the criterion of quasi-admissibility and raisability discussed in Section 3, we have the following theorem.

Theorem 4.6. *Let $\overline{G}$ be any of the covering $\overline{\mathrm{GL}}_r^{(n)}$, $\overline{\mathrm{SO}}_{2r+1}^{(n)}$, $\overline{\mathrm{Sp}}_{2r}^{(n)}$, $\overline{\mathrm{SO}}_{2r}^{(n)}$, $\overline{G}_2^{(n)}$, $\overline{F}_4^{(n)}$ and $\overline{E}_r^{(n)}$, $6\leqslant r\leqslant 8$ discussed above. Assume that $\overline{G}$ is persistent. Consider the F-split orbit $\mathcal{O}_\Theta$ of type $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W\varepsilon_\nu)$ of G.*

Table 7 $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$ for $\overline{G}_2^{(n)}$

n	Φ_χ	$j_{W_\nu}^W \varepsilon_\nu$	$\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$	$\dim \mathcal{O}$
1	G_2	$\phi_{1,6}$	$\{0\}$	0
2	$\tilde{A}_1 + A_1$	$\phi_{2,2}$	$\tilde{A}_1$	8
3	$\tilde{A}_2$	$\phi''_{1,3}$	A_1	6
$4, 5, 6, 9$	$\tilde{A}_1$	$\phi_{2,1}$	$G_2(a_1)$	10
$7, 8$ or $\geqslant 10$	$\emptyset$	$\phi_{1,0}$	G_2	12

Table 8 $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$ for $\overline{F}_4^{(n)}$

n	Φ_χ	$j_{W_\nu}^W \varepsilon_\nu$	$\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$	$\dim \mathcal{O}$
1	F_4	$\phi_{1,24}$	$\{0\}$	0
2	C_4	$\phi''_{2,16}$	A_1	16
3	$\tilde{A}_2 + A_2$	$\phi_{6,6}$	$\tilde{A}_2 + A_1$	36
4	$A_3 + A_1$	$\phi''_{4,7}$	$A_2 + \tilde{A}_1$	34
$5, 6$	$\tilde{A}_2 + A_1$	$\phi_{12,4}$	$F_4(a_3)$	40
$7, 10$	$\tilde{A}_1 + A_1$	$\phi_{9,2}$	$F_4(a_2)$	44
8	$\tilde{A}_2$	$\phi''_{8,3}$	B_3	42
$9, 11, 12, 14, 16$	$\tilde{A}_1$	$\phi_{4,1}$	$F_4(a_1)$	46
$13, 15$ or $\geqslant 17$	$\emptyset$	$\phi_{1,0}$	F_4	48

Table 9 $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$ for $\overline{E}_6^{(n)}$

n	Φ_χ	$j_{W_\nu}^W \varepsilon_\nu$	$\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$	$\dim \mathcal{O}$
1	E_6	$\phi_{1,36}$	$\{0\}$	0
2	$A_5 + A_1$	$\phi_{15,16}$	$3A_1$	40
3	$3A_2$	$\phi_{10,9}$	$2A_2 + A_1$	54
4	$2A_2 + A_1$	$\phi_{80,7}$	$D_4(a_1)$	60
5	$A_2 + 2A_1$	$\phi_{60,5}$	$A_4 + A_1$	62
$6, 7$	$3A_1$	$\phi_{30,3}$	$E_6(a_3)$	66
8	$2A_1$	$\phi_{20,2}$	D_5	68
$9, 10, 11$	A_1	$\phi_{6,1}$	$E_6(a_1)$	70
$\geqslant 12$	$\emptyset$	$\phi_{1,0}$	E_6	72

Table 10 $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$ for $\overline{E}_7^{(n)}$

n	Φ_χ	$j_{W_\nu}^W \varepsilon_\nu$	$\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$	$\dim \mathcal{O}$
1	E_7	$\phi_{1,63}$	$\{0\}$	0
2	A_7	$\phi_{15,28}$	$4A_1$	70
3	$A_5 + A_2$	$\phi_{70,18}$	$2A_2 + A_1$	90
4	$A_4 + A_2$	$\phi_{210,13}$	$A_3 + A_2 + A_1$	100
5	$A_3 + A_2 + A_1$	$\phi_{210,10}$	$A_4 + A_2$	106
6	$A_2 + A_2 + A_1$	$\phi_{315,7}$	$E_7(a_5)$	112
7	$A_2 + 3A_1$	$\phi_{105,6}$	A_6	114
8	$A_2 + 2A_1$	$\phi_{189,5}$	$E_7(a_4)$	116
9	$(4A_1)''$	$\phi_{120,4}$	$E_6(a_1)$	118
$10, 11$	$(3A_1)'$	$\phi_{56,3}$	$E_7(a_3)$	120
$12, 13$	$2A_1$	$\phi_{27,2}$	$E_7(a_2)$	122
$14, 15, 16, 17$	A_1	$\phi_{7,1}$	$E_7(a_1)$	124
$\geqslant 18$	$\emptyset$	$\phi_{1,0}$	E_7	126

Table 11 $\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$ for $\overline{E}_8^{(n)}$

n	Φ_χ	$j_{W_\nu}^W \varepsilon_\nu$	$\mathcal{O}_{\mathrm{Spr}}(j_{W_\nu}^W \varepsilon_\nu)$	$\dim \mathcal{O}$
1	E_8	$\phi_{1,120}$	$\{0\}$	0
2	D_8	$\phi_{50,56}$	$4A_1$	128
3	A_8	$\phi_{175,36}$	$2A_2+2A_1$	168
4	D_5+A_3	$\phi_{840,26}$	$2A_3$	188
5	A_4+A_4	$\phi_{420,20}$	A_4+A_3	200
6	A_4+A_3	$\phi_{4480,16}$	$E_8(a_7)$	208
7	$A_4+A_2+A_1$	$\phi_{2835,14}$	A_6+A_1	212
8	$A_3+A_2+2A_1$	$\phi_{1400,11}$	A_7	218
9	$A_3+A_2+A_1$	$\phi_{2240,10}$	$E_8(b_6)$	220
$10, 11$	$2A_2+2A_2$	$\phi_{1400,8}$	$E_8(a_6)$	224
$12, 13$	A_2+3A_1	$\phi_{700,6}$	$E_8(a_5)$	228
14	A_2+2A_1	$\phi_{560,5}$	$E_8(b_4)$	230
$15, 16, 17$	$(4A_1)'$	$\phi_{210,4}$	$E_8(a_4)$	232
$18, 19$	$3A_1$	$\phi_{112,3}$	$E_8(a_3)$	234
$20, 21, 22, 23$	$2A_1$	$\phi_{35,2}$	$E_8(a_2)$	236
$24, 25, 26, 27, 28, 29$	A_1	$\phi_{8,1}$	$E_8(a_1)$	238
$\geqslant 30$	$\emptyset$	$\phi_{1,0}$	E_8	240

(i) *The orbit $\mathcal{O}_\Theta$ is quasi-admissible and non-raisable.*

(ii) *If the orbit $\mathcal{O}_\Theta$ is the regular orbit of a Levi subgroup of G, then it supports certain generalized Whittaker models of the theta representation $\Theta(\overline{G}, \nu)$.*

Proof. First, (i) follows from comparing Tables 6–11 with Theorems 3.1, 3.3 and 3.4 and Tables 1–5.

For (ii), recall from [32] that for any $\pi \in \mathrm{Irr}_{\mathrm{gen}}(\overline{G})$ and any Whittaker pair (S, u) of $\mathcal{O}_u$, there is a degenerate Whittaker model $\pi_{S,u}$ such that one has a $\overline{G}$-equivariant surjection $\pi_{N_\mathcal{O},\psi_\mathcal{O}} \twoheadrightarrow \pi_{S,u}$ (see [32, Theorem A]). If $\mathcal{O}_u$ is the split regular orbit of a Levi subgroup L, then one can pick (S, u) such that $\pi_{S,u}$ is equal to the semi-Whittaker models of π, i.e., the Whittaker model of the Jacquet model $J_U(\pi)$ with respect to the unipotent radical U of the parabolic subgroup $P = LU$.

Applying the above to $\Theta(\overline{G}, \nu)$, by the periodicity of theta representations (see the argument in [9, Theorem 2.3], [14] or [46], which actually applies to general $\overline{G}$), one has that $J_U(\Theta(\overline{G}, \nu))$ is a theta representation on the Levi subgroup $\overline{L}$. Thus, it suffices to show that every theta representation $\Theta(\overline{L}_\mathcal{O})$ on the Levi subgroup $\overline{L}_\mathcal{O}$ associated with such $\mathcal{O}_\Theta$ is generic. It follows from [25, Proposition 6.2] that $\dim \mathrm{Wh}_\psi(\Theta(\overline{L}_\mathcal{O})) = \langle \varepsilon_{W(L_\mathcal{O})}, \sigma^{\mathscr{X}} \rangle_{W(L_\mathcal{O})}$, where $W(L_\mathcal{O})$ denotes the Weyl group of $L_\mathcal{O}$. Moreover, an explicit numerical criterion on the non-vanishing of $\dim \mathrm{Wh}_\psi(\Theta(\overline{L}_\mathcal{O}))$ is given in [27, §3]. The result then follows from a direct check by using results in [27, §3] and Tables 6–11. We illustrate this by considering $\overline{\mathrm{Sp}}_{2r}^{(n)}$ and $\overline{E}_8^{(n)}$ as examples.

The cover $\overline{\mathrm{Sp}}_{2r}^{(n)}$ is persistent if and only if n is odd or $n = 2m$ with m even. If n is odd, then by Example 3.5, we have (for $2r = an + b$ and $0 \leqslant b < n$) the following:

• If a and b are both even, then $\mathcal{O}_\Theta = (n^a b)$ is the principal orbit of a Levi with $L_\mathcal{O} = \mathrm{Sp}_b \times \prod_{j=1}^{a/2} \mathrm{GL}_n$. In this case, $\Theta(\overline{L}_\mathcal{O})$ is generic by [24, Proposition 5.1] or [27, Lemma 3.3].

• If a and b are both odd, then $\mathcal{O}_\Theta$ is a principal orbit of a Levi only when $b = n-2$, and in this case, $L_\mathcal{O} = \mathrm{GL}_{n-1} \times \prod_{j=1}^{(a-1)/2} \mathrm{GL}_n$. Again, $\Theta(\overline{L}_\mathcal{O})$ is generic in this case.

Now, if $n = 2m$ with m even, then the orbit $\mathcal{O}_\Theta = (m^a b)$ with $2r = am + b$ is always the principal orbit of the Levi subgroup $L_\mathcal{O} = \mathrm{Sp}_b \times \prod_{i=1}^{a/2} \mathrm{GL}_m$. In this case, $\Theta(\overline{L}_\mathcal{O})$ is generic by results in [24, 27] as well.

For $\overline{E}_8^{(n)}$, it follows that the non-trivial orbits $\mathcal{O}_\Theta$, which are regular orbits of Levi subgroups, are $4A_1$, $2A_2+2A_1$, $2A_3$, A_4+A_3, A_6+A_1 and A_7, which are associated with $n = 2, 3, 4, 5, 7, 8$, respectively. It follows that $\Theta(\overline{L}_\mathcal{O})$ is always generic in this case.

Other groups can be checked in the same way, and this completes the proof. □

5 The coefficient $c_{\mathcal{O}}$ for $\Theta(\overline{\mathrm{GL}}_r^{(n)})$

In this section, we verify the equality (4.4) in Conjecture 4.1 regarding the leading coefficient $c_{\mathcal{O}}$ for the theta representation of covers of GL_r. Note that the equality (4.3) for $\overline{\mathrm{GL}}_r^{(n)}$ is due to Savin[2)] for the so-called Savin's coverings and Cai [14, Theorem 1.2] for Kazhdan-Patterson coverings [49] (for a comparison of the two families of coverings, see [28, § 5.1]).

Theorem 5.1. *Consider a Kazhdan-Patterson cover $\overline{\mathrm{GL}}_r^{(n)}$. Assume $p \nmid n$. Then for every unramified theta representation $\Theta(\pi^\dagger, \nu)$, one has $c_{\mathcal{O}} = \langle j_{W_\nu}^W(\varepsilon_{W_\nu}), \varepsilon_W \otimes \sigma^{\mathscr{X}} \rangle_W$, where $\mathcal{O} = (n^a b)$ is the unique orbit in $\mathcal{N}_{\mathrm{tr}}^{\max}(\Theta(\pi^\dagger, \nu))$ with $r = na + b, 0 \leqslant b < n$.*

Proof. We first note that every $\overline{\mathrm{GL}}_r$ is saturated and thus persistent. For any partition $\mu \in \mathscr{P}(r)$ of r, we denote by $M_\mu \subset \mathrm{GL}_r$ the standard Levi subgroup. The Weyl group of M_μ is denoted by $W_\mu \subset W$. We write $\lambda := (n^a b) \in \mathscr{P}(r)$. Since $\mathcal{N}_{\mathrm{tr}}^{\max}(\Theta(\pi^\dagger, \nu)) = \{\lambda\}$, we see that [59] and [33, Theorem 1.5] together with the periodicity of theta representations (see [14, Proposition 3.21]) give the first equality in $c_{\mathcal{O}} = \dim \mathrm{Wh}_\psi(\Theta(\overline{M}_\lambda^{(n)})) = \langle \varepsilon_{W_\lambda}, \sigma^{\mathscr{X}} \rangle_{W_\lambda} = \langle \mathrm{Ind}_{W_\lambda}^W(\varepsilon_{W_\lambda}), \sigma^{\mathscr{X}} \rangle_W$, where the theta representation $\overline{M}_\lambda^{(n)}$ is associated with $\pi^\dagger$ and ν, and the second equality follows from [25, Proposition 6.2]. On the other hand, the Weyl subgroup W_ν is a parabolic Weyl subgroup associated with $\lambda^\top = ((a+1)^b a^{n-b})$. Thus,

$$\begin{aligned} \langle j_{W_\nu}^W(\varepsilon_{W_\nu}), \varepsilon_W \otimes \sigma^{\mathscr{X}} \rangle_W &= \langle j_{W_{\lambda^\top}}^W(\varepsilon_{W_{\lambda^\top}}), \varepsilon_W \otimes \sigma^{\mathscr{X}} \rangle_W \\ &= \langle \varepsilon_W \otimes j_{W_{\lambda^\top}}^W(\varepsilon_{W_{\lambda^\top}}), \sigma^{\mathscr{X}} \rangle_W = \langle j_{W_\lambda}^W(\varepsilon_{W_\lambda}), \sigma^{\mathscr{X}} \rangle_W, \end{aligned}$$

where the last equality follows from [29, Corollary 5.4.9].

We have

$$\mathrm{Ind}_{W_\lambda}^W(\varepsilon_{W_\lambda}) = j_{W_\lambda}^W(\varepsilon_{W_\lambda}) + \sum_{\substack{\lambda \leqslant \mu \\ \mu \neq \lambda}} j_{W_\mu}^W(\varepsilon_{W_\mu})$$

(see [29, Theorem 5.4.7]). Also, for any $\mu > \lambda$,

$$\langle j_{W_\mu}^W(\varepsilon_{W_\mu}), \sigma^{\mathscr{X}} \rangle_W \leqslant \langle \mathrm{Ind}_{W_\mu}^W(\varepsilon_{W_\mu}), \sigma^{\mathscr{X}} \rangle_W = \langle \varepsilon_{W_\mu}, \sigma^{\mathscr{X}} \rangle_{W_\mu} = \dim \mathrm{Wh}_\psi(\Theta(\overline{M}_\mu)),$$

where the last equality follows from [25]. Since $\mu > \lambda$, there is a component in the partition of μ strictly greater than n, and thus one has $\dim \mathrm{Wh}_\psi(\Theta(\overline{M}_\mu)) = 0$. All the above together give $c_{\mathcal{O}} = \langle j_{W_\nu}^W(\varepsilon_{W_\nu}), \varepsilon_W \otimes \sigma^{\mathscr{X}} \rangle_W$. This completes the proof. □

Note that our restriction to the Kazhdan-Patterson cover [49] of GL_r is more for convenience only. Results in Theorem 5.1 and other parts of the paper could be extended to general Brylinski-Deligne covers of GL_r without much difficulty, where we anticipate that the only essential alternation in the statement is to replace n by n_α.

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