INTEGRAL EQUATIONS WITH NORMAL KERNELS*

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I. INTRODUCTION

By a normal kernel, we mean a kernel $K(x, y)$ satisfying the condition

$$KK^*[x, y], = K^*K[x, y], \tag{1.1}$$

i.e.,

$$\int_a^b K(x, \xi)\, \overline{K(y, \xi)}\, d\xi = \int_a^b \overline{K(\xi, x)}\, K(\xi, y)\, d\xi .$$

Evidently real symmetric kernels, real skew-symmetric kernels, Hermitian kernels, skew-Hermitian kernels, etc. are normal kernels. In this paper, we discuss the properties and solutions of the integral equations with normal kernels, especially the properties of the singular values (E. Schmidt's characteristic values), the characteristic values of such kernels and their relations. The main results are given by the following lemmas 2.1, 2.2; theorems 2.1, 2.2, 2.3, 4.1, 5.1, 6.2, 6.3, 6.4 and 7.1. Our starting points are chiefly based upon the properties of algebraic kernels, i.e., kernels of the form

$$\sum_{i=1}^{m} X_i(x)\, Y_i(y)\,,$$

where each $X_i(x)$ is a function of x alone, and each $Y_i(y)$ is a function of y alone, and m is a finite integer. We shall show that all normal kernels and their iterated kernels and the solutions of the integral equations with normal kernels are expressible in terms of the characteristic functions of such kernels. As applications of these results, we give new proofs to some classical theorems, for example: (i) the existence theorem of characteristic values of normal kernels, and (ii) the singular points of the resolvent kernel of any L^2 normal

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kernel are all simple poles, etc. In what follows, we shall use the notations

$$(f, g) = \int_a^b f(s)\,\overline{g(s)}\,ds\,, \qquad \|f\| = (f, f)^{\frac{1}{2}}\,,$$

$$AB[x, y] = \int_a^b A(x, \xi)\,B(\xi, y)\,d\xi\,, \qquad ABC[x, y] = AB \cdot C[x, y]\,,$$

$$A^n[x, y] = A^{n-1}A[x, y]\,, \qquad (n = 1, 2, 3, \cdots)$$

$$K f[x] = \int_a^b K(x, \xi)\,f(\xi)\,d\xi\,, \qquad K^* f[x] = \int_a^b \overline{K(\xi, x)}\,f(\xi)\,d\xi\,,$$

$$K' f[x] = \int_a^b K(\xi, x)\,f(\xi)\,d\xi\,, \qquad \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

We shall use $K[\varphi_h, \psi_h; \lambda_h]$ to denote a kernel $K(x, y)$ of which the set of singular values is $\{\lambda_h\}$ $(h=1, 2, \ldots)$ and the complete orthonormal system of pairs of adjoint singular functions is $\{\varphi_h(x), \psi_h(y)\}$, so that

$$\varphi_h(x) = \lambda_h K\psi_h[x]\,, \qquad \psi_h(x) = \lambda_h K^* \varphi_h[x]\,. \qquad (h = 1, 2, \cdots)$$

The Lebesgue integral is used throughout, and equalities between functions will generally be understood as holding almost everywhere.

II. Properties of the Singular Values of Normal Kernels

Lemma 2. 1. *Suppose* $K(x, y) = K[\varphi_h, \psi_h; \lambda_h]$ *is a* L^2 *kernel so that*

$$\int_a^b \int_a^b |K(x, y)|^2\,dx\,dy \quad < +\infty$$

and is normal, then

$$K(x, y) = K[\psi_h, \lambda_h K^* \psi_h; \lambda_h]\,.$$

Proof. (i) Let $\lambda_h K^* \psi_h[x] = \chi_h(x)$, then by (1.1), we have

$$\lambda_h K \chi_h[x] = \lambda_h^2 KK^* \psi_h[x] = \lambda_h^2 K^* K \psi_h[x] = \psi_h(x)\,,$$

i.e., $\psi_h(x)$, $\chi_h(y)$ is a pair of adjoint singular functions of $K(x, y)$.

(ii) By hypothesis $(\psi_i, \psi_j) = \delta_{ij}$, therefore we have

$$\begin{aligned}(\chi_i, \chi_h) &= \lambda_i \lambda_h (K^* \psi_i, K^* \psi_h) = \lambda_i \lambda_h (\psi_i, KK^* \psi_h) \\ &= \lambda_i \lambda_h (\psi_i, K^* K \psi_h) = \lambda_i \lambda_h (\psi_i, \psi_h / \lambda_h^2) \\ &= \lambda_i \delta_{ih} / \lambda_h = \delta_{ih},\end{aligned}$$

i.e., the system $\{\psi_h(x), \chi_h(y)\}$ $(h = 1, 2, \cdots)$ is orthonormal.

(iii) By hypothesis the set $\{\psi_h(x)\}$ is a complete orthonormal system of characteristic functions of the kernel (1.1). Therefore assume λ_h be a singular value of rank h_ρ of the kernel $K(x,y)$, i.e., in the complete orthonormal system of adjoint singular functions $\{\varphi_h(x), \psi_h(y)\}$, there are h_ρ and only h_ρ pairs of adjoint singular functions

$$\varphi_{hi}(x), \psi_{hi}(y) \qquad (i = 1, 2, \cdots, h_\rho)$$

belonging to the same singular value λ_h such that

$$(\varphi_{hi}, \varphi_{hj}) = \delta_{ij}, \quad (\psi_{hi}, \psi_{hj}) = \delta_{ij}, \quad (i, j=1, 2, \cdots, h_\rho) \tag{2.1}$$

then for any pair of adjoint singular functions $\varphi(x),\psi(y)$ of $K(x,y)$ belonging to the singular value λ_h, there exists a finite linear combination

$$\varphi(x) = \sum_{i=1}^{h_\rho} c_i \psi_{hi}(x).$$

Whence

$$\begin{aligned}\psi(y) &= \lambda_h K^* \varphi[y] = \lambda_h K^* \left\{\sum_i c_i \psi_{hi}\right\}[y] = \lambda_h \sum_i c_i K^* \psi_{hi}[y] \\ &= \lambda_h \sum_i c_i \chi_{hi}'(y) / \lambda_h = \sum_i c_i \chi_{hi}(y).\end{aligned}$$

It means that the system $\psi_h(x)$, $\chi_h(y)$ is complete. Thus our lemma is established.

Lemma 2. 2. *A L^2 kernel $K(x,y)$ of which the n-th iterated kernel $K^n[x,y]$ may have a canonical decomposition*[1] *in n equal factors for any integer n, is necessarily a normal kernel, and vice versa.*

1) If $K(x,y)=AB[x,y]$ where $A = A[\varphi_h, \psi_h; \alpha_h]$, $B = B[\psi_h, \chi_h; \beta_h]$, then we say that $K(x,y)$ has a decomposition in 2 factors. Similarly, for the definition of canonical decomposition in n factors. Cf. [1].

Proof. Suppose $K(x,y)$ is a L^2 normal kernel, then by lemma 2.1, we have

$$K^n[x,y] = \prod_{i=1}^{n} K[\varphi_h^{(i)}, \psi_h^{(i)}; \lambda_h] \ (\varphi_h^{(i+1)}(x) = \psi_h^{(i)}(x);\ i=1,2,\cdots,n-1),$$

i.e., $K^n[x,y]$ has a canonical decomposition with n equal factors.

Conversely, suppose $K^n[x,y]$ has a canonical decomposition in n equal factors for any integer n, then in particular $K^2[x,y]$ has a canonical decomposition in two equal factors

$$K^2[x,y] = K[\varphi_h, \psi_h; \lambda_h] \cdot K[\psi_h, \chi_h; \lambda_h]. \qquad (2.2)$$

From $K(x,y) = K[\varphi_h, \psi_h; \lambda_h]$, we have $K^*K[x,y] = \sum_{h=1}^{\infty} \psi_h(x)\, \psi_h(y)/\lambda_h^2$; and from $K(x,y) = K[\psi_h, \chi_h; \lambda_h]$, we have $KK^*[x,y] = \sum_{h=1}^{\infty} \psi_h(x)\, \psi_h(y)/\lambda_h^2$. Therefore we have $KK^*[x,y] = K^*K[x,y]$, i.e. $K(x,y)$ is normal.

Corollary. *A L^2 kernel is normal if and only if the second iterated kernel $K^2[x,y]$ has a canonical decomposition in two equal factors.*

Theorem 2. 1. *If $K(x,y)$ is a L^2 normal kernel of which the set of singular values is $\{\lambda_h\}$, then the set of singular values of the n-th iterated kernel $K^n[x,y]$ is $\{\lambda_h^n\}$.*

Proof. Suppose $K(x,y)$ is a L^2 normal kernel, then by lemma 2.1, we have[3], for any L^2 function $f(x)$,

$$Kf[x] \sim \sum_{h=1}^{\infty} \varphi_h(x)\,(f, \psi_h)/\lambda_h = \sum_{h=1}^{\infty} \psi_h(x)\,(f, \lambda_h K^* \psi_h)/\lambda_h^2,$$

where the symbol $\sim$ denotes the convergence in the mean square. From the last equation we have

$$K^2[x,y] = K^2[\varphi_h, \lambda_h K^* \psi_h; \lambda_h^2] = K[\varphi_h, \psi_h; \lambda_h] \cdot K[\psi_h, \lambda_h K^* \psi_h; \lambda_h].$$

By induction we get

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2) This theorem is an analogue of a known theorem that if λ is a characteristic value of a L^2 kernel $K(x,y)$ then λ^n is a characteristic value of $K^n[x,y]$. See [5] p. 119.

3) See [3] pp. 266-267.

$$K^n[x,y] = K^n[\varphi_h, \lambda_h^{n-1} K^{*n-1} \psi_h; \lambda_h^n], \qquad (n=1,2,3,\cdots) \tag{2.3}$$

and the theorem follows.

Theorem 2. 2. *Let* λ_h *be a singular value of rank* h_ρ *of the kernel* $K(x,y)$ *and*

$$\varphi_{hi}(x), \psi_{hi}(y) \qquad (i=1,2,\cdots,h_\rho)$$

be the h_ρ *pairs of adjoint singular functions of* $K(x,y)$ *belonging to the same singular value* λ_h *such that*

$$(\varphi_{hi}, \varphi_{hj}) = \delta_{ij}, \qquad (\psi_{hi}, \psi_{hj}) = \delta_{ij}, \qquad (i,j=1,2,\cdots,h_\rho) \tag{2.1}$$

then a necessary and sufficient condition for $K(x,y)$ *to be normal is that for each h, we should have*

$$\varphi_{hi}(x) = \sum_{j=1}^{h_\rho} a_{h,ij}\, \psi_{hj}(x), \qquad (i=1,2,\cdots,h_\rho) \tag{2.4}$$

where the $a_{h,ij}$*'s are constants such that the matrix*

$$\Delta = (a_{h,ij}) \qquad (i,j=1,2,\cdots,h_\rho) \tag{2.5}$$

is unitary.

Proof. If $K(x,y)$ is a L^2 normal kernel, then each of the two sets

$$\varphi_{h1}(x), \varphi_{h2}(x), \cdots, \varphi_{hh_\rho}(x); \qquad \psi_{h1}(x), \psi_{h2}(x), \cdots, \psi_{hh_\rho}(x)$$

represents a fundamental set of linearly independent characteristic functions of the same kernel $KK^*[x,y]=K^*K[x,y]$ and belonging to the same characteristic value λ_h^2. Whence each $\varphi_{hi}(x)$ must be a linear form of the $\psi_{hi}(x)$'s as shown in (2.4). Then by the conditions (2.1), we have

$$\sum_{j=1}^{h_\rho} |a_{h,ij}|^2 = 1, \ \sum_{j=1}^{h_\rho} a_{h,ij}\, \bar{a}_{h,kj} = 0 \qquad (i \neq k;\ i,k = 1,2,\cdots,h_\rho). \tag{2.6}$$

But it is the condition that the matrix Δ is unitary.

Conversely, if (2.4) and (2.6) are true for $h=1,2,\ldots$, then each of the sets

$$\{\varphi_{hi}(x)\},\ \{\psi_{hi}(x)\} \qquad (h=1, 2, \cdots; i=1, 2, \cdots, h_\rho)$$

represents a complete orthonormal system of characteristic functions of the same kernel

$$\sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho} \varphi_{hi}(x)\,\overline{\varphi_{hi}(y)}\,/\,\lambda_h^2 = \sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho} \psi_{hi}(x)\,\overline{\psi_{hi}(y)}\,/\,\lambda_h^2, \tag{2.7}$$

i.e. $KK^*[x,y]=K^*K[x,y]$. Thus the proof is completed.

Theorem 2. 3. *If $K(x,y)$ is a L^2 normal kernel, then*

$$\psi_{hi}(x) = \sum_{j=1}^{h_\rho} \bar{a}_{h,ji}\,\varphi_{hj}(x), \tag{2.8}$$

$$K^n[x,y] = K^n[\varphi_{hi}(x),\ \sum_{j=1}^{h_\rho} \bar{a}^{(n)}_{h,ji}\,\varphi_{hj}(y);\ \lambda_h^n], \tag{2.9}$$

$$K^n[x,y] = K^n\left[\sum_{j=1}^{h_\rho} a^{(n)}_{h,ij}\,\psi_{hj}(x),\ \psi_{hi}(y);\ \lambda_h^n\right], \tag{2.9^1}$$

where

$$a^{(1)}_{h,ij} = a_{h,ij},\ (a^{(n)}_{h,ij}) = (a_{h,ij})^n = (\bar{a}_{h,ji})^{-n} = \overline{\Delta}'^{-n}.$$

Proof. Since $\Delta^{-1}=\overline{\Delta}'$, where Δ' and $\overline{\Delta}$ denote respectively the transpose and conjugate of Δ. Therefore from (2.4), we get (2.8). Whence we get

$$\lambda_h K^*\psi_{hi}[y] = \lambda_h \sum_{j=1}^{h_\rho} \bar{a}_{h,ji} K^*\varphi_{hj}[y] = \sum_{j=1}^{h_\rho} \bar{a}_{h,ji}\,\psi_{hj}(y)$$

$$= \sum_{j=1}^{h_\rho} \bar{a}_{h,ji} \sum_{t=1}^{h_\rho} \bar{a}_{h,tj}\,\varphi_{ht}(y) = \sum_{t=1}^{h_\rho} \bar{a}^{(2)}_{h,ti}\,\varphi_{ht}(y).$$

By induction, we have

$$\lambda_h^{n-1} K^{*n-1}\psi_{hi}[y] = \sum_{j=1}^{h_\rho} \bar{a}^{(n)}_{h,ji}\,\varphi_{hj}(y).$$

Substituting this result into (2.3), we get (2.9). Further, by lemma 2.1,

$$K(x,y) = K[\lambda_h^{n-2} K^{*n-2}\psi_{hi}[x],\ \lambda_h^{n-1} K^{*n-1}\psi_{hi}[y];\ \lambda_h], \qquad (n=2, 3, \cdots)$$

therefore we may consider $\sum_{j=1}^{h_\rho} \bar{a}_{h,ji}^{(n)} \varphi_{hj}(x)$ as $\psi_{hi}(x)$ and substituting $\varphi_{hi}(x)$ by $\sum_{j=1}^{h_\rho} a_{h,ij}^{(n)} \psi_{hj}(x)$ in (2.9). Then we get (2.9′) and the proof is completed.

III. Existence Theorem of Characteristic Values of Normal Kernels

Theorem 3. 1. *Any L^2 normal kernel $K(x,y)$ has at least one characteristic value.*

Proof. Let

$$K_h(x,y) = \sum_{i=1}^{h_\rho} \varphi_{hi}(x) \left\{ \sum_{j=1}^{h_\rho} a_{h,ji}\, \bar{\varphi}_{hj}(y) \right\} / \lambda_h, \qquad (h=1,2,3,\cdots) \tag{3.1}$$

then

$$K(x,y) \sim \sum_{h=1}^{\infty} K_h(x,y).$$

Evidently

$$K_i K_j\,[x,y] = 0 \quad (i \neq j), \quad \| K_h \| = \frac{h_\rho}{\lambda_h^2},$$

and

$$\| K \|^2 = \sum_{h=1}^{\infty} \| K_h \|^2 = \sum_{h=1}^{\infty} h_\rho / \lambda_h^2 \qquad < +\infty,$$

where

$$\| K \|^2 = \int_a^b \int_a^b |K(x,y)|^2\, dx\, dy, \quad \| K_h \|^2 = \int_a^b \int_a^b |K_h(x,y)|^2\, dx\, dy. \quad (h=1,2,\cdots)$$

Whence, by a known result[2], we have

$$D_K^*(\lambda) = \prod_{h=1}^{\infty} D_{K_h}^*(\lambda), \tag{3.2}$$

where $D_K^*(\lambda)$ denotes the generalized Fredholm determinant in Carleman's sense of the kernel $K(x,y)$. Therefore in order to prove our theorem, it is sufficient to prove that there is at least one characteristic value of the kernel $K_h(x,y)$, where h denotes a positive integer. Now, since $K_h(x,y)$ is a L^2

algebraic kernel, its Fredholm determinant $D_{K_h}(\lambda)$ exist, and is identical with $D^*_{K_h}(\lambda)$, i.e.

$$D_{K_h}(\lambda) = D^*_{K_h}(\lambda) = \begin{vmatrix} 1-\lambda\alpha_{11}, & -\lambda\alpha_{12}, & \cdots, & -\lambda\alpha_{1h_\rho} \\ -\lambda\alpha_{21}, & 1-\lambda\alpha_{22}, & \cdots, & -\lambda\alpha_{2h_\rho} \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda\alpha_{h_\rho 1}, & -\lambda\alpha_{h_\rho 2}, & \cdots, & 1-\lambda\alpha_{h_\rho h_\rho} \end{vmatrix}, \quad (3.3)$$

where

$$\alpha_{il} = (\varphi_{hl}, \sum_{j=1}^{h_\rho} \bar{a}_{h,ji}\,\varphi_{hj}/\lambda_h) = a_{h,li}/\lambda_h = (\varphi_{hl}, \psi_{hi})/\lambda_h. \quad (3.4)$$

By (2.6), each α_{ij} is finite in value. Since the matrix (2.5) is unitary, therefore the coefficient of the highest term of $D_{K_h}(\lambda)$ is

$$\{(-1)^{h_\rho}/\lambda_h^{h_\rho}\} \operatorname*{Det}_{l,i} |a_{h,li}| = \left\{\frac{-1}{\lambda_h}\right\}^{h_\rho} \varepsilon \neq 0. \qquad (|\varepsilon| = 1)$$

And since the constant term of $D_{K_h}(\lambda)$ is 1, therefore $D_{K_h}(\lambda)$ has h_ρ roots, which are the characteristic values of $K_h(x, y)$. Thus our theorem follows.

IV. Relations Between the Characteristic Values and Singular Values of Normal Kernels

Theorem 4. 1. *For any L^2 normal kernel $K(x, y)$, there exists a one to one correspondence between the set of characteristic values $\{\mu_h\}$ and the set of singular values $\{\lambda_h\}$ each arranged in order of non-decreasing absolute value such that*

$$|\mu_h| = \lambda_h. \qquad (h = 1, 2, 3, \cdots; |\mu_1| \leqslant |\mu_2| \leqslant \cdots; 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots)$$

Proof. Denoting the characteristic values of $K_h(x, y)$ by $\mu_{hi}(i = 1, 2, \cdots, h_\rho)$, and substituting λ_h/μ for λ in the determinant (3.3), then it is easy to see that the roots of the equation

$$\begin{vmatrix} a_{h,11}-\mu, & a_{h,21}, & \cdots, & a_{h,h_\rho 1} \\ a_{h,12} & a_{h,22}-\mu, & \cdots, & a_{h,h_\rho 2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{h,1h_\rho}, & a_{h,2h_\rho}, & \cdots, & a_{h,h_\rho h_\rho}-\mu \end{vmatrix} = 0 \quad (4.1)$$

are $\lambda_h/\mu_{hi}(i = 1, 2, \cdots, h_\rho)$. It means that $\lambda_h/\mu_{hi}(i = 1, 2, \cdots, h_\rho)$ are the latent roots of the unitary matrix $(a_{h,ij})$. But it is known[5] that all the latent roots of a unitary matrix are of modulus 1, whence we get $|\lambda_h/\mu_{hi}|=1$, $(i=1, 2, \cdots, h_\rho)$ or $\lambda_h = |\mu_{hi}|$, $(i = 1, 2, \cdots, h_\rho)$ as we wished to prove.

V. Relations Between Characteristic Functions and Adjoint Singular Functions of the Normal Kernels

Theorem 5. 1. *Suppose $K(x,y)$ be a L^2 normal kernel, and*

$$K(x, y) = K\left[\varphi_{hi}(x), \sum_{j=1}^{h_\rho} \bar{a}_{h,ji}\, \varphi_{hj}(y)\,; \lambda_h\right] = K\left[\sum_{j=1}^{h_\rho} a_{h,ij}\, \psi_{hj}(x), \psi_{hi}(y)\,; \lambda_h\right],$$

$$(h=1, 2, \cdots; \ i=1, 2, \cdots, h_\rho)$$

then (i) *The rank of each characteristic value of $K(x,y)$ is equal to its multiplicity;*

(ii) *Corresponding to the characteristic values $\mu_{hi}(i = 1, 2, \cdots, h_\rho)$, there exist h_ρ linearly independent characteristic functions $u_{hi}(x)$ $(i = 1, 2, \cdots, h_\rho)$ and h_ρ linearly independent transpose characteristic functions $v_{hi}(x)$ $(i = 1, 2, \cdots, h_\rho)$ so that*

$$u_{hi}(x) = \mu_{hi}\, K\, u_{hi}[x]\,, \qquad v_{hi}(x) = \mu_{hi}\, K'\, v_{hi}[x]\,; \tag{5.1}$$

(iii) *Each of the $u_{hi}(x)$ and each of the $v_{hi}(x)$ is a linear combination of the $\varphi_{hi}(x)$'s, and therefore is also a linear combination of the $\psi_{hi}(x)$'s;*

(iv) *We can choose suitable coefficients of these linear combinations, so that the characteristic functions $u_{hi}(x)$'s and the transpose characteristic functions $v_{hi}(x)$'s satisfy the relations*

$$u_{hi}(x) = \bar{v}_{hi}(x)\,. \qquad (i=1, 2, \cdots, h_\rho) \tag{5.2}$$

Proof. Since

$$(\varphi_{hl}, \psi_{hi}) = (\lambda_h\, K\, \psi_{hl}, \psi_{hi}) = \lambda_h\, J(\bar{\psi}_{hi}, \psi_{hl})\,,$$

where

$$J(\bar{\psi}_{hi}, \psi_{hl}) = \int_a^b \int_a^b K(x, y)\, \bar{\psi}_{hi}(x)\, \psi_{hl}(y)\, dx\, dy\,,$$

therefore the determinant (3.3) can be written in the following form

[5] See [4] p. 107, ex. 3.

$$D_{K_h}(\lambda) = \begin{vmatrix} 1-\lambda J(\bar{\psi}_{h1}, \psi_{h1}), & -\lambda J(\bar{\psi}_{h1}, \psi_{h2}), & \cdots, & -\lambda J(\bar{\psi}_{h1}, \psi_{hh_\rho}) \\ -\lambda J(\bar{\psi}_{h2}, \psi_{h1}), & 1-\lambda J(\bar{\psi}_{h2}, \psi_{h2}), & \cdots, & -\lambda J(\bar{\psi}_{h2}, \psi_{hh_\rho}) \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda J(\bar{\psi}_{hh_\rho}, \psi_{h1}), & -\lambda J(\bar{\psi}_{hh_\rho}, \psi_{h2}), & \cdots, & 1-\lambda J(\bar{\psi}_{hh_\rho}, \psi_{hh_\rho}) \end{vmatrix}. \quad (3.3^*)$$

Now let

$$f_{ht}(x) = \sum_{j=1}^{h_\rho} b_{h,tj}\, \psi_{hj}(x), \qquad (t=1, 2, \cdots, h_\rho) \tag{5.3}$$

where $(b_{h,tj}) = B$ $(t, j=1, 2, \cdots, h_\rho)$ denotes a unitary matrix, consequently the functions $f_{ht}(x)$ $(t = 1, 2, \cdots, h_\rho)$ constitute an orthonormal system of functions satisfying the condition $(f_{hi}, f_{hj}) = \delta_{ij}$. We can easily deduce the following relation in matrices

$$(J(\bar{f}_{hi}, f_{hl})) = \bar{B}(J(\bar{\psi}_{hs}, \psi_{ht}))\, B' = \bar{B}(J(\bar{\psi}_{hs}, \psi_{ht}))\, \bar{B}^{-1},$$

where $\bar{B}$, B' and B^{-1} represent respectively the conjugate, transpose and inverse of the matrix B. Since the matrix $(\lambda_h\ J\ (\bar{\psi}_{hs}, \psi_{ht}))$ is unitary, therefore by a known theorem[6], we can choose a suitable unitary matrix B, such that the matrix $\bar{B}\ (\lambda_h\ J\ (\bar{\psi}_{hs}, \psi_{ht}))\ \bar{B}^{-1}$ is a diagonal matrix, and consequently the matrix $\bar{B}\ (J\ (\bar{\psi}_{hs}, \psi_{ht})\ \bar{B}^{-1}$ is a diagonal matrix

$$\begin{pmatrix} J(\bar{f}_{h1}, f_{h1}), & 0, & \cdots, & 0 \\ 0, & J(\bar{f}_{h2}, f_{h2}), & \cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \cdots, & J(\bar{f}_{hh_\rho}, f_{hh_\rho}) \end{pmatrix}.$$

Whence we get

$$\bar{B}(I - \lambda J(\bar{\psi}_{hs}, \psi_{ht}))\, \bar{B}^{-1} = I - \lambda \bar{B}(J(\bar{\psi}_{hs}, \psi_{ht}))\, \bar{B}^{-1} =$$

$$= \begin{pmatrix} 1-\lambda J(\bar{f}_{h1}, f_{h1}), & 0, & \cdots, & 0 \\ 0, & 1-\lambda J(\bar{f}_{h2}, f_{h2}), & \cdots, & \\ \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \cdots, & 1-\lambda J(\bar{f}_{hh_\rho}, f_{hh_\rho}) \end{pmatrix}. \tag{5.4}$$

6) See [4], p. 108, ex. 11.

Now since the characteristic values of a matrix are invariant under a linear transformation with non-singular matrix[7], therefore the characteristic values of $K_h(x,y)$ are $\mu_{hi} = 1/J(\bar{f}_{hi}, f_{hi})$, i.e.

$$J(\bar{f}_{hi}, f_{hi}) = \int_a^b \int_a^b K(x,y)\, \bar{f}_{hi}(x)\, f_{hi}(y)\, dx\, dy = 1/\mu_{hi}. \quad (i=1,2,\cdots,h_\rho) \quad (5.5)$$

But since the functions $f_{hi}(x)$ $(i=1,2,\cdots,h_\rho)$ denote an orthonormal set of functions, we must have

$$Kf_{hi}[x] = f_{hi}(x)/\mu_{hi}, \qquad K'\bar{f}_{hi}[y] = \bar{f}_{hi}(y)/\mu_{hi}, \qquad (i=1,2,\cdots,h_\rho)$$

it means that the functions $f_{hi}(x)$ are characteristic functions of $K(x,y)$ and the functions $\bar{f}_{hi}(x)$ are the transpose characteristic functions of $K(x,y)$ belonging to the same characteristic value μ_{hi}. Replacing $f_{hi}(x)$ by $u_{hi}(x)$ and $\bar{f}_{hi}(x)$ by $v_{hi}(x)$, we get (5.1) and (5.2).

Further, if μ_{ht} is a r-tiple root of the equation $D_{K_h}(\lambda)=0$, then on the diagonal of the matrix (5.4) there must be exactly r elements equal to $1-\lambda/\mu_{ht}$. Thus when $\lambda=\mu_{ht}$, the rank of the matrix (5.4) is $h_\rho - r$, and consequently when $\lambda=\mu_{ht}$, the rank of the determinant $D_{K_h}(\lambda)$ is $h_\rho - r$. Therefore[8] μ_{ht} is a characteristic value of rank r of $K_h(x,y)$ and consequently μ_{ht} is a characteristic value of rank r of $K(x,y)$. Thus the parts (i) and (ii) in our theorem have been proved.

From (i), it follows that to the h_ρ (distinct or equal) characteristic values μ_{hi} $(i=1,2,\cdots,h_\rho)$ of $K_h(x,y)$ (and therefore of $K(x,y)$), there exist correspondingly h_ρ linearly independent characteristic functions $u_{hi}(x)$ $(i=1,2,\cdots,h_\rho)$ and h_ρ linearly independent transpose characteristic functions $v_{hi}(x)$ $(i=1,2,\cdots,h_\rho)$ of $K_h(x,y)$ (and therefore of $K(x,y)$); so we have proved (ii). It remains to prove (iii).

Suppose $u(x)$ be any characteristic function of $K(x,y)$ belonging to the characteristic value μ_{ht} then $u(x)$ must be a linear combination of the fundamental solutions of just one kernel $K_h(x,y)$ having μ_{ht} as a characteristic value. Therefore there exists one and only one integer h, such that $u(x) = \mu_{ht} K_h u[x]$. Thus from (5.1), we get

$$u_{ht}(x) = \mu_{ht} K_h u_{ht}[x] = \frac{\mu_{ht}}{\lambda_h} \sum_{i=1}^{h_\rho} \varphi_{hi}(x)\,(u_{ht}, \psi_{hi})$$

$$= \frac{\mu_{ht}}{\lambda_h} \sum_{i=1}^{h_\rho} \sum_{j=1}^{h_\rho} a_{h,ij}\, \psi_{hj}(x)\,(u_{ht}, \psi_{hi}). \qquad (t=1,2,\cdots,h_\rho)$$

7) See [4], p. 42, ex. 2.

8) See [5], p. 122.

9) *Ibid.*, pp. 129-130.

Similarly, from $v_{ht}(x) = \mu_{ht} K'v_{ht}[x]$, we get

$$v_{ht}(x) = \frac{\mu_{ht}}{\lambda_h} \sum_{i=1}^{h_\rho} \bar{\psi}_{hi}(x)\,(v_{ht}, \bar{\varphi}_{hi})$$

$$= \frac{\mu_{ht}}{\lambda_h} \sum_{i=1}^{h_\rho} \sum_{j=1}^{h_\rho} a_{h,ji}\, \bar{\varphi}_{hj}(x)\,(v_{ht}, \bar{\varphi}_{hi}). \qquad (t=1, 2, \cdots, h_\rho)$$

Thus theorem 5.1 is completely established.

Corollary 1. *Each complete system of characteristic functions of a L^2 normal kernel can be transformed linearly into a complete orthonormal system of characteristic functions of the same kernel.*

Proof. Let $u_{hi}(x)\,(i=1,2,\cdots,h_\rho)$ and $v_{hi}(x)\,(i=1,2,\cdots,h_\rho)$ denote respectively a complete orthonormal system of characteristic functions and a complete orthonormal system of transpose characteristic functions of $K_h(x,y)$ such that $\bar{u}_{hi}(x) = v_{hi}(x)$. Then by a known theorem[10], if $h \neq s$, we have

$$\int_a^b u_{hi}(x)\, v_{st}(x)\, dx = 0 \qquad \text{or} \qquad (u_{hi}, u_{st}) = 0,$$

therefore

$$(u_{hi}, u_{st}) = \delta_{hs} \cdot \delta_{it},$$

i.e.

$$\{u_{hi}(x)\} \qquad (h=1, 2, \cdots;\ i=1, 2, \cdots, h_\rho) \tag{5.6}$$

is an orthonormal system of functions. Since $u_{h1}(x), u_{h2}(x), \cdots, u_{hh_\rho}(x)$ denote a complete orthonormal system of characteristic functions of $K_h(x,y)$, therefore the system (5.6) denote a complete orthonormal system of characteristic functions of $K(x,y)$.

VI. Expansion Theorem for Normal Kernels

Theorem 6.1. *The poles of the resolvent kernel of any L^2 normal kernel are all simple.*

Proof. In order to prove this theorem, it is sufficient[11] to prove that to each characteristic function $u_{ht}(x)$ there exists a transpose characteristic function $v_{hs}(x)$ belonging to the same characteristic value μ_{ht} such that

10) See [5], p. 105.

11) *Ibid.*, p. 144.

$$\int_a^b u_{ht}(x)\, v_{hs}(x)\, dx \neq 0\,. \tag{6.1}$$

Now we choose a complete orthonormal system of characteristic functions $\{u_{hi}(x)\}$ $(h=1,2,\cdots,i=1,2,\cdots,h_\rho)$ of $K(x,y)$ such that the set $\{\bar{u}_{hi}(x)\}$ $(h=1,2,\cdots;i=1,2,\cdots,h_\rho)$ denotes a complete orthonormal system of transpose characteristic functions of $K(x,y)$. Therefore substituting $\bar{u}_{ht}(x)$ for $v_{hs}(x)$, we get the relation (6.1) and the theorem follows.

Theorem 6.2. *Suppose* $K(x,y) \sim \sum_{h=1}^{\infty} K_h(x,y)$ *be a* L^2 *normal kernel, where* $K_h(x,y)$ *are given by* (3.1). *Let* $u_{h1}(x),u_{h2}(x),\cdots,u_{hh_\rho}(x)$ *denote a complete orthonormal system of characteristic functions satisfying the condition* (5.2) *of* $K_h(x,y)$ $(h=1,2,\cdots)$ *and let* $\{\mu_{hi}\}$ $(i=1,2,\cdots,h_\rho)$ *denote the set of characteristic values of* $K_h(x,y)$, *then*

$$K(x,y) \sim \sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho} \frac{u_{hi}(x)\,\bar{u}_{hi}(y)}{\mu_{hi}}\,. \tag{6.2}$$

Proof. Since the poles of the resolvent kernel of $K_h(x,y)$ are all simple, therefore by a known theorem[12] we have

$$K_h(x,y) = \sum_{i=1}^{h_\rho} \frac{u_{hi}(x)\, v_{hi}(y)}{\mu_{hi}}\,, \tag{6.3}$$

where $\{u_{hi}(x)\}$, $\{v_{hi}(y)\}$ $(i=1,2,\cdots,h_\rho)$ denote respectively a complete orthonormal system of characteristic functions and transpose characteristic functions of $K_h(x,y)$ such that

$$\int_a^b u_{hi}(x)\, v_{hj}(x)\, dx = \delta_{ij}\,.$$

But this condition is satisfied, when we set $\bar{u}_{hi}(x) = v_{hi}(x)$. Thus we have

$$K_h(x,y) = \sum_{i=1}^{h_\rho} \frac{u_{hi}(x)\,\bar{u}_{hi}(y)}{\mu_{hi}}\,. \tag{6.3*}$$

Substituting into $K(x,y) \sim \sum_{h=1}^{\infty} K_h(x,y)$, we get (6.2) as we wished to prove.

12) See [5], p. 135.

From the properties of convergence in the mean square, it is easy to deduce the following corollaries:

Corollary 1. *For* $n=2, 3, 4, \ldots\ldots$, *we have*

$$K^n[x, y] = \sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho}\frac{u_{hi}(x)\,\bar{u}_{hi}(y)}{\mu_{hi}^n} = \sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho}\sum_{j=1}^{h_\rho}\frac{a_{h,ji}^{(n)}\,\varphi_{hi}(x)\,\bar{\varphi}_{hj}(y)}{\lambda_h^n}. \quad (6.4)$$

Corollary 2. *If* $q(x)\in L^2$, $g(x)=Kq\,[x]$, *then*

$$g(x)\sim\sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho}u_{hi}(x)\,(g, u_{hi}) = \sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho}u_{hi}(x)\,(q, u_{hi})\,/\,\mu_{hi}. \quad (6.5)$$

Theorem 6.3. *Suppose* $p(x)\in L^2$, $q(x)\in L^2$, *then*

$$\int_a^b\int_a^b K(x, y)\,p(x)\,q(y)\,dx\,dy = \sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho}\int_a^b u_{hi}(x)\,p(x)\,dx\int_a^b \bar{u}_{hi}(x)\,q(x)\,dx\,/\,\mu_{hi}. \quad (6.6)$$

(It is a generalization of a formula of Hilbert)

Proof. Let $g(x)$ be a function satisfying the relation (6.5), then from the properties of the convergence in the mean square, we have

$$\int_a^b\int_a^b K(x, y)\,p(x)\,q(y)\,dx\,dy = \int_a^b p(x)\,g(x)\,dx = \sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho}\frac{(u_{hi}, \bar{p})\,(q, u_{hi})}{\mu_{hi}}.$$

Theorem 6.4. *For any* L^2 *normal kernel, we have*

$$\sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho}\frac{1}{|\mu_{hi}|^2} = \sum_{h=1}^{\infty}\frac{1}{\lambda_h^2} = \int_a^b\int_a^b |K(x, y)|^2\,dx\,dy. \quad (6.7)$$

(It is only known previously that

$$\sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho}\frac{1}{|\mu_{hi}|^2}\leqslant\|K(x, y)\|^2\Big)$$

Proof. Since

$$K(x, y)\sim\sum_{h=1}^{\infty}K_h(x, y) = \sum_{h=1}^{\infty}\frac{\varphi_h(x)\,\bar{\psi}_h(y)}{\lambda_h},$$

therefore we have

$$\lim_{n\to\infty}\int_a^b\int_a^b \left| K(x,y)-\sum_{h=1}^{n}\frac{\varphi_h(x)\,\bar{\psi}_h(y)}{\lambda_h}\right|^2 dx\,dy=0\,,$$

i.e.

$$\lim_{n\to\infty}\int_a^b\int_a^b\left\{K(x,y)-\sum_{h=1}^{n}\frac{\varphi_h(x)\,\bar{\psi}_h(y)}{\lambda_h}\right\}\left\{\bar{K}(x,y)-\sum_{h=1}^{n}\frac{\bar{\varphi}_h(x)\,\psi_h(y)}{\lambda_h}\right\}dx\,dy=0\,.$$

Whence we have

$$\|K(x,y)\|^2-\lim_{n\to\infty}\left\{\sum_{h=1}^{n}\frac{\int_a^b\psi_h(y)\,\bar{\psi}_h(y)\,dy}{\lambda_h^2}+\sum_{h=1}^{n}\frac{\int_a^b\varphi_h(x)\,\bar{\varphi}_h(x)\,dx}{\lambda_h^2}-\sum_{h=1}^{n}\frac{1}{\lambda_h^2}\right\}=0\,,$$

and consequently

$$\sum_{h=1}^{\infty}\frac{1}{\lambda_h^2}=\|K(x,y)\|^2\,.$$

Finally, from the relation $|\mu_{hi}|=\lambda_h$, we get (6.7) as we wished to prove.

VII. The General Solution of Integral Equations with Normal Kernels

Theorem 7.1. *Suppose* $K(x,y)\sim\sum_{h=1}^{\infty}K_h(x,y)$ *be a* L^2 *normal kernel, where* $K_h(x,y)$ *are given by* (3.1). Let $\{u_{hi}(x)\}$ $(i=1,2,\cdots,h_\rho)$ *be a complete orthonormal system of characteristic functions of* $K_h(x,y)$ *and* $q(x)\in L^2$.

Then (i) *the solution of the integral equation*

$$q(x)=\varphi(x)-\lambda K\varphi[x] \tag{7.1}$$

is given by

$$\varphi(x)=\lambda\sum_{h=1}^{\infty}\sum_{i=1}^{h_\rho}u_{hi}(x)\,(q,u_{hi})/(\mu_{hi}-\lambda)+q(x)\,, \tag{7.2}$$

if λ *is not a characteristic value of* $K(x,y)$.

(ii) *If* $\lambda=\mu_{ts}$ *is a characteristic value of* $K(x,y)$, *then a necessary and sufficient condition for the solvability of the integral equation* (7.1) *is that* $q(x)$ *should be orthogonal to all transpose characteristic functions of*

$K(x,y)$ belonging to μ_{ts}, and when this condition is satisfied, then the solution is given by

$$\varphi(x) = \lambda \sum_{h=1}^{\infty}{}' \sum_{i=1}^{h_\rho}{}' u_{hi}(x)(q, u_{hi})/(\mu_{hi}-\lambda) + q(x), \tag{7.2*}$$

the accents indicating that for those values of h and i for which $\mu_{hi} = \mu_{ts}$ the coefficient of $u_{hi}(x)$ may take any arbitrary value.

Proof. Suppose λ is not a characteristic value of $K(x,y)$. Let the solution of the integral equation (7.1) be

$$\varphi(x) = q(x) + \sum_{h=1}^{\infty} \sum_{t=1}^{h_\rho} x_{ht} u_{ht}(x), \tag{7.3}$$

where the unknowns x_{ht} are to be determined. Substituting (7.3) into (7.1), we get

$$\sum_{h=1}^{\infty} \sum_{t=1}^{h_\rho} x_{ht} u_{ht}(x) = \lambda Kq[x] + \lambda \sum_{h=1}^{\infty} \sum_{t=1}^{h_\rho} x_{ht} Ku_{ht}[x].$$

But $\mu_{ht} Ku_{ht}[x] = u_{ht}(x)$, therefore by (6.5) we have

$$\sum_{h=1}^{\infty} \sum_{t=1}^{h_\rho} x_{ht} u_{ht}(x) = \lambda \sum_{h=1}^{\infty} \sum_{i=1}^{h_\rho} \frac{u_{hi}(x)}{\mu_{hi}} (q, u_{hi}) + \lambda \sum_{h=1}^{\infty} \sum_{t=1}^{h_\rho} \frac{x_{ht}}{\mu_{ht}} u_{ht}(x).$$

Multiplying both sides by $\bar{u}_{ht}(x)$ and integrating both sides term by term, then since $\{u_{hi}(x)\}$ $(h=1,2,\cdots; i=1,2,\cdots,h_\rho)$ is an orthonormal system of characteristic functions of $K(x,y)$, we have

$$x_{ht} = \frac{\lambda}{\mu_{ht}}(q, u_{ht}) + \lambda x_{ht}/\mu_{ht}.$$

Whence we get

$$x_{ht} = \lambda (q, \mu_{ht})/(\mu_{ht}-\lambda).$$

Substituting this value into (7.3), we get (7.2).

(ii) The proof is obvious.

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