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Uniqueness of the decomposition of Lie superalgebras and quadratic Lie superalgebras

ZHU Linsheng (朱林生)[1] & MENG Daoji (孟道骥)[2]

1. Department of Mathematics, Nanjing University, Nanjing 210093, China; Department of Mathematics, BcChangshu College, Changshu 215500, China;

2. Department of Mathematics, Nankai University, Tianjin 300071, China

Correspondence should be addressed to Zhu Linsheng (email: lszhu@pub.sz.jsinfo.net)

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Abstract A Lie superalgebra endowed with a non-degenerate super-symmetric and invariant bilinear form is called a quadratic Lie superalgebra. In this paper, we consider the decomposition of a Lie superalgebra and the first main result is that the decomposition of a Lie superalgebra into indecomposable graded ideals is unique up to an isomorphism. Next, we obtain the uniqueness of the decomposition of an arbitrary quadratic Lie superalgebra into irreducible graded ideals up to an isometry.

Keywords: quadratic Lie superalgebra, decomposition, graded ideal, isotropic.

A Lie superalgebra is a Z_2-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, together with a graded Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following conditions

$$[a,b] = -(-1)^{xy}[b,a], \quad \text{(skew symmetry)}$$

$$[a,[b,c]] = [[a,b],c] + (-1)^{xy}[b,[a,c]], \quad \text{(Jocobian identity)}$$

for all elements $a \in \mathfrak{g}_x,\ b \in \mathfrak{g}_y$, and $c \in \mathfrak{g}_z$ $(x, y, z, \in Z_2)$. For simplicity, we call the elements in $\mathfrak{g}_{\bar{0}}$ or $\mathfrak{g}_{\bar{1}}$ homogeneous.

It is well known that the study of the Killing form plays a key role in the theory of simple Lie superalgebras. It is desirable to extend the investigation to Lie superalgebras with a non-degenerate super-symmetric invariant bilinear form. For simplicity, we call such a Lie superalgebra a quadratic Lie superalgebra and the form an invariant scalar product. Quadratic Lie superalgebras, in particular, Lie bi-superalgebras occur naturally in physics[1,2]. Besides semi-simple Lie superalgebras, quadratic Lie superalgebras include some solvable superalgebras[3,4]. The notion of double extension of Lie algebras[5] was generalized to quadratic Lie superalgebras and a sufficient condition for a quadratic Lie superalgebra to be a double extension was given by Benamor et al.[3]. In ref. [4], Benayadi studied the quadratic Lie superalgebras with the completely reducible action of even part on the odd part.

It is well known that there is much work on simple Lie superalgebras[6−9]. But the theory of quadratic Lie superalgebras does not seem to be well developed.

This paper considers the decomposition of a Lie superalgebra and a quadratic Lie superalgebra. Our main results are Theorem 1 and Theorem 2. For a quadratic Lie algebra, the uniqueness of the decomposition was obtained by Zhu et al.[1)]

1) Zhu, F. H., Zhu, L. S., The uniqueness of the decomposition of quadratic Lie algebras, to appear in Comm. Alg.

All Lie superalgebras mentioned in this paper are over a field $\mathbf{F}$ of characteristic 0 and finite-dimensional.

Definition 1[3]. Let $\mathfrak{g}$ be a Lie superalgebra. A bilinear form B on $\mathfrak{g}$ is called super-symmetric if $B(x,y)=(-1)^{\alpha\beta}B(y,x)$ for any $x\in\mathfrak{g}_\alpha,\ y\in\mathfrak{g}_\beta$, called $\mathfrak{g}$-invariant if $B([x,y],z)=B(x,[y,z])$ for all $x,y,z\in\mathfrak{g}$, called consistent if $B(x,y)=0$ for all $x\in\mathfrak{g}_{\bar{0}}$ and $y\in\mathfrak{g}_{\bar{1}}$.

From now on we shall consider only consistent bilinear forms.

Definition 2[3]. (1) Let $\mathfrak{g}$ be a Lie superalgebra with a bilinear form B. Then $(\mathfrak{g},B)$ is called quadratic if B is super-symmetric, non-degenerate and $\mathfrak{g}$-invariant. In this case, B is called an invariant scalar product on $\mathfrak{g}$.

(2) Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra. A graded ideal I of $\mathfrak{g}$ is called non-degenerate (resp. degenerate) if the restriction of B on I is non-degenerate (resp. degenerate).

(3) We call a quadratic Lie superalgebra $(\mathfrak{g},B)$ B-irreducible (or irreducible for short) if $\mathfrak{g}$ contains no nontrivial non-degenerate graded ideal.

Let $\mathfrak{g}$ be a Lie superalgebra. Clearly, $\mathfrak{g}$ has the following decomposition

$$\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_n \tag{1}$$

as Lie superalgebras, where $\mathfrak{g}_i$ $(i=1,2,\cdots,n)$ are indecomposable graded ideals of $\mathfrak{g}$.

Similarly, any quadratic Lie superalgebra can be decomposed into a direct sum of irreducible non-degenerate graded ideals by the following Lemma 2.

Theorem 1. With the above notations, the decomposition of $\mathfrak{g}$ as in (1) is unique up to an isomorphism. Furthermore, if $C(\mathfrak{g})=0$, then the decomposition is unique up to the order of ideals $\mathfrak{g}_i$ $(i=1,2,\cdots,n)$.

Theorem 2. Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra. Then the decomposition of $\mathfrak{g}$ into irreducible non-degenerate graded ideals is unique up to an isometry. Furthermore, if $C(\mathfrak{g})=0$, then $\mathfrak{g}$ is just the direct sum of all its irreducible non-degenerate graded ideals (coinciding with all its indecomposable graded ideals) and so the decomposition is unique up to the order of these graded ideals.

Following Theorem 2, the problem of studying quadratic Lie superalgebras is reduced to the problem of studying irreducible quadratic Lie superalgebras.

Proof of main results

The proof of Theorem 1. Assume that a homogeneous element x is in the center $C(\mathfrak{g})$ of $\mathfrak{g}$. Then we have the following fact:

F: $x\notin[\mathfrak{g},\mathfrak{g}]$ if and only if the 1-dimensional graded subspace $\mathbf{F}x$ is an abelian direct factor of $\mathfrak{g}$ as Lie superalgebra.

Let

$$\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2, \tag{2}$$

where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are graded ideals of $\mathfrak{g}$ and $\mathfrak{g}_2$ is the maximal abelian direct factors of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{g}_1'\oplus\mathfrak{g}_2'$ be another such decomposition of $\mathfrak{g}$. Then by fact **F**, one may easily check that $\mathfrak{g}_1\cap\mathfrak{g}_2'=0$ and $\mathfrak{g}_1'\cap\mathfrak{g}_2=0$. Consider the projection:

$$\pi_1:\ \mathfrak{g}_1\to\mathfrak{g}_1'.$$

Obviously, π_1 is homomorphism of Lie superalgebras and $\mathrm{Ker}\pi_1\subseteq\mathfrak{g}_2'\cap\mathfrak{g}_1=0$, so π_1 is injective. Thus we have $\dim\mathfrak{g}_1\leqslant\dim\mathfrak{g}_1'$. Similarly, we have $\dim\mathfrak{g}_1\geqslant\dim\mathfrak{g}_1'$. Therefore $\dim\mathfrak{g}_1=\dim\mathfrak{g}_1'$. Setting $\pi_2:\mathfrak{g}_2\to\mathfrak{g}_2'$ to be the projection, one may easily check that $\pi=(\pi_1,\pi_2)$ is an isomorphism of Lie superalgebra $\mathfrak{g}$ and such decomposition as in (2) is unique up to an isomorphism. Hence we may assume that $\mathfrak{g}$ has no abelian graded direct factor.

For $\mathfrak{g}$, we have the decomposition (1). Assume

$$\mathfrak{g} = \mathfrak{g}_1' \oplus \cdots \oplus \mathfrak{g}_m' \tag{3}$$

is another such decomposition of $\mathfrak{g}$. Then

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \oplus_{j=1}^m [\mathfrak{g}_1, \mathfrak{g}_j']$$

and $[\mathfrak{g}_1, \mathfrak{g}_1] \neq 0$, which implies $[\mathfrak{g}_1, \mathfrak{g}_j'] \neq 0$ for some j. We assume $[\mathfrak{g}_1, \mathfrak{g}_1'] \neq 0$ for simplicity. Denote $A_i = \mathfrak{g}_1 \cap \mathfrak{g}_i'$ $(i = 1, 2, \cdots, m)$. Define

$$\mathfrak{M} = \{(B_1, B_2, \cdots, B_m) \mid A_i \subseteq B_i, \quad B_i \bigcap \sum_{j \neq i} B_j = 0, \text{ any } B_i \text{ is a graded ideal of } \mathfrak{g}_1\}.$$

Clearly $(A_1, A_2, \cdots, A_m) \in \mathfrak{M}$.

Claim 1. $A_2 = A_3 = \cdots = A_m = 0$.

Case 1. If $\mathfrak{g}_1 = A_1 \oplus A_2 \oplus \cdots \oplus A_m$, then $A_2 = A_3 = \cdots = A_m = 0$ since $\mathfrak{g}_1$ is an indecomposable graded ideal of $\mathfrak{g}$ and $0 \neq [\mathfrak{g}_1, \mathfrak{g}_1'] \subseteq A_1$.

Case 2. If $\mathfrak{g}_1 \neq A_1 \oplus A_2 \oplus \cdots \oplus A_m$, we may choose a homogeneous element $d \in \mathfrak{g}_1$ with $d \notin A_1 \oplus A_2 \oplus \cdots \oplus A_m$. Assume

$$d = d_1' + d_2' + \cdots + d_m' \tag{4}$$

by (3), where $d_i' \in \mathfrak{g}_i'$ $(i = 1, 2, \cdots, m)$ are homogeneous. Furthermore, by (1), we assume

$$d_i' = d_{i1} + d_{i2} + \cdots + d_{in}, \ i = 1, 2, \cdots, m, \tag{5}$$

where $d_{ij} \in \mathfrak{g}_j$ $(j = 1, 2, \cdots, n)$ are homogeneous. So

$$d = \sum_{i=1}^m d_{i1} + \sum_{i=1}^m d_{i2} + \cdots + \sum_{i=1}^m d_{in}.$$

Since $d \in \mathfrak{g}_1$, we have $d = \sum\limits_{i=1}^m d_{i1}$ and $\sum\limits_{j=2}^n \sum\limits_{i=1}^m d_{ij} = 0$. By (5), we have

$$[d_{i1}, \mathfrak{g}_1] = \left[\sum_{j=1}^n d_{ij}, \mathfrak{g}_1\right] = [d_i', \mathfrak{g}_1] \subseteq [\mathfrak{g}_1, \mathfrak{g}_i'], \quad i = 1, 2, \cdots, m. \tag{6}$$

Assume $d_{11} \notin A_1 \oplus \cdots \oplus A_m$. Then we let $C_1 = A_1 \dot{+} \mathbf{F} d_{11}$. Furthermore, let $C_2 = A_2 \dot{+} \mathbf{F} d_{21}$ if $d_{21} \notin C_1 \dot{+} A_2$ and let $C_2 = A_2$ otherwise. Generally, we let $C_j = A_j \dot{+} \mathbf{F} d_{j1}$ if $d_{j1} \notin C_1 \dot{+} \cdots \dot{+} C_{j-1} \dot{+} A_j$ and let $C_j = A_j$ $(j = 3, 4, \cdots, m)$ otherwise. Clearly, C_j $(j = 1, 2, \cdots, m)$ are graded ideals of $\mathfrak{g}_1$ by (6). Since $\dim \mathfrak{g}_1 < \infty$, repeating the above discussion, there exists an element $(B_1, B_2, \cdots, B_m) \in \mathfrak{M}$ such that

$$B_1 \oplus B_2 \oplus \cdots \oplus B_m = \mathfrak{g}_1.$$

Similar to Case 1, we have $B_i = 0$ $(i = 2, 3, \cdots, m)$, which implies $A_i = 0$ $(i = 2, 3, \cdots, m)$.

Claim 2. With the above notations, we have $n = m$, $\dim \mathfrak{g}_j = \dim \mathfrak{g}_j'$ (up to a permutation of index), $[\mathfrak{g}_j, \mathfrak{g}_j] = [\mathfrak{g}_j, \mathfrak{g}_j'] = [\mathfrak{g}_j', \mathfrak{g}_j']$ and $[\mathfrak{g}_j, \mathfrak{g}_k'] = 0$ for $j \neq k$.

Since $A_i = 0$ $(i = 2, 3, \cdots, m)$, we have $[\mathfrak{g}_1, \mathfrak{g}_i'] = 0$. So $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_1, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1']$. Similarly, we have $[\mathfrak{g}_k, \mathfrak{g}_1'] = 0$ for $k = 2, 3, \cdots, n$ and $[\mathfrak{g}_1'\mathfrak{g}_1'] = [\mathfrak{g}_1, \mathfrak{g}_1']$. Consider the projection

$$\pi_1 : \mathfrak{g}_1 \to \mathfrak{g}_1'.$$

Obviously, π_1 is a homomorphism of Lie superalgebras and $\ker\pi_1 \subseteq \sum_{i=2}^{m} \mathfrak{g}_1 \cap \mathfrak{g}_i' = 0$, so π_1 is injective. Thus we have $\dim\mathfrak{g}_1 \leqslant \dim\mathfrak{g}_1'$. Similarly, we have $\dim\mathfrak{g}_1 \geq \dim\mathfrak{g}_1'$. Therefore $\dim\mathfrak{g}_1 = \dim\mathfrak{g}_1'$. Repeating the above discussion for $j = 2, 3, \cdots, n$, we obtain Claim 2.

By Claim 2 and its proof, we know that all $\pi_i : \mathfrak{g}_i \to \mathfrak{g}_i'$ $(i = 1, 2, \cdots, n)$ are isomorphisms. Define $\pi = (\pi_1, \pi_2, \cdots, \pi)$ such that $\pi|_{\mathfrak{g}_i} = \pi_i$. Then π is an isomorphism of $\mathfrak{g}$ we want.

Now we assume $C(\mathfrak{g}) = 0$. Then $C(\mathfrak{g}_i) = C(\mathfrak{g}_i') = 0$ for $i = 1, 2, \cdots, n$. Define $\pi_{ij} : \mathfrak{g}_i \to \mathfrak{g}_j'$ $i \neq j$; $i, j = 1, 2, \cdots, n$ to be the projections. For any homogeneous element $x \in \mathfrak{g}_i$, we assume $x = \sum_{j=1}^{n} x_j$, where $x_j \in \mathfrak{g}_j'$ are homogeneous. Clearly $\pi_{ij}(x) = x_j$ and $[x_j, \mathfrak{g}_j'] = [x, \mathfrak{g}_j'] \subseteq \mathfrak{g}_i \cap \mathfrak{g}_j' = 0$. Thus $x_j \in C(\mathfrak{g}_j') = 0$. It follows that $\pi_i = id_{\mathfrak{g}_i}$, i.e. $\mathfrak{g}_i = \mathfrak{g}_i'$.

Remark. The latter assertion of Theorem 1 is the main result in ref. [10].

In order to prove Theorem 2, we first establish some lemmas.

Lemma 1. Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra, and let $\mathfrak{a}$ and $\mathfrak{b}$ be graded subspaces of $\mathfrak{g}$. Then the following assertions are equivalent:

(1) $x \in [\mathfrak{a}, \mathfrak{b}]^\perp$; (2) $[x, \mathfrak{a}] \subseteq \mathfrak{b}^\perp$; (3) $[x, \mathfrak{b}] \subseteq \mathfrak{a}^\perp$.

Lemma 2[11]. Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra, and let $\mathfrak{a}$ be an graded ideal of $\mathfrak{g}$. Then we have the following:

(1) $[\mathfrak{a}, \mathfrak{g}]^\perp$ is the centralizer $C_{\mathfrak{g}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{g}$;

(2) $\mathfrak{a}^\perp$ is an graded ideal of $\mathfrak{g}$ and centralizes $\mathfrak{a}$. Therefore, if $[\mathfrak{g}, \mathfrak{a}] = \mathfrak{a}$, then $\mathfrak{a}^\perp = C_{\mathfrak{g}}(\mathfrak{a})$;

(3) $[\mathfrak{g}, \mathfrak{g}]^\perp = C(\mathfrak{g})$;

(4) Assume $\mathfrak{a}$ is non-degenerate. Then $\mathfrak{a}^\perp$ is also non-degenerate and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$.

Proof. (1) By Lemma 1, $x \in [\mathfrak{a}, \mathfrak{g}]^\perp$ if and only if $[x, \mathfrak{a}] \subseteq \mathfrak{g}^\perp = 0$. Hence $[\mathfrak{a}, \mathfrak{g}]^\perp = C_{\mathfrak{g}}(\mathfrak{a})$.

(2) Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{a}_\beta$, $z \in \mathfrak{a}_\gamma^\perp$. Then $B([x, z], y) = -(-1)^{\alpha\gamma} B(z, [x, y]) = 0$ by the invariance of B. Hence $[x, z] \in \mathfrak{a}^\perp$. Consequently, $\mathfrak{a}^\perp$ is a graded ideal of $\mathfrak{g}$.

Since $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$, we have $\mathfrak{a}^\perp \subseteq [\mathfrak{a}, \mathfrak{g}]^\perp = C_{\mathfrak{g}}(\mathfrak{a})$. By (1), $[\mathfrak{a}, \mathfrak{g}] = \mathfrak{a}$ implies $\mathfrak{a}^\perp = C_{\mathfrak{g}}(\mathfrak{a})$.

(3) follows from (1).

(4) Let $x \in \mathfrak{a} \cap \mathfrak{a}^\perp$. Then $B(x, \mathfrak{a}) = 0$. Since $\mathfrak{a}$ is a non-degenerate graded ideal of $\mathfrak{g}$, $x = 0$. It follows that $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$. Note that $\dim\mathfrak{g} = \dim\mathfrak{a} + \dim\mathfrak{a}^\perp$. We have $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$.

Lemma 3. Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra. Then any perfect graded direct factor $\mathfrak{s}$ of $\mathfrak{g}$ is non-degenerate.

Proof. Assume that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{g}_1$ as Lie superalgebras and $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$. Then $0 = \mathfrak{g}^\perp = \mathfrak{s}^\perp \cap \mathfrak{g}_1{}^\perp$. Since $B(\mathfrak{g}_1, \mathfrak{s}) = B(\mathfrak{g}_1, [\mathfrak{s}, \mathfrak{s}]) = B([\mathfrak{g}_1, \mathfrak{s}], \mathfrak{s}) = 0$, $\mathfrak{s} \subseteq \mathfrak{g}_1{}^\perp$. Hence $\mathfrak{s} \cap \mathfrak{s}^\perp = 0$. It follows that $\mathfrak{s}$ is non-degenerate.

Lemma 4. Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra with the center $C(\mathfrak{g})$ being isotropic. Then any graded direct factor of $\mathfrak{g}$ is non-degenerate. In particular, $\mathfrak{g}$ has no abelian graded direct factor.

Proof. Assume that $\mathfrak{g}_1$ is a graded direct factor of $\mathfrak{g}$. Then there exists another graded ideal $\mathfrak{g}_2$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Assume $\mathfrak{g}_1$ is not quadratic. Then the radical of $\mathfrak{a} = \mathrm{rad}(B|_{\mathfrak{g}_1}) \neq 0$ is a graded ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is quadratic and $B([\mathfrak{g}_1, \mathfrak{g}_1], \mathfrak{g}_2) = B(\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_2]) = 0$, we have $\mathfrak{a} \cap [\mathfrak{g}_1, \mathfrak{g}_1] = 0$. Therefore $\mathfrak{a} \cap C(\mathfrak{g}) = 0$ since $C(\mathfrak{g}) \subseteq C(\mathfrak{g})^\perp = [\mathfrak{g}, \mathfrak{g}]$, i.e. $[\mathfrak{a}, \mathfrak{g}_1] \neq 0$. Clearly $B([\mathfrak{a}, \mathfrak{g}_1], \mathfrak{g}) = B(\mathfrak{a}, [\mathfrak{g}_1, \mathfrak{g}]) = 0$, which contradicts the non-degeneracy of B.

The latter assertion of the lemma is clear.

Proof of Theorem 2.

Case I. The center $C(\mathfrak{g})$ of $\mathfrak{g}$ is isotropic.

Let

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}'_1 \oplus \cdots \oplus \mathfrak{g}'_m$$

be two decompositions of $\mathfrak{g}$. Here $\mathfrak{g}_i$, $\mathfrak{g}'_j$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, are irreducible non-degenerate graded ideals of $\mathfrak{g}$ and $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$, $B(\mathfrak{g}'_i, \mathfrak{g}'_j) = 0$ for $i \neq j$. By Lemma 4, we have

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \bigoplus_{j=1}^{m} [\mathfrak{g}_1, \mathfrak{g}'_j] \neq 0.$$

Hence $[\mathfrak{g}_1, \mathfrak{g}'_j] \neq 0$ for some j. Assume $[\mathfrak{g}_1, \mathfrak{g}'_1] \neq 0$ for simplicity. Let $A_i = \mathfrak{g}_1 \cap \mathfrak{g}'_i$, $i = 1, 2, \cdots, m$. Clearly, A_i $(i = 1, 2, \cdots, m)$ are graded ideals of $\mathfrak{g}_1$. If $\mathfrak{g}_1 = A_1 \oplus A_2 \oplus \cdots \oplus A_m$, then A_i is non-degenerate for any i by Lemma 4. Since $\mathfrak{g}_1$ is irreducible non-degenerate and $A_1 \neq 0$, we have $A_i = 0$ for $i = 2, 3, \cdots, m$. If $\mathfrak{g}_1 \neq A_1 \oplus A_2 \oplus \cdots \oplus A_m$, as in the proof of Theorem 1, we define

$$\mathfrak{M} = \{(B_1, B_2, \cdots, B_m)\},$$

where B_i $(i = 1, 2, \cdots, m)$ are graded ideals of $\mathfrak{g}_1$, $A_i \subseteq B_i$ and $B_i \cap \sum_{j \neq i} B_j = 0$. By Lemmas 2 and 4, similar to the discussion in the proof of Theorem 1, we have $B_i = 0$, consequently, $A_i = 0$ $(i = 2, 3, \cdots, m)$. Thus $[\mathfrak{g}_1, \mathfrak{g}_i] = 0$ $(i = 2, 3, \cdots, m)$, which implies that $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_1, \mathfrak{g}'_1]$. Similarly, we have $[\mathfrak{g}_1, \mathfrak{g}'_1] = [\mathfrak{g}'_1, \mathfrak{g}'_1]$ and $[\mathfrak{g}_j, \mathfrak{g}'_1] = 0$ for all $j = 2, 3, \cdots, n$. Therefore $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}'_1, \mathfrak{g}'_1]$. By Lemma 2, we have $\dim \mathfrak{g}_1 = \dim [\mathfrak{g}_1, \mathfrak{g}_1] + \dim C(\mathfrak{g}_1)$. It is clear that $C(\mathfrak{g}_1) = C(\mathfrak{g}'_1)$. It follows that $\dim \mathfrak{g}_1 = \dim \mathfrak{g}'_1$. Repeating the above discussion for $j = 2, 3, \cdots, n$, we have $n = m$, $\dim \mathfrak{g}_j = \dim \mathfrak{g}'_j$ (up to a permutation of index), $[\mathfrak{g}_j, \mathfrak{g}_j] = [\mathfrak{g}_j, \mathfrak{g}'_j] = [\mathfrak{g}'_j, \mathfrak{g}'_j]$ and $[\mathfrak{g}_j, \mathfrak{g}'_k] = 0$ for $j \neq k$.

It is easy to check that all the projections $\pi_j : \mathfrak{g}_j \to \mathfrak{g}'_j$ $(1 \leqslant j \leqslant n)$ are isomorphisms of Lie superalgebras and preserve the bilinear form. Define $\pi = (\pi_1, \ldots, \pi_n)$ such that $\pi|_{\mathfrak{g}_i} = \pi_i$. Then π is an isometry of $\mathfrak{g}$, i.e. the decomposition is unique up to an isometry, furthermore, π is an automorphism of $\mathfrak{g}$.

Case II. $C(\mathfrak{g})$ is not isotropic.

If $\mathfrak{g}$ is abelian, it is clear that any irreducible graded ideal $\mathfrak{a}$ of $\mathfrak{g}$ is 1-dimensional or 2-dimensional. Following the super-symmetry of B, we must have $\mathfrak{a} = Fx \subseteq \mathfrak{g}_{\bar{0}}$ with $B(x, x) \neq 0$ or $\mathfrak{a} = Fx_1 \oplus Fx_2 \subseteq \mathfrak{g}_{\bar{1}}$ with $B(x_1, x_1) = B(x_2, x_2) = 0$ and $B(x_1, x_2) \neq 0$. So the assertion is clear.

Now assume that $\mathfrak{g}$ is not abelian. Since $C(\mathfrak{g})$ is not isotropic, there exists an element $x \in C(\mathfrak{g})_{\bar{0}}$ such that $B(x, x) \neq 0$, in this case, $\mathfrak{a} = Fx$ is an irreducible graded ideal of $\mathfrak{g}$, or there exist two elements $x_1, x_2 \in C(\mathfrak{g})_{\bar{1}}$ such that $\mathfrak{a} = Fx_1 \oplus Fx_2$ is an irreducible graded ideal of $\mathfrak{g}$. Define $\mathfrak{b} = \mathfrak{a}^{\perp}$. By Lemma 2, we have that $\mathfrak{b}$ is an irreducible graded ideal of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. Since $\dim \mathfrak{g} < \infty$, repeating the above discussion for $\mathfrak{b}$, we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are irreducible graded ideals of $\mathfrak{g}, \mathfrak{g}_2 \subseteq C(\mathfrak{g})$ and the center of $\mathfrak{g}_1 (\neq 0)$ is isotropic.

Let $\mathfrak{g} = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2$ be another such decomposition. Then we have

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}'_1, \mathfrak{g}'_1] = [\mathfrak{g}_1, \mathfrak{g}'_1].$$

Similar to Case I, we get $\dim \mathfrak{g}_1 = \dim \mathfrak{g}'_1$, consequently, $\dim \mathfrak{g}_2 = \dim \mathfrak{g}'_2$.

Since the center $C(\mathfrak{g}_1)$ of $\mathfrak{g}_1$ is isotropic, by Lemma 2, we have

$$C(\mathfrak{g}_1) \subseteq C(\mathfrak{g}_1)^{\perp} = [\mathfrak{g}_1, \mathfrak{g}_1]. \tag{7}$$

Note that $\mathfrak{g}_1$ is consistent. By (7), we may choose a basis

$$\{x_1,\dots,x_k,y_1,\dots,y_{k_1};x_{k+1},\dots,x_n,y_{k_1+1},\dots,y_{n_1};x_{n+1},\dots,x_{n+k},y_{n_1+1},\dots,y_{n_1+k_1}\}$$

of $\mathfrak{g}_1$ such that $x_i\in(\mathfrak{g}_1)_{\bar 0}$ for $i=1,2,\dots,n+k$, $y_j\in(\mathfrak{g}_1)_{\bar 1}$ for $j=1,2,\dots,n_1+k_1$ such that $\{x_1,\dots,x_k,y_1,\dots,y_{k_1}\}$ is a basis of $C(\mathfrak{g}_1)$ and $\{x_1,\dots,x_n,y_1,\dots,y_{n_1}\}$ is a basis of $[\mathfrak{g}_1,\mathfrak{g}_1]=[\mathfrak{g}'_1,\mathfrak{g}'_1]$, and

$$\begin{array}{lll} B(x_i,x_j)=\delta_{ij}, & B(y_l,y_m)=\delta_{lm} & k+1\leqslant i,j\leqslant n;\ k_1+1\leqslant l,m\leqslant n_1\\ B(x_i,x_{n+j})=\delta_{ij}, & B(y_l,y_{n_1+m})=\delta_{lm} & 1\leqslant i,j\leqslant k; 1\leqslant l,m\leqslant k_1\\ B(x_i,x_j)=0, & B(y_l,y_m)=0 & 1\leqslant i,j\leqslant k; 1\leqslant l,m\leqslant k_1\\ B(x_i,x_j)=0, & B(y_l,y_m)=0 & n+1\leqslant i,j\leqslant n+k;\ n_1+1\leqslant l,m\leqslant n_1+k_1. \end{array}$$

Let $\pi_i:\mathfrak{g}_i\to\mathfrak{g}'_i\quad i=1,2$ be two projections. Clearly they are isomorphisms of Lie superalgebras. For $x,y\in\mathfrak{g}_2\subset C(\mathfrak{g})$, we assume that $x=x_1+x_2$ and $y=y_1+y_2$, where $x_i\in\mathfrak{g}'_{i_{\bar 0}}$, $y_i\in\mathfrak{g}'_{i_{\bar 0}}$ $(i=1,2)$ and $x_1,y_1\in C(\mathfrak{g}'_1)$. Since $C(\mathfrak{g}'_1)$ is isotropic, we have $B(\pi_2(x),\pi_2(y))=B(x_2,y_2)=B(x,y)$.

Now consider the projection π_1. Noticing that $x_j,y_m\in[\mathfrak{g}_1,\mathfrak{g}_1]=[\mathfrak{g}'_1,\mathfrak{g}'_1]$, by definition of π_1, we get $\pi_1|_{[\mathfrak{g}_1,\mathfrak{g}_1]}=\mathrm{id}$, $B(\pi_1(x_i),\pi_1(x_j))=B(x_i,x_j)$ and $B(\pi_1(y_l),\pi_1(y_m))=B(y_l,y_m)$ for $1\leqslant i\leqslant n+k$, $1\leqslant j\leqslant n$, $1\leqslant l\leqslant n_1+k_1$ and $1\leqslant m\leqslant n_1$.

Assume that $x_p=x'_{p1}+x'_{p2}$ and $y_i=y'_{i1}+y'_{i2}$ for $n+1\leqslant p\leqslant n+k$ and $n_1+1\leqslant i\leqslant n_1+k_1$ and $x'_{pj},y'_{ij}\in\mathfrak{g}'_j$ $(j=1,2)$. Then $\pi_1(x_p)=x'_{p1}$ and $\pi_1(y_i)=y'_{i1}$. For $n+1\leqslant q\leqslant n+k$ and $n_1+1\leqslant j\leqslant n_1+k_1$, we have

$$0=B(x_p,x_q)=B(x'_{p1},x'_{q1})+B(x'_{p2},x'_{q2})$$

and

$$0=B(y_i,y_j)=B(y'_{i1},y'_{j1})+B(y'_{i2},y'_{j2}).$$

Let $b_{pq}=B(x'_{p2},x'_{q2})$ for $p\neq q$, $c_{ij}=B(y'_{i2},y'_{j2})$ for $i\neq j$, $2b_{pp}=B(x'_{p2},x'_{p2})$ and $2c_{ii}=B(y'_{i2},y'_{i2})$. Define

$$x'_p=x'_{p1}+\sum_{l=p}^{n+k}b_{pl}x_{l-n}\text{ and }y'_i=y'_{i1}+\sum_{m=i}^{n_1+k_1}c_{im}x_{m-n_1}.$$

Note that $B(x'_{p1},x_{q-n})=B(x_p,x_{q-n})=\delta_{pq}$ for $n+1\leqslant p,q\leqslant n+k$ and $B(y'_{i1},y_{j-n_1})=B(y_i,y_{j-n_1})=\delta_{ij}$ for $n_1+1\leqslant i,j\leqslant n_1+k_1$. One may check that

$$\begin{array}{ll} B(x'_p,x'_p)=B(x'_{p1},x'_{p1})+2b_{pp}=0, & n+1\leqslant p\leqslant n+k;\\ B(x'_p,x'_q)=B(x'_{p1},x'_{q1})+b_{pq}=0, & n+1\leqslant p<q\leqslant n+k;\\ B(y'_i,y'_i)=B(y'_{i1},y'_{i1})+2c_{ii}=0, & n_1+1\leqslant i\leqslant n_1+k_1;\\ B(y'_i,y'_j)=B(y'_{i1},y'_{j1})+c_{ij}=0, & n_1+1\leqslant i<j\leqslant n_1+k_1. \end{array}$$

Define $\pi'_1:\mathfrak{g}_1\to\mathfrak{g}'_1$ such that

$$\begin{array}{llll} \pi'_1(x_i)=x_i, & 1\leqslant i\leqslant n; & \pi'_1(x_i)=x'_i, & n+1\leqslant i\leqslant n+k;\\ \pi'_1(y_j)=y_j, & 1\leqslant j\leqslant n_1; & \pi'_1(y_j)=y'_j, & n_1+1\leqslant j\leqslant n_1+k_1. \end{array}$$

One can easily check that π'_1, which preserves the bilinear forms, is also an isomorphism from $\mathfrak{g}_1$ onto $\mathfrak{g}'_1$. Let $\pi=(\pi'_1,\pi_2)$. Then it is easy to see that π is an isometry of $\mathfrak{g}$.

Assume $C(\mathfrak{g}) = 0$. Then by Lemma 2, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, i.e. $\mathfrak{g}$ is perfect. Note that any direct factor of a perfect Lie superalgebra is also perfect. By Lemmas 2–4, $\mathfrak{g}$ has a decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n,$$

where any $\mathfrak{g}_i$ is a perfect irreducible non-degenerate graded ideal of $\mathfrak{g}$. Similar to the discussion in the proof of Theorem 1, we get the latter assertion of Theorem 2.

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