紧致广义 Hermite 流形上纯型的实 d_H - 闭形式的局部延拓

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收稿日期: 2019-08-30; 接受日期: 2020-02-12; 网络出版日期: 2020-03-20

摘要 关于满足广义 $\partial_H\bar{\partial}_H$ - 引理的紧致广义 Hermite 流形的形变, 本文给出一个纯型的实 d_H - 闭形式的局部延拓公式.

关键词 复结构的形变 Hodge 理论 Hermite 和 Kähler 流形

MSC (2020) 主题分类 32G05, 58A14, 53C55

1 引言

光滑形式关于形变的局部延拓是复结构形变理论中的一个重要问题. 文献 [1, 2] 给出了关于紧致 Hermite 流形的形变中 p -Kähler 结构的一个局部延拓. 本文要利用与之平行的方法, 将上述结果推广到紧致广义 Hermite 流形的情形. 具体地, 我们有下面的定理:

定理 1.1 设 (X, G) 为一个紧致广义 Hermite 流形, 其中 G 为 X 上广义 Riemann 度量 [3]. 假定 X 满足广义 $\partial_H\bar{\partial}_H$ - 引理 (参见定义 2.8). 记 $X_{\epsilon(t)}$ 为对应于广义 Beltrami 微分 $\epsilon(t)$ 的广义复流形 (参见第 2.2 小节), 且 $\epsilon(t)$ 的 Hölder 范数 $|\epsilon(t)|_{s+\theta}$ 充分小, 则对于任意满足 $d_H\Omega = 0$ 的 X 上纯型的实光滑形式 $\Omega \in U_{X_0}^0(X_0) \subset \wedge^\bullet T_{X_0}^*(X_0)$, 我们可以构造

$$\Omega(t) := \Omega_0 + \sum_{\alpha \geq 1} \Omega_\alpha t^\alpha \in U_{X_0}^0(X_0),$$

使得 $\Omega_0 = \Omega$, $e^{\epsilon(t)|\overline{\epsilon(t)}}\Omega(t) \in U_{X_{\epsilon(t)}}^0(X_{\epsilon(t)})$ 为 $X_{\epsilon(t)}$ 上的实光滑形式, 且 $d_H e^{\epsilon(t)|\overline{\epsilon(t)}}\Omega(t) = 0$, 其中 $U_{X_0}^0(X_0)$ 和 $U_{X_{\epsilon(t)}}^0(X_{\epsilon(t)})$ 以及纯型的定义参见第 2.1 小节中定义 2.4, 局部延拓算子 $e^{\epsilon(t)|\overline{\epsilon(t)}}$ 的定义参见第 3.1 小节中定义 3.1.

本文余下内容的结构如下: 第 2 节简要介绍紧致广义 Hermite 流形及其形变的预备知识, 并且求解一个广义的 $\bar{\partial}_H$ - 方程. 第 3 节简要介绍关于形变的局部延拓算子及其相关的性质, 并进一步给出本文需要用到的记号与计算. 第 4 节证明本文的主要结果 (定理 1.1).

英文引用格式: Wei K. Local extensions of d_H -closed real forms of pure type on compact generalized Hermitian manifolds (in Chinese). Sci Sin Math, 2021, 51: 1163–1190, doi: 10.1360/SSM-2019-0209

2 预备知识

2.1 广义 Hermite 流形

本小节介绍紧致广义 Hermite 流形的基本定义和性质, 具体内容可参见文献 [3–7]. 首先, 设 X^{2n} 为一个光滑流形, 其实维数为 $\dim_{\mathbb{R}} X = 2n$. 在 X 的切空间直和余切空间 $T_X \oplus T_X^*$ 上, 我们介绍以下配对 $\langle \cdot, \cdot \rangle$.

定义 2.1 ^[3] 设 $a := Y + \xi, b := Z + \eta \in T_X \oplus T_X^*$, 其中 $Y, Z \in T_X, \xi, \eta \in T_X^*$. 定义

$$\langle a, b \rangle := \frac{1}{2}(\xi(Z) + \eta(Y)).$$

其次, 我们介绍 X 上的广义复结构.

定义 2.2 ^[5] 设 X^{2n} 为一个光滑流形, 其实维数为 $\dim_{\mathbb{R}} X = 2n$. 称 $T_X \oplus T_X^*$ 上满足

$$J^2 = -1, \quad \langle a, b \rangle = \langle Ja, Jb \rangle, \quad \text{其中 } a, b \in T_X \oplus T_X^*$$

的自同构 J 为 X 上的一个广义近复结构. 为了介绍广义复结构, 我们还需要以下的说明. 由于 $J^2 = -1$, 我们可以将 $T_X \oplus T_X^*$ 分解为关于 J 的特征值为 $\pm i$ 的特征子空间:

$$T_X \oplus T_X^* = L + \bar{L},$$

其中 L 为特征值为 $+i$ 的特征子空间. 对于某个 X 上的上同调元 $H \in H^3(M, \mathbb{R})$, 定义以下 H - 扭 Courant 括号 $[\cdot, \cdot]_H$: 设 $a = Y + \xi, b = Z + \eta \in T_X \oplus T_X^*$, 其中 $Y, Z \in T_X, \xi, \eta \in T_X^*$,

$$\begin{aligned} [a, b]_H &:= [Y + \xi, Z + \eta]_H \\ &:= [Y, Z] + \mathcal{L}_Y \eta - \mathcal{L}_Z \xi - \frac{1}{2}d(i_Y \eta - i_Z \xi) + i_Z i_Y H, \end{aligned}$$

其中 $[Y, Z] := YZ - ZY, \mathcal{L}_Y \eta := d \circ i_Y \eta + i_Y \circ d\eta, i_Y$ 表示关于切向量 Y 的缩并. 若 J 为 X 上一个近复结构, 且关于 J 特征值为 $+i$ 的特征子空间 L 满足 $[L, L]_H \subset L$, 则称 J 为 X 上的一个广义复结构.

再次, 我们要介绍 $T_X \oplus T_X^*$ 到 X 上的光滑形式丛 $\wedge^\bullet T_X^*$ 的 Clifford 作用.

定义 2.3 ^[5] 设 $\sigma \in \wedge^\bullet T_X^*, a := Y + \xi \in T_X \oplus T_X^*$, 其中 $Y \in T_X, \xi \in T_X^*$, 定义以下 Clifford 作用:

$$a\sigma := (Y + \xi)\sigma := i_Y \sigma + \xi \wedge \sigma.$$

通过直接计算, 有

$$ab\sigma + ba\sigma = 2\langle a, b \rangle \sigma, \tag{2.1}$$

其中 $a, b \in L \subset T_X \oplus T_X^*$. 若 J 为 X 上广义复结构, 则 $\langle \bar{L}, \bar{L} \rangle = 0$. 所以,

$$ab\sigma + ba\sigma = 0.$$

所以, 我们可以将关于 $\bar{L}$ 的 Clifford 作用写成外代数的形式, 记作 $\wedge^\bullet \bar{L}$, 且有

$$ab\sigma = (-1)^{pq}ba\sigma,$$

其中 $a \in \wedge^p \bar{L}$, $b \in \wedge^q \bar{L}$. 更进一步, 还是因为 $\langle L, L \rangle = 0$, 我们还可以将 $\bar{L}$ 与 L 关于上述配对 $\langle \cdot, \cdot \rangle$ 的对偶空间 L^* 等同. Courant 括号也可以通过以下定义推广到 $\wedge^\bullet L^* = \wedge^\bullet \bar{L}$ 上:

$$[a, b]_H := \sum_{i,j} (-1)^{i+j} [a_i, b_j]_H a_1 \cdots \hat{a}_i \cdots a_p b_1 \cdots \hat{b}_j \cdots b_q,$$

其中 $a = a_1 \cdots a_p \in \wedge^p \bar{L}$, $b = b_1 \cdots b_q \in \wedge^q \bar{L}$, $\hat{a}_i$ 表示省略 a_i .

利用 Clifford 作用, 以下介绍 $\wedge^\bullet T_X^*$ 上 Clifford 分解.

定义 2.4 (参见文献 [4, 第 108 页]) 利用 Clifford 作用, 令

$$\begin{aligned} U_X^{-n} &:= \{\rho \in \wedge^\bullet T_X^* \mid L\rho = 0\}, \\ U_X^p &:= \wedge^{p+n} \bar{L} \cdot U_X^{-n}. \end{aligned}$$

我们还知道 U_X^p 是关于 J 特征值为 $(-p)i$ 的特征子空间, 并且有以下关于 $\wedge^\bullet T_X^*$ 的 Clifford 分解:

$$\wedge^\bullet T_X^* = U_X^{-n} \oplus U_X^{-n+1} \oplus \cdots \oplus U_X^n,$$

其中 $n = \dim_{\mathbb{C}} X$. 我们又记 $U_X^p(X)$ 为 U_X^p 在 X 上的整体截面. 对于任意的 $\sigma \in \wedge^\bullet T_X^*$, 如果 $\sigma \in U_X^p$ 仅属于上述某个 U_X^p , 则称 σ 为纯型的.

接下来, 我们要定义以下 $\wedge^\bullet T_X^*$ 上关于某个 $H \in H^3(X, \mathbb{R})$ 的 H -扭 de Rham 微分算子 d_H .

定义 2.5 [5, 8] 定义

$$\begin{aligned} d_H &: \wedge^\bullet T_X^* \rightarrow \wedge^\bullet T_X^*, \\ \sigma &\mapsto d\sigma - H \wedge \sigma. \end{aligned}$$

另外, 我们还有 H -扭 Dolbeault 微分算子 ∂_H 和 $\bar{\partial}_H$:

$$\begin{aligned} \partial_H &:= \pi_{p-1} \circ d_H : U_X^p \rightarrow U_X^{p-1}, \\ \bar{\partial}_H &:= \pi_{p+1} \circ d_H : U_X^p \rightarrow U_X^{p+1}, \end{aligned}$$

其中 π_p 为到 U_X^p 上的投射. 已经知道 J 为广义复结构当且仅当 $d_H = \partial_H + \bar{\partial}_H$ (参见文献 [5, 第 51 页]). 我们还知道 $(\wedge^\bullet T_X^*, \bar{\partial}_H)$ 为一个复形. 还需要介绍以下复形 $(\wedge^\bullet \bar{L}, d_L)$.

定义 2.6 [9] 定义 Lie 导数 d_L 如下:

$$\begin{aligned} d_L &: \wedge^p \bar{L} \rightarrow \wedge^{p+1} \bar{L}, \\ a &\mapsto d_L a, \end{aligned}$$

其中

$$\begin{aligned} (d_L a)(x_0, \dots, x_p) &:= \sum_k (-1)^k \pi_T(x_k) a(x_0, \dots, \hat{x}_k, \dots, x_p) \\ &\quad + \sum_{k < l} (-1)^{k+l} a([x_k, x_l]_H, x_0, \dots, \hat{x}_k, \dots, \hat{x}_l, \dots, x_p), \end{aligned}$$

$a \in \wedge^p \bar{L}$, $x_k \in L$ ($0 \leq k \leq p$), $\pi_T : L \rightarrow T_X$ 为投射. 关于上述两个复形 $(\wedge^\bullet T_X^*, \bar{\partial}_H)$ 和 $(\wedge^\bullet \bar{L}, d_L)$, 有以下关系:

$$\bar{\partial}_H(a\sigma) = (d_L a)\sigma + (-1)^p a(\bar{\partial}_H \sigma),$$

其中 $a \in \wedge^p \bar{L}$, $\sigma \in \wedge^\bullet T_X^*$.

以下, 我们再介绍一些作用在 $\wedge^\bullet T_X^*(X)$ 上的算子. 在 $\wedge^\bullet T_X^*(X)$ 上, 有由广义 Hermite 度量 G 所诱导的以下 Born-Infeld 内积 $(\cdot, \cdot)$ (参见文献 [3]):

$$\begin{aligned}(\alpha, \beta) &:= \int_X \alpha \wedge \sigma(*)\bar{\beta}, \\ |\alpha|^2 &:= (\alpha, \alpha),\end{aligned}$$

其中 $* := a_1 \cdots a_{2n}$ 为广义 Hodge- $*$ 算子, $\{a_i\}_{i=1}^{2n}$ 为关于 G 特征值为 $+1$ 的特征子空间 C_+ 的一组定向基, 且这组基关于 G 标准正交, $\sigma(a_1 \cdots a_{2n}) := a_{2n} \cdots a_1$. 记 d_H^* 、 ∂_H^* 和 $\bar{\partial}_H^*$ 分别为 d_H 、 ∂_H 和 $\bar{\partial}_H$ 关于上述 Born-Infeld 内积 $(\cdot, \cdot)$ 的伴随算子, 即对任意的 $\alpha, \beta \in \wedge^\bullet T_X^*(X)$, 有

$$\begin{aligned}(\partial_H \alpha, \beta) &= (\alpha, \partial_H^* \beta), \\ (\bar{\partial}_H \alpha, \beta) &= (\alpha, \bar{\partial}_H^* \beta), \\ (d_H \alpha, \beta) &= (\alpha, d_H^* \beta).\end{aligned}$$

定义以下广义 Laplace 算子:

$$\begin{aligned}\Delta_{d_H} &:= d_H d_H^* + d_H^* d_H, \\ \Delta_{\partial_H} &:= \partial_H \partial_H^* + \partial_H^* \partial_H, \\ \Delta_{\bar{\partial}_H} &:= \bar{\partial}_H \bar{\partial}_H^* + \bar{\partial}_H^* \bar{\partial}_H.\end{aligned}$$

我们知道 $\Delta_{\bar{\partial}_H}$ 关于上述内积 $(\cdot, \cdot)$ 是自伴随的, 且有以下广义 Hodge 分解:

$$1 = \mathbb{H}_{\bar{\partial}_H} + G_{\bar{\partial}_H} \Delta_{\bar{\partial}_H} = \mathbb{H}_{\bar{\partial}_H} + \Delta_{\bar{\partial}_H} G_{\bar{\partial}_H},$$

其中 $\mathbb{H}_{\bar{\partial}_H}$ 为到 $H_{\bar{\partial}_H}^\bullet(X)$ 上的投射, $G_{\bar{\partial}_H}$ 为对应于 $\Delta_{\bar{\partial}_H}$ 的 Green 算子. 对于 Δ_{d_H} 和 Δ_{∂_H} , 也有类似的结论. 最后, 我们还有以下广义 Bott-Chern Laplace 算子^[10–13].

定义 2.7 定义广义 Bott-Chern Laplace 算子为

$$\begin{aligned}\Delta_{H,BC} &:= (\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* + (\partial_H \bar{\partial}_H)^*(\partial_H \bar{\partial}_H) + (\bar{\partial}_H^* \partial_H)(\bar{\partial}_H^* \partial_H)^* + (\bar{\partial}_H^* \partial_H)^*(\bar{\partial}_H^* \partial_H) \\ &\quad + \bar{\partial}_H^* \bar{\partial}_H + \partial_H^* \partial_H.\end{aligned}$$

2.2 紧致广义 Hermite 流形形变的性质

本小节介绍紧致广义 Hermite 流形形变的性质, 具体内容可参见文献 [3, 5, 9, 14–17]. 首先, 关于广义复结构的形变, 由 Kodaira-Spencer-Kuranishi 理论^[18], 对于任意的紧致广义复流形 X , 存在一个包含 X 上所有广义复结构形变的完备全纯族 $\pi: \mathcal{X} \rightarrow \mathcal{T}$, 其中 $\pi^{-1}(0) := X_0 := X$. 任意 X 的局部形变 X_ϵ 都可以由一个可积的广义 Beltrami 微分 $\epsilon \in \wedge^2 \bar{L}(X_0)$ 来表示. 此外, $\mathcal{T}$ 在 $t=0$ 处的全纯切空间有以下同构关系:

$$\begin{aligned}T_{\mathcal{T},0}^{1,0} &\cong H_{d_L}^2(X, \wedge^\bullet \bar{L}), \\ \frac{\partial}{\partial t^i} \Big|_{t=0} &\mapsto \epsilon_i,\end{aligned}$$

其中 $T_{\mathcal{T},0}^{1,0}$ 为 $\mathcal{T}$ 在 $t=0$ 处的全纯切空间, $H_{d_L}^2(X, \wedge^\bullet \bar{L})$ 为复形 $(\wedge^\bullet \bar{L}, d_L)$ 的上同调群, $\{\epsilon_i\}_{i=1}^N$ 为 $H_{d_L}^2(X, \wedge^\bullet \bar{L})$ 的一组基. 为简便起见且不失一般性, 我们可以假设 $\dim_{\mathbb{C}} H_{d_L}^2(X, \wedge^\bullet \bar{L}) = 1$. 所以, 任意 X 的局部形变所对应的广义 Beltrami 微分 $\epsilon \in \wedge^2 \bar{L}(X)$ 可以写作

$$\epsilon = \epsilon(t) := \sum_{\alpha=1}^{\infty} \epsilon_{\alpha} t^{\alpha}, \quad \epsilon_1 \in H_{d_L}^2(X_0, \wedge^\bullet \bar{L}).$$

2.3 广义 $\bar{\partial}_H$ -方程的求解

本小节要利用广义 Hodge 理论, 求解一个本文中需要用到的广义 $\bar{\partial}_H$ -方程. 首先, 我们列出广义 $\partial_H \bar{\partial}_H$ -引理^[3,4]:

定义 2.8 若

$$\text{Im}(\partial_H) \cap \text{Ker}(\bar{\partial}_H) = \text{Im}(\bar{\partial}_H) \cap \text{Ker}(\partial_H) = \text{Im}(\partial_H \bar{\partial}_H),$$

则称 (X, J) 满足广义 $\partial_H \bar{\partial}_H$ -引理.

接着, 受到文献 [1,2] 的启发, 我们要求解一个本文需要用到的广义 $\bar{\partial}_H$ -方程.

引理 2.1 假设 X 满足广义 $\partial_H \bar{\partial}_H$ -引理. 对任意的 $a \in U_X^{-n+p-2}(X)$, $b \in U_X^{n-p-2}(X)$, 若 $\partial_H \bar{\partial}_H a = \partial_H \bar{\partial}_H b = 0$, 则方程组

$$\begin{cases} \partial_H \sigma = \bar{\partial}_H a, \\ \bar{\partial}_H \sigma = \partial_H b \end{cases} \quad (2.2)$$

有解

$$\sigma = \bar{\partial}_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H a - \partial_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H b \in U_X^{-n+p}(X), \quad (2.3)$$

其中 $G_{H,BC}$ 为对应于广义 Bott-Chern Laplace 算子 $\Delta_{H,BC}$ 的 Green 算子.

证明 由广义 Bott-Chern Laplace $\Delta_{H,BC}$ 的定义, 以及 $\partial_H^2 = \bar{\partial}_H^2 = \partial_H \bar{\partial}_H + \bar{\partial}_H \partial_H = 0$, 有

$$\Delta_{H,BC}(\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* = (\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^*(\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* = (\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* \Delta_{H,BC}.$$

由于 $I = \mathbb{H}_{H,BC} + \Delta_{H,BC} G_{H,BC} = \mathbb{H}_{H,BC} + G_{H,BC} \Delta_{H,BC}$ (参见文献 [19, 引理 2.13]), 所以,

$$\begin{aligned} G_{H,BC} \Delta_{H,BC}(\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* G_{H,BC} \\ &= G_{H,BC}(\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* \Delta_{H,BC} G_{H,BC} \\ &= (I - \mathbb{H}_{H,BC})(\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* G_{H,BC} \\ &= G_{H,BC}(\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^*(I - \mathbb{H}_{H,BC}), \end{aligned}$$

其中 I 为恒同映射, $\mathbb{H}_{H,BC}$ 为到广义 Bott-Chern 上同调群的投射. 又由于广义 Bott-Chern Laplace $\Delta_{H,BC}$ 的定义, 我们有 $\text{Im} \partial_H \bar{\partial}_H \subset \text{Im} \Delta_{H,BC}$ (参见文献 [19, 引理 2.13]), 所以, 在 $\text{Im} \partial_H \bar{\partial}_H$ 上, 有

$$(\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* G_{H,BC} \big|_{\text{Im} \partial_H \bar{\partial}_H} = G_{H,BC}(\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* \big|_{\text{Im} \partial_H \bar{\partial}_H}.$$

仍然由广义 Bott-Chern Laplace 的定义, 有

$$G_{H,BC} \Delta_{H,BC} \big|_{\text{Im} \partial_H \bar{\partial}_H} = G_{H,BC}(\partial_H \bar{\partial}_H)(\partial_H \bar{\partial}_H)^* \big|_{\text{Im} \partial_H \bar{\partial}_H}.$$

由于 $\partial_H \bar{\partial}_H a = \partial_H \bar{\partial}_H b = 0$, X 满足广义 $\partial_H \bar{\partial}_H$ - 引理, 我们可以得到

$$\bar{\partial}_H a, \bar{\partial}_H b \in \text{Ker} \partial_H \cap \text{Im} \bar{\partial}_H = \text{Im} \partial_H \bar{\partial}_H.$$

所以,

$$\begin{aligned} \partial_H \sigma &= \partial_H \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H a - \partial_H \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \bar{b} \\ &= \partial_H \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H a \\ &= G_{H,BC} \partial_H \bar{\partial}_H (\partial_H \bar{\partial}_H)^* \bar{\partial}_H a \\ &= G_{H,BC} \Delta_{H,BC} \bar{\partial}_H a \\ &= (1 - \mathbb{H}_{H,BC}) \bar{\partial}_H a \\ &= \bar{\partial}_H a, \end{aligned}$$

其中第 4 个和最后一个等式成立是因为 $\bar{\partial}_H a \in \text{Im} \partial_H \bar{\partial}_H$, 第 5 个等式成立是因为广义 Hodge 分解, $\mathbb{H}_{H,BC}$ 表示到 $\text{Ker} \Delta_{H,BC}$ 上的投射. 类似地,

$$\begin{aligned} \bar{\partial}_H \sigma &= \bar{\partial}_H \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H a - \bar{\partial}_H \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \bar{b} \\ &= \partial_H \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \bar{b} \\ &= G_{H,BC} \partial_H \bar{\partial}_H (\partial_H \bar{\partial}_H)^* \partial_H \bar{b} \\ &= G_{H,BC} \Delta_{H,BC} \partial_H \bar{b} \\ &= (1 - \mathbb{H}_{H,BC}) \partial_H \bar{b} \\ &= \partial_H \bar{b}. \end{aligned}$$

证毕. □

3 一些记号与引理

3.1 局部延拓算子

本小节将定义对应于定义 2.4 中 Clifford 分解, 关于形变的局部延拓算子, 给出它的一些性质. 首先, 我们要定义关于形变的以下局部延拓算子.

定义 3.1 ^[19–23] 定义局部延拓算子为

$$\begin{aligned} e^{\epsilon|\bar{\epsilon}} : U_{X_0}^{-n+p}(X_0) &\rightarrow U_{X_\epsilon}^{-n+p}(X_\epsilon), \quad p = 0, \dots, 2n, \\ \sigma &\mapsto e^{\epsilon|\bar{\epsilon}}(\sigma), \end{aligned}$$

其中

$$\begin{aligned} \sigma &:= \frac{1}{p!} \sigma_{i_1 \dots i_p} l^{i_1} \dots l^{i_p} \rho, \quad l^{i_\alpha} \in \bar{L}(X_0), \quad \rho \in U_{X_0}^{-n}(X_0), \\ e^{\epsilon|\bar{\epsilon}}(\sigma) &:= \frac{1}{p!} \sigma_{i_1 \dots i_p} (1 + \bar{\epsilon})(l^{i_1}) \dots (1 + \bar{\epsilon})(l^{i_p})(e^\epsilon \rho), \\ e^\epsilon \rho &:= \sum_{k \geq 0} \frac{1}{k!} \epsilon^k \rho, \quad \epsilon^k := \overbrace{\epsilon \epsilon \dots \epsilon}^k. \end{aligned}$$

由于 $\epsilon\bar{\epsilon} \neq \bar{\epsilon}\epsilon$, 换句话说, $e^{\epsilon|\bar{\epsilon}} \neq e^{\bar{\epsilon}\epsilon}$, 所以, 我们有以下引理:

引理 3.1 ^[2, 19] 对于任意的广义 Beltrami 微分 ϵ , 以及光滑微分形式的整体截面 σ , 有

$$e^{-\epsilon}e^{\epsilon|\bar{\epsilon}}\sigma = e^{\bar{\epsilon}-\epsilon\bar{\epsilon}}\sigma,$$

其中

$$\begin{aligned} e^{\bar{\epsilon}-\epsilon\bar{\epsilon}}\left(\frac{1}{p!}\sigma_{i_1\dots i_p}l^{i_1}\dots l^{i_p}\rho\right) &:= \frac{1}{p!}\sigma_{i_1\dots i_p}(1+\bar{\epsilon}-\epsilon\bar{\epsilon})(l^{i_1})\dots(1+\bar{\epsilon}-\epsilon\bar{\epsilon})(l^{i_p})(e^{\bar{\epsilon}-\epsilon\bar{\epsilon}}\rho) \\ &= \frac{1}{p!}\sigma_{i_1\dots i_p}(1+\bar{\epsilon}-\epsilon\bar{\epsilon})(l^{i_1})\dots(1+\bar{\epsilon}-\epsilon\bar{\epsilon})(l^{i_p})\rho \quad (\text{因为 } \bar{\epsilon}\rho = 0). \end{aligned}$$

若假定 $|\epsilon|_{L^\infty} < 1$, 则 $(1-\epsilon\bar{\epsilon})^{-1}$ 存在且有以下引理:

引理 3.2 ^[2, 19] 对于任意充分小的广义 Beltrami 微分 ϵ , 以及光滑微分形式的整体截面 σ , 有

$$e^{-\epsilon}e^{-\bar{\epsilon}}e^{\epsilon}\sigma = e^{-1-\bar{\epsilon}+(1-\epsilon\bar{\epsilon})^{-1}}\sigma,$$

其中

$$\begin{aligned} e^{-1-\bar{\epsilon}+(1-\epsilon\bar{\epsilon})^{-1}}\left(\frac{1}{p!}\sigma_{i_1\dots i_p}l^{i_1}\dots l^{i_p}\rho\right) \\ &:= \frac{1}{p!}\sigma_{i_1\dots i_p}(-\bar{\epsilon}+(1-\epsilon\bar{\epsilon})^{-1})(l^{i_1})\dots(-\bar{\epsilon}+(1-\epsilon\bar{\epsilon})^{-1})(l^{i_p})(e^{-1-\bar{\epsilon}+(1-\epsilon\bar{\epsilon})^{-1}}\rho) \\ &= \frac{1}{p!}\sigma_{i_1\dots i_p}(-\bar{\epsilon}+(1-\epsilon\bar{\epsilon})^{-1})(l^{i_1})\dots(-\bar{\epsilon}+(1-\epsilon\bar{\epsilon})^{-1})(l^{i_p})\rho \quad (\text{因为 } \bar{\epsilon}\rho = 0). \end{aligned}$$

3.2 记号与引理

本小节进一步给出一些本文需要用到的记号和引理. 对任意的 X 上光滑形式 σ , 令

$$\tilde{\sigma} := e^{-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}}e^{-\epsilon}e^{\epsilon|\bar{\epsilon}}(\sigma).$$

我们可以说明, 若 $\sigma \in U_{X_0}^{-n+p}(X_0)$, 则仍然有 $\tilde{\sigma} \in U_{X_0}^{-n+p}(X_0)$. 事实上,

$$\begin{aligned} \tilde{\sigma} &:= e^{-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}}e^{-\epsilon}e^{\epsilon|\bar{\epsilon}}(\sigma) \\ &= e^{-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}}e^{-\epsilon}e^{\epsilon|\bar{\epsilon}}\left(\frac{1}{p!}\sigma_{i_1\dots i_p}l^{i_1}\dots l^{i_p}\rho\right) \\ &= e^{-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}}e^{-\epsilon}\left(\frac{1}{p!}\sigma_{i_1\dots i_p}(1+\bar{\epsilon})l^{i_1}\dots(1+\bar{\epsilon})l^{i_p}e^{\epsilon}\rho\right) \\ &= e^{-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}}\left(\frac{1}{p!}\sigma_{i_1\dots i_p}(1-\epsilon)(1+\bar{\epsilon})l^{i_1}\dots(1-\epsilon)(1+\bar{\epsilon})l^{i_p}e^{-\epsilon}e^{\epsilon}\rho\right) \\ &= e^{-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}}\left(\frac{1}{p!}\sigma_{i_1\dots i_p}(1-\epsilon\bar{\epsilon}+\bar{\epsilon})l^{i_1}\dots(1-\epsilon\bar{\epsilon}+\bar{\epsilon})l^{i_p}\rho\right) \\ &= \frac{1}{p!}\sigma_{i_1\dots i_p}(1-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1})(1-\epsilon\bar{\epsilon}+\bar{\epsilon})l^{i_1}\dots(1-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1})(1-\epsilon\bar{\epsilon}+\bar{\epsilon})l^{i_p}e^{-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}}\rho \\ &= \frac{1}{p!}\sigma_{i_1\dots i_p}(1-\epsilon\bar{\epsilon})l^{i_1}\dots(1-\epsilon\bar{\epsilon})l^{i_p}\rho \\ &= e^{-\epsilon\bar{\epsilon}}\sigma, \end{aligned} \tag{3.1}$$

其中第 5 个等式成立是因为 $(1 - \epsilon)(1 + \bar{\epsilon}) = (1 - \epsilon\bar{\epsilon} + \bar{\epsilon}) + \epsilon$, $\epsilon\bar{L} = 0$; 第 7 个等式成立是因为 $\bar{\epsilon}\bar{L} = 0$, $(1 - \epsilon\bar{\epsilon})^{-1}\bar{\epsilon} = \bar{\epsilon}(1 - \epsilon\bar{\epsilon})^{-1}$, $\bar{\epsilon}\rho = 0$. 又令

$$e^{\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}}\tilde{\sigma} := \sum_{k=0}^{\infty} A_k,$$

其中

$$A_k := \frac{1}{k!}(\bar{\epsilon}(1 - \epsilon\bar{\epsilon})^{-1})^k \tilde{\sigma} = \frac{1}{k!} \left(\sum_{i=0}^{\infty} \bar{\epsilon}(\epsilon\bar{\epsilon})^i \right)^k \tilde{\sigma} \in U_{X_0}^{-n+p-2k}(X_0).$$

引理 3.3 假定 $\sigma \in U_{X_0}^{-n+p}(X_0)$, 则

$$d_H e^{\epsilon|\bar{\epsilon}} \sigma = e^{\epsilon} \left(\bar{\partial}_{H,\epsilon} A_0 + \sum_{k=0}^{\infty} (\partial_H A_k + \bar{\partial}_{H,\epsilon} A_{k+1}) \right),$$

其中 $\bar{\partial}_{H,\epsilon} := \bar{\partial}_H + \{\partial_H, \epsilon\}$, $\{\partial_H, \epsilon\} := \partial_H \circ \epsilon - \epsilon \circ \partial_H$.

证明 本文中, 我们均假定 $\epsilon^{-k} := 0$, $A_{-k} := 0$ ($k > 0$), $\epsilon^0 = 1$, $0! = 1$. 通过直接计算, 有

$$\begin{aligned} d_H e^{\epsilon|\bar{\epsilon}} \sigma &= d_H e^{\epsilon} e^{\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}} e^{-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}} e^{-\epsilon} e^{\epsilon|\bar{\epsilon}} \sigma \\ &= e^{\epsilon} (\bar{\partial}_H + \{\partial_H, \epsilon\} + \partial_H) e^{\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}} \tilde{\sigma} \\ &= e^{\epsilon} (\bar{\partial}_{H,\epsilon} + \partial_H) e^{\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}} \tilde{\sigma} \\ &= e^{\epsilon} (\bar{\partial}_{H,\epsilon} + \partial_H) \sum_{k=0}^{\infty} \frac{1}{k!} (\bar{\epsilon}(1 - \epsilon\bar{\epsilon})^{-1})^k \tilde{\sigma} \\ &= e^{\epsilon} (\bar{\partial}_{H,\epsilon} + \partial_H) \sum_{k=0}^{\infty} A_k \\ &= e^{\epsilon} \left(\bar{\partial}_{H,\epsilon} A_0 + \sum_{k=0}^{\infty} (\partial_H A_k + \bar{\partial}_{H,\epsilon} A_{k+1}) \right), \end{aligned}$$

其中第 2 个等式成立是因为 $e^{-\epsilon} d_H e^{\epsilon} = d_H + \{\partial_H, \epsilon\} + \frac{1}{2}[\epsilon, \epsilon]_H$ (参见文献 [16, 22]) 以及可积性条件 $\{\bar{\partial}_H, \epsilon\} + \frac{1}{2}[\epsilon, \epsilon]_H = 0$ (参见文献 [5, 第 62 页, (5.2)]). $\square$

因此, 通过比较在 Clifford 分解下光滑形式的类型, 有

$$d_H e^{\epsilon|\bar{\epsilon}} \sigma = 0 \Leftrightarrow \begin{cases} \bar{\partial}_{H,\epsilon} A_0 = 0 \in U_{X_0}^{-n+p+1}(X_0), \\ \partial_H A_k + \bar{\partial}_{H,\epsilon} A_{k+1} = 0 \in U_{X_0}^{-n+p-1-2k}(X_0), \quad k \geq 0. \end{cases} \quad (3.2)$$

若 $\sigma := \sigma(t)$, 则 $\epsilon := \epsilon(t)$ 写作关于 t 的幂级数形式, 即

$$\begin{aligned} \sigma(t) &:= \sum_{\alpha=0}^{\infty} \sigma_{\alpha} t^{\alpha} \in U_{X_0}^{-n+p}(X_0), \\ d_H e^{\epsilon(t)|\bar{\epsilon}(t)} \sigma(t) &:= \sum_{\alpha=0}^{\infty} (d_H e^{\epsilon(t)|\bar{\epsilon}(t)} \sigma(t))_{\alpha} t^{\alpha}. \end{aligned}$$

我们可以将上述引理改写为以下幂级数形式的版本:

引理 3.4 对于任意的 $N \in \mathbb{N} \cup \{0\}$, 若

$$(d_H e^{\epsilon(t) | \bar{\epsilon}(t)} \sigma(t))_\alpha = 0 \quad (\alpha = 0, 1, \dots, N),$$

则

$$\begin{aligned} (\bar{\partial}_{H, \epsilon(t)} A_0(t))_\alpha &= 0 \in U_{X_0}^{-n+p+1}(X_0), \\ (\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t))_\alpha &= 0 \in U_{X_0}^{-n+p+1-2k}(X_0) \quad (k \geq 0, \quad \alpha = 0, 1, \dots, N). \end{aligned}$$

证明 由上述引理 3.3, 有

$$e^{-\epsilon(t)} d_H e^{\epsilon(t) | \bar{\epsilon}(t)} \sigma(t) = \bar{\partial}_{H, \epsilon(t)} A_0(t) + \sum_{k=0}^{\infty} (\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t)).$$

所以, 对任意的 $\alpha = 0, 1, \dots, N$, 有

$$\begin{aligned} (e^{-\epsilon(t)} d_H e^{\epsilon(t) | \bar{\epsilon}(t)} \sigma(t))_\alpha &= \sum_{\alpha_1 + \alpha_2 = \alpha, \alpha_1, \alpha_2 \geq 0} (e^{-\epsilon(t)})_{\alpha_1} (d_H e^{\epsilon(t) | \bar{\epsilon}(t)} \sigma(t))_{\alpha_2} \\ &= (\bar{\partial}_{H, \epsilon(t)} A_0(t))_\alpha + \sum_{k=0}^{\infty} (\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t))_\alpha. \end{aligned}$$

所以, 若

$$(d_H e^{\epsilon(t) | \bar{\epsilon}(t)} \sigma(t))_\alpha = 0 \quad (\alpha = 0, 1, \dots, N),$$

则通过比较 t 的次数 α , 有

$$\begin{aligned} (\bar{\partial}_{H, \epsilon(t)} A_0(t))_\alpha &= 0, \\ (\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t))_\alpha &= 0 \quad (\alpha = 0, 1, \dots, N, \quad k \geq 0). \end{aligned}$$

证毕. □

引理 3.5 假设 $\sigma \in U_{X_0}^{-n+p}(X_0)$, 则

$$0 = \sum_{r,s=0}^{\infty} \left(\partial_H \frac{\epsilon^s}{s!} + \bar{\partial}_H \frac{\epsilon^{s-1}}{(s-1)!} \right) A_r \Rightarrow d_H e^{\epsilon | \bar{\epsilon}} \sigma = 0.$$

证明 直接计算, 可得

$$\begin{aligned} d_H e^{\epsilon | \bar{\epsilon}} \sigma &= d_H e^{\epsilon} e^{\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}} e^{-\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}} e^{-\epsilon} e^{\epsilon | \bar{\epsilon}} \sigma \\ &= d_H e^{\epsilon} e^{\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}} \tilde{\sigma} \\ &= \partial_H e^{\epsilon} e^{\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}} \tilde{\sigma} + \bar{\partial}_H e^{\epsilon} e^{\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1}} \tilde{\sigma} \\ &= \sum_{k_1, k_2=0}^{\infty} \partial_H \frac{\epsilon^{k_1}}{k_1!} \frac{(\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1})^{k_2}}{k_2!} \tilde{\sigma} + \sum_{l_1, l_2=0}^{\infty} \bar{\partial}_H \frac{\epsilon^{l_1}}{l_1!} \frac{(\bar{\epsilon}(1-\epsilon\bar{\epsilon})^{-1})^{l_2}}{l_2!} \tilde{\sigma} \\ &= \sum_{k_1, k_2=0}^{\infty} \partial_H \frac{\epsilon^{k_1}}{k_1!} A_{k_2} + \sum_{l_1, l_2=0}^{\infty} \bar{\partial}_H \frac{\epsilon^{l_1}}{l_1!} A_{l_2} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k_1, k_2=0}^{\infty} \left(\partial_H \frac{\epsilon^{k_1}}{k_1!} A_{k_2} + \sum_{l=0}^{\infty} \bar{\partial}_H \frac{\epsilon^l}{l!} A_{l+1+(k_2-k_1)} \right) \\
 &= \sum_{k_1, k_2=0}^{\infty} \left(\partial_H \frac{\epsilon^{k_1}}{k_1!} + \bar{\partial}_H \frac{\epsilon^{k_1-1}}{(k_1-1)!} \right) A_{k_2} \quad (\text{令 } l := k_1 - 1),
 \end{aligned}$$

其中第 6 个等式成立是因为比较光滑形式的类型且

$$\partial_H \frac{\epsilon^{k_1}}{k_1!} A_{k_2} + \sum_{l=0}^{\infty} \bar{\partial}_H \frac{\epsilon^l}{l!} A_{l+1+(k_2-k_1)} \in U_{X_0}^{-n+p-2(k_2-k_1)-1} \quad (\forall k_1, k_2 \geq 0).$$

所以, 结论得证. $\square$

4 主要定理

本节证明定理 1.1. 我们将使用平行于文献 [1, 2, 21] 中的迭代方法构造出我们需要的 $\Omega(t)$. 该方法起源于文献 [24], 并且发展于文献 [25–33]. 这里需要假定 $\Omega(t) \in U_{X_0}^{-p+n}(X_0)$ 为实的, 即 $\Omega(t) = \overline{\Omega(t)}$. 因为 $U_{X_0}^p = \overline{U_{X_0}^{-p}}$, 所以, $\Omega(t) \in U_{X_0}^0(X_0)$. 令

$$\begin{aligned}
 \Omega(t) &:= \sum_{\alpha=0}^{\infty} \Omega_{\alpha} t^{\alpha} \in U_{X_0}^0(X_0), \quad d_H \Omega_0 = 0, \\
 \widetilde{\Omega(t)} &:= e^{-\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} e^{-\epsilon(t)} e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) := \sum_{\alpha=0}^{\infty} \widetilde{\Omega}_{\alpha} t^{\alpha}, \\
 d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) &:= \sum_{\alpha=0}^{\infty} (d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t))_{\alpha} t^{\alpha}, \\
 A_k(t) &:= \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \\
 &:= \sum_{\alpha=0}^{\infty} (A_k(t))_{\alpha} t^{\alpha}.
 \end{aligned} \tag{4.1}$$

由上述引理 3.4 的证明, 有

$$d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) = \sum_{k_1, k_2=0}^{\infty} \partial_H \frac{\epsilon(t)^{k_1}}{k_1!} A_{k_2}(t) + \sum_{l_1, l_2=0}^{\infty} \bar{\partial}_H \frac{\epsilon(t)^{l_1}}{l_1!} A_{l_2}(t),$$

通过比较在 Clifford 分解下光滑形式的类型, 有

$$\begin{aligned}
 (d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t))|_{U_{X_0}^1(X_0)} &= \sum_{k_1=0}^{\infty} \partial_H \frac{\epsilon(t)^{k_1}}{k_1!} A_{k_1-1}(t) + \sum_{l_1=0}^{\infty} \bar{\partial}_H \frac{\epsilon(t)^{l_1}}{l_1!} A_{l_1}(t) \\
 &= \sum_{k=0}^{\infty} \left(\partial_H \frac{\epsilon(t)^k}{k!} + \bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} \right) A_{k-1}(t) \quad (\text{令 } k := k_1 = l_1 + 1), \\
 (d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t))|_{U_{X_0}^{-1}(X_0)} &= \sum_{k_1=0}^{\infty} \partial_H \frac{\epsilon(t)^{k_1}}{k_1!} A_{k_1}(t) + \sum_{l_1=0}^{\infty} \bar{\partial}_H \frac{\epsilon(t)^{l_1}}{l_1!} A_{l_1+1}(t) \\
 &= \sum_{k=0}^{\infty} \left(\partial_H \frac{\epsilon(t)^k}{k!} + \bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} \right) A_k(t) \quad (\text{令 } k := k_1 = l_1 + 1),
 \end{aligned}$$

其中 $\epsilon(t)^0 = 1$, $\epsilon(t)^{-k} := 0$, $A_{-k}(t) := 0$ ($k > 0$). 以下要证明在 $\Omega(t) \in U_{X_0}^0(X_0)$ 情形下, 我们可以减弱上述引理 3.4 的条件, 得到相同的结论.

引理 4.1 若

$$\begin{cases} (d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^1(X_0)} = \sum_{k=0}^{\infty} \left(\bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} + \partial_H \frac{\epsilon(t)^k}{(k)!} \right) A_{k-1}(t) = 0, \\ (d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^{-1}(X_0)} = \sum_{k=0}^{\infty} \left(\bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} + \partial_H \frac{\epsilon(t)^k}{(k)!} \right) A_k(t) = 0, \end{cases} \quad (4.2)$$

则

$$d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) = 0. \quad (4.3)$$

证明 由引理 3.3, 可得

$$d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) = e^{\epsilon(t)} \left(\bar{\partial}_{H, \epsilon} A_0(t) + \sum_{k=0}^{\infty} (\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t)) \right),$$

则将右端第 2 项移到左端, 可得

$$e^{\epsilon(t)} (\bar{\partial}_{H, \epsilon(t)} A_0(t)) = d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) - e^{\epsilon(t)} \sum_{k=0}^{\infty} (\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t)),$$

其中 $\bar{\partial}_{H, \epsilon(t)} := \bar{\partial}_H + \{\partial_H, \epsilon(t)\}$. 因为 $e^{\epsilon(t)} : U_{X_0}^{-n+p}(X_0) \rightarrow \bigoplus_{i \geq 0} U_{X_0}^{-n+p+2i}(X_0)$ 且 $A_k(t) \in U_{X_0}^{-2k}(X_0)$, 所以通过比较在 Clifford 分解下光滑形式的类型, 有

$$\bar{\partial}_{H, \epsilon(t)} A_0(t) = (d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^1(X_0)} - \sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} (\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t)). \quad (4.4)$$

又因为

$$\begin{aligned} d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) &= e^{\epsilon(t) | \overline{\epsilon(t)}} e^{-\epsilon(t) | -\overline{\epsilon(t)}} e^{\epsilon(t)} e^{-\epsilon(t)} d_H e^{\epsilon(t)} e^{-\epsilon(t)} e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) \\ &= e^{\epsilon(t) | \overline{\epsilon(t)}} e^{-1-\epsilon(t)+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} e^{-\epsilon(t)} d_H e^{\epsilon(t)} e^{-\epsilon(t)\overline{\epsilon(t)}+\overline{\epsilon(t)}} \Omega(t), \end{aligned} \quad (4.5)$$

其中第 2 个等式成立是因为引理 3.1 和 3.2, 且

$$\begin{aligned} e^{-1+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} : U_{X_0}^{-n+p} &\rightarrow U_{X_0}^{-n+p}, \\ e^{-1-\overline{\epsilon(t)}+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} : U_{X_0}^{-n+p} &\rightarrow \bigoplus_{i \geq 0} U_{X_0}^{-n+p-2i}, \\ e^{-\epsilon(t)\overline{\epsilon(t)}} : U_{X_0}^{-n+p} &\rightarrow U_{X_0}^{-n+p}, \\ e^{\overline{\epsilon(t)}} : U_{X_0}^{-n+p} &\rightarrow \bigoplus_{i \geq 0} U_{X_0}^{-n+p-2i}, \\ \bar{\partial}_{H, \epsilon(t)} : U_{X_0}^{-n+p} &\rightarrow U_{X_0}^{-n+p+1}, \\ \partial_H : U_{X_0}^{-n+p} &\rightarrow U_{X_0}^{-n+p-1}, \end{aligned}$$

所以, 有

$$\bar{\partial}_H |_{X_{\epsilon(t)}} e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) \in U_{X_{\epsilon(t)}}^1(X_{\epsilon(t)})$$

$$\begin{aligned}
 &= (d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_{\epsilon(t)}}^1(X_{\epsilon(t)})} \\
 &= e^{\epsilon(t) | \overline{\epsilon(t)}} (e^{-1 - \overline{\epsilon(t)} + (1 - \epsilon(t) \overline{\epsilon(t)})^{-1}} \\
 &\quad \times (\bar{\partial}_{H, \epsilon(t)} + \partial_H) e^{-\epsilon(t) \overline{\epsilon(t)} + \overline{\epsilon(t)}} \Omega(t) |_{U_{X_0}^1(X_0)}) \\
 &= e^{\epsilon(t) | \overline{\epsilon(t)}} (e^{-1 + (1 - \epsilon(t) \overline{\epsilon(t)})^{-1}} \bar{\partial}_{H, \epsilon(t)} e^{-\epsilon(t) \overline{\epsilon(t)}} \Omega(t)) \\
 &= e^{\epsilon(t) | \overline{\epsilon(t)}} e^{-1 + (1 - \epsilon(t) \overline{\epsilon(t)})^{-1}} \bar{\partial}_{H, \epsilon(t)} A_0(t) \\
 &= e^{\epsilon(t) | \overline{\epsilon(t)}} e^{-1 + (1 - \epsilon(t) \overline{\epsilon(t)})^{-1}} \left((d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^1(X_0)} \right. \\
 &\quad \left. - \sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} (\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t)) \right),
 \end{aligned}$$

其中 $\bar{\partial}_H |_{X_{\epsilon(t)}}$ 表示 $X_{\epsilon(t)}$ 上的 $\bar{\partial}_H$ - 算子, 第 2 个等式成立是因为 (4.5), 第 3 个等式成立是因为上述对在 Clifford 分解下光滑形式类型的比较, 第 4 个等式成立是因为 (3.1) 和 $\widetilde{\Omega(t)} := A_0(t)$, 第 5 个等式成立是因为 (4.4). 由于假定 $\Omega(t)$ 是实的, $e^{\epsilon(t) | \overline{\epsilon(t)}}$ 和 $e^{-1 + (1 - \epsilon(t) \overline{\epsilon(t)})^{-1}}$ 均为实算子, 所以将上式复共轭, 可得

$$\begin{aligned}
 &\partial_H |_{X_{\epsilon(t)}} e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t) \in U_{X_{\epsilon(t)}}^{-1}(X_{\epsilon(t)}) \\
 &:= e^{\epsilon(t) | \overline{\epsilon(t)}} e^{-1 + (1 - \epsilon(t) \overline{\epsilon(t)})^{-1}} \left((d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^{-1}(X_0)} \right. \\
 &\quad \left. - \sum_{k=0}^{\infty} \frac{\overline{\epsilon(t)}^{k+1}}{(k+1)!} (\overline{\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t)}) \right).
 \end{aligned}$$

我们又假定

$$\begin{aligned}
 (d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^1(X_0)} &:= \sum_{\alpha=0}^{\infty} ((d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^1(X_0)})_{\alpha} t^{\alpha}, \\
 (d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^{-1}(X_0)} &:= \sum_{\alpha=0}^{\infty} ((d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^{-1}(X_0)})_{\alpha} t^{\alpha}.
 \end{aligned}$$

以下通过对 t 的次数 α 做归纳来证明结论. 对于 $\alpha = 0$, 有

$$(d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t))_0 = d_H \Omega_0 = 0.$$

对于 $\alpha \leq N$, 由

$$\begin{aligned}
 ((d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^1(X_0)})_{\alpha} &= 0, \\
 ((d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^{-1}(X_0)})_{\alpha} &= 0 \quad (\alpha = 0, 1, \dots, N),
 \end{aligned}$$

可推出

$$(d_H e^{\epsilon(t) | \overline{\epsilon(t)}} \Omega(t))_{\alpha} = 0 \quad (\alpha = 0, 1, \dots, N).$$

所以, 由引理 3.4, 有

$$(\bar{\partial}_{H, \epsilon} A_0(t))_{\alpha} = 0 \in U_{X_0}^{-n+p+1}(X_0),$$

$$(\partial_H A_k(t) + \bar{\partial}_{H,\epsilon(t)} A_{k+1}(t))_\alpha = 0 \in U_{X_0}^{-n+p-1-2k}(X_0) \quad (k \geq 0, \quad \alpha = 0, 1, \dots, N).$$

所以, 对于 $\alpha = N + 1$, 如果加上条件

$$\begin{aligned} ((d_H e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t)) |_{U_{X_0}^1(X_0)}_{N+1} &= 0, \\ ((d_H e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t)) |_{U_{X_0}^{-1}(X_0)}_{N+1} &= 0, \end{aligned}$$

则

$$\begin{aligned} & (\bar{\partial}_H |_{X_{\epsilon(t)}} e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t))_{N+1} \in U_{X_{\epsilon(t)}}^1(X_{\epsilon(t)}) \\ &= \left(e^{\epsilon(t)} | \overline{\epsilon(t)} e^{-1+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} \left((d_H e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t) \right) |_{U_{X_0}^1(X_0)} \right. \\ & \quad \left. - \sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon(t)} A_{k+1}(t)) \right)_{N+1} \\ &= \sum_{\alpha_1+\alpha_2=N+1, \alpha_1, \alpha_2 \geq 0} (e^{\epsilon(t)} | \overline{\epsilon(t)} e^{-1+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}})_{\alpha_1} ((d_H e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t)) |_{U_{X_0}^1(X_0)}_{\alpha_2} \\ & \quad - \sum_{k=0}^{\infty} \sum_{\beta_1+\beta_2+\beta_3=N+1, \beta_1, \beta_3 \geq 0, \beta_2 \geq 1} (e^{\epsilon(t)} | \overline{\epsilon(t)} e^{-1+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}})_{\beta_1} \left(\frac{\epsilon(t)^{k+1}}{(k+1)!} \right)_{\beta_2} \\ & \quad \times (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon(t)} A_{k+1}(t))_{\beta_3} \\ &= \sum_{\alpha_1+\alpha_2=N+1, \alpha_1, \alpha_2 \geq 0} (e^{\epsilon(t)} | \overline{\epsilon(t)} e^{-1+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}})_{\alpha_1} ((d_H e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t)) |_{U_{X_0}^1(X_0)}_{\alpha_2} \\ &= 0, \end{aligned}$$

其中第 3 个等式成立是因为第 2 项中 $\epsilon(t)$ 关于 t 的次数 $k+1 \geq 1$, 最后一个等式成立是因为题设 $((d_H e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t)) |_{U_{X_0}^1(X_0)}_\alpha = 0$ ($\alpha = 1, \dots, N+1$). 类似地,

$$\begin{aligned} & (\partial_H |_{X_{\epsilon(t)}} e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t))_{N+1} \in U_{X_{\epsilon(t)}}^{-1}(X_{\epsilon(t)}) \\ &= \left(e^{\epsilon(t)} | \overline{\epsilon(t)} e^{-1+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} \left((d_H e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t) \right) |_{U_{X_0}^{-1}(X_0)} \right. \\ & \quad \left. - \sum_{k=0}^{\infty} \frac{\overline{\epsilon(t)}^{k+1}}{(k+1)!} (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon(t)} A_{k+1}(t)) \right)_{N+1} \\ &= \sum_{\alpha_1+\alpha_2=N+1, \alpha_1, \alpha_2 \geq 0} (e^{\epsilon(t)} | \overline{\epsilon(t)} e^{-1+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}})_{\alpha_1} ((d_H e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t)) |_{U_{X_0}^{-1}(X_0)}_{\alpha_2} \\ & \quad - \sum_{k=0}^{\infty} \sum_{\beta_1+\beta_2+\beta_3=N+1, \beta_1, \beta_3 \geq 0, \beta_2 \geq 1} (e^{\epsilon(t)} | \overline{\epsilon(t)} e^{-1+(1-\epsilon(t)\overline{\epsilon(t)})^{-1}})_{\beta_1} \left(\frac{\overline{\epsilon(t)}^{k+1}}{(k+1)!} \right)_{\beta_2} \\ & \quad \times (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon(t)} A_{k+1}(t))_{\beta_3} \\ &= 0. \end{aligned}$$

所以, $(d_H e^{\epsilon(t)} | \overline{\epsilon(t)}) \Omega(t))_{N+1} = 0$. □

通过以上讨论, 我们将求解满足 $d_H e^{\epsilon(t)} | \overline{\epsilon(t)} \Omega(t) = 0$ 的 $\Omega(t)$ 问题转化为求解满足方程组 (4.2) 的

$\Omega(t)$. 以下需要对问题再一次地转化. 令

$$\begin{aligned}\widetilde{\Omega(t)'} &:= \sum_{k=0}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} = \sum_{k=0}^{\infty} \frac{\epsilon(t)^k}{k!} A_k(t) \\ &= \widetilde{\Omega(t)} + \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)},\end{aligned}$$

则

$$\begin{aligned}&\sum_{k=0}^{\infty} \left(\bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} + \partial_H \frac{\epsilon(t)^k}{(k)!} \right) \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)} \\ &= \sum_{k=0}^{\infty} \bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)} + \sum_{k=0}^{\infty} \partial_H \frac{\epsilon(t)^k}{(k)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)} \\ &= \bar{\partial}_H \widetilde{\Omega(t)'} + \sum_{k=0}^{\infty} \partial_H \frac{\epsilon(t)^k}{(k)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)}\end{aligned}$$

且

$$\begin{aligned}&\sum_{k=0}^{\infty} \left(\bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} + \partial_H \frac{\epsilon(t)^k}{(k)!} \right) \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} \\ &= \sum_{k=0}^{\infty} \bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} + \sum_{k=0}^{\infty} \partial_H \frac{\epsilon(t)^k}{(k)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} \\ &= \sum_{k=0}^{\infty} \bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} + \partial_H \widetilde{\Omega(t)'}. \end{aligned}$$

因此, 可以将方程组 (4.2) 重新写作

$$\begin{cases} \bar{\partial}_H \widetilde{\Omega(t)'} = - \sum_{k=0}^{\infty} \partial_H \frac{\epsilon(t)^k}{(k)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)}, \\ \partial_H \widetilde{\Omega(t)'} = - \sum_{k=0}^{\infty} \bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)}. \end{cases} \quad (4.6)$$

所以, 我们将求解满足方程组 (4.2) 的 $\Omega(t)$ 问题转化为求解满足上述方程组 (4.6) 的 $\widetilde{\Omega(t)'}$. 又由引理 2.1, 只需要证明 $\widetilde{\Omega(t)'}$ 满足条件

$$\begin{cases} \sum_{k=0}^{\infty} \bar{\partial}_H \partial_H \frac{\epsilon(t)^k}{(k)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)} = 0, \\ \sum_{k=0}^{\infty} \partial_H \bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} = 0. \end{cases} \quad (4.7)$$

下面验证 (4.7) 成立. 事实上, 因为

$$\partial_H \bar{\partial}_H e^{\epsilon(t)} e^{\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1}} \widetilde{\Omega(t)} = \sum_{k,l=0}^{\infty} \partial_H \bar{\partial}_H \frac{\epsilon(t)^k}{k!} \frac{(e^{\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1}})^l}{l!} \widetilde{\Omega(t)},$$

所以, 通过比较在 Clifford 分解下光滑形式的类型, 有

$$\begin{aligned}
 & \sum_{k=0}^{\infty} \partial_H \bar{\partial}_H \frac{\epsilon(t)^k}{k!} \frac{(\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)} \\
 &= (\partial_H \bar{\partial}_H e^{\epsilon(t)} e^{\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} \widetilde{\Omega(t)})|_{U_{X_0}^2(X_0)} \\
 &= (\partial_H \bar{\partial}_H e^{\epsilon(t)} | \overline{\epsilon(t)} \Omega(t))|_{U_{X_0}^2(X_0)} \quad (\text{由定义直接得到}) \\
 &= (\partial_H d_H e^{\epsilon(t)} | \overline{\epsilon(t)} \Omega(t))|_{U_{X_0}^2(X_0)} \\
 &= \partial_H (d_H e^{\epsilon(t)} | \overline{\epsilon(t)} \Omega(t))|_{U_{X_0}^3(X_0)}, \tag{4.8}
 \end{aligned}$$

其中第 2 个等式成立是因为 $\widetilde{\Omega(t)}$ 的定义. 下面通过对 t 的次数 α 进行归纳来证明结论. 对 $\alpha = 0$, 有

$$\begin{cases} \left(\sum_{k=0}^{\infty} \partial_H \bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{(\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} \right)_0 = \partial_H \bar{\partial}_H \Omega_0 = 0, \\ \left(\sum_{k=0}^{\infty} \partial_H \bar{\partial}_H \frac{\epsilon(t)^k}{k!} \frac{(\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)} \right)_0 = \partial_H \bar{\partial}_H \Omega_0 = 0. \end{cases}$$

Ω_0 即为所求. 若对于 $\alpha \leq N$,

$$\begin{cases} \left(\sum_{k=0}^{\infty} \partial_H \bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{(\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} \right)_{\alpha} = 0, \\ \left(\sum_{k=0}^{\infty} \bar{\partial}_H \partial_H \frac{\epsilon(t)^k}{(k)!} \frac{(\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)} \right)_{\alpha} = 0 \end{cases}$$

成立且我们可以构造满足

$$\begin{cases} \left(\sum_{k=0}^{\infty} \left(\bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} + \partial_H \frac{\epsilon(t)^k}{(k)!} \right) \frac{(\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} \right)_{\alpha} = 0, \\ \left(\sum_{k=0}^{\infty} \left(\bar{\partial}_H \frac{\epsilon(t)^{k-1}}{(k-1)!} + \partial_H \frac{\epsilon(t)^k}{(k)!} \right) \frac{(\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)} \right)_{\alpha} = 0 \quad (\forall \alpha = 0, 1, \dots, N) \end{cases}$$

的 $\widetilde{\Omega(t)} := \sum_{\alpha=0}^N (\widetilde{\Omega(t)})_{\alpha} t^{\alpha}$, 则由引理 4.1 的证明, 有

$$(d_H e^{\epsilon(t)} | \overline{\epsilon(t)} \Omega(t))_{\alpha} = 0 \quad (\forall \alpha = 0, 1, \dots, N).$$

所以, 由引理 3.4, 有

$$\begin{aligned}
 & (\bar{\partial}_{H, \epsilon(t)} A_0(t))_{\alpha} = 0 \in U_{X_0}^1(X_0), \\
 & (\partial_H A_k(t) + \bar{\partial}_{H, \epsilon(t)} A_{k+1}(t))_{\alpha} = 0 \in U_{X_0}^{-1-2k}(X_0) \quad (\alpha = 0, 1, \dots, N, \quad k \geq 0).
 \end{aligned}$$

所以,

$$\begin{aligned}
 & \left(\sum_{k=0}^{\infty} \partial_H \bar{\partial}_H \frac{\epsilon(t)^k}{k!} \frac{(\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1})^{k-1}}{(k-1)!} \widetilde{\Omega(t)} \right)_{N+1} \\
 &= \partial_H (d_H e^{\epsilon(t)} | \overline{\epsilon(t)} \Omega(t))|_{U_{X_0}^3(X_0)}_{N+1}
 \end{aligned}$$

$$\begin{aligned}
 &= \partial_H \left(\sum_{k=-1}^{\infty} e^{\epsilon(t)} (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon} A_{k+1}(t)) |_{U_{X_0}^3(X_0)} \right)_{N+1} \\
 &= \partial_H \left(\sum_{k=-1}^{\infty} \sum_{l=0}^{\infty} \frac{\epsilon(t)^l}{l!} (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon} A_{k+1}(t)) |_{U_{X_0}^3(X_0)} \right)_{N+1} \\
 &= \partial_H \left(\sum_{k=-1}^{\infty} \frac{\epsilon(t)^{k+2}}{(k+2)!} (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon} A_{k+1}(t)) \right)_{N+1} \\
 &= 0,
 \end{aligned}$$

其中第 1 个等式成立是因为 (4.8), 第 2 个等式成立是因为引理 3.3, 最后一个等式成立是因为 $\epsilon(t)$ 中 t 的次数至少为 1 且假定 $A_k(t) = 0$ ($k < 0$). 类似地, 由于 $\Omega(t), \widetilde{\Omega(t)} \in U_{X_0}^0(X_0)$ 为实的, 因此,

$$\begin{aligned}
 \overline{e^{\epsilon(t)} e^{\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} \widetilde{\Omega(t)}} &= \overline{e^{\epsilon(t)} e^{\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} \widetilde{\Omega(t)}} \\
 &= \overline{e^{\epsilon(t)} e^{\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} e^{-\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} e^{-\epsilon(t)} e^{\epsilon(t)} |_{\overline{\epsilon(t)}} \Omega(t)} \\
 &= \overline{e^{\epsilon(t)} |_{\overline{\epsilon(t)}} \Omega(t)} \\
 &= e^{\epsilon(t)} |_{\overline{\epsilon(t)}} \Omega(t).
 \end{aligned}$$

所以,

$$\begin{aligned}
 & - \sum_{k=0}^{\infty} \partial_H \bar{\partial}_H \frac{\overline{\epsilon(t)}^{k-1}}{(k-1)!} \frac{\overline{(\epsilon(t)(1-\epsilon(t)\overline{\epsilon(t)})^{-1})}^k}{k!} \widetilde{\Omega(t)} \\
 &= -(\partial_H \bar{\partial}_H e^{\epsilon(t)} e^{\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} \widetilde{\Omega(t)}) |_{U_{X_0}^2(X_0)} \\
 &= -\partial_H (d_H e^{\epsilon(t)} e^{\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1}} \widetilde{\Omega(t)}) |_{U_{X_0}^3(X_0)} \\
 &= -\partial_H (d_H e^{\epsilon(t)} |_{\overline{\epsilon(t)}} \Omega(t)) |_{U_{X_0}^3(X_0)} \\
 &= -\partial_H \sum_{k=-1}^{\infty} e^{\epsilon(t)} (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon} A_{k+1}(t)) |_{U_{X_0}^3(X_0)} \\
 &= -\partial_H \sum_{k=-1}^{\infty} \sum_{l=0}^{\infty} \frac{\epsilon(t)^l}{l!} (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon} A_{k+1}(t)) |_{U_{X_0}^3(X_0)} \\
 &= -\partial_H \sum_{k=-1}^{\infty} \frac{\epsilon(t)^{(k+2)}}{(k+2)!} (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon} A_{k+1}(t)).
 \end{aligned}$$

因此,

$$\begin{aligned}
 & - \left(\sum_{k=0}^{\infty} \partial_H \bar{\partial}_H \frac{\overline{\epsilon(t)}^{k-1}}{(k-1)!} \frac{\overline{(\epsilon(t)(1-\epsilon(t)\overline{\epsilon(t)})^{-1})}^k}{k!} \widetilde{\Omega(t)} \right)_{N+1} \\
 &= \left(-\partial_H \sum_{k=-1}^{\infty} \frac{\epsilon(t)^{(k+2)}}{(k+2)!} (\partial_H A_k(t) + \bar{\partial}_{H,\epsilon} A_{k+1}(t)) \right)_{N+1} \\
 &= 0.
 \end{aligned}$$

至此, 我们证明了 $\widetilde{\Omega}(t)$ 满足条件 (4.7). 将

$$\begin{aligned}\sigma_\alpha &:= (\widetilde{\Omega}(t))'_\alpha = \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^k}{k!} A_k(t) \right)_\alpha, \\ a_\alpha &:= - \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} A_k(t) \right)_\alpha, \\ b_\alpha &:= - \left(\sum_{k=0}^{\infty} \frac{\overline{\epsilon(t)}^{k+1}}{(k+1)!} A_k(t) \right)_\alpha \quad (\alpha = 0, 1, \dots, N+1)\end{aligned}$$

代入引理 2.1, 得到

$$\begin{aligned}(\widetilde{\Omega}(t))_{N+1} &:= (\widetilde{\Omega}(t))'_{N+1} - \left(\sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} A_k(t) \right)_{N+1} \\ &= - \left(\sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} A_k(t) \right)_{N+1} - \left(\bar{\partial}_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} A_k(t) \right) \right)_{N+1} \\ &\quad + \left(\partial_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} A_k(t) \right) \right)_{N+1} \\ &= - \left(\sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega}(t) \right)_{N+1} \\ &\quad - \left(\bar{\partial}_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega}(t) \right) \right)_{N+1} \\ &\quad + \left(\partial_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega}(t) \right) \right)_{N+1}.\end{aligned}$$

因为上式右端中 $\epsilon(t)$ 关于 t 的次数大于等于 1, 所以, 我们可以通过对 t 的次数 α 进行归纳, 通过 $(\widetilde{\Omega}(t))_\alpha$ ($\alpha = 0, 1, \dots, N$) 构造 $(\widetilde{\Omega}(t))_{N+1}$, 进而构造出 $\widetilde{\Omega}(t)$.

下面需要对 $\widetilde{\Omega}(t)$ 正则化并证明其光滑性. 我们将使用与文献 [2, 34–36] 中类似的方法. 首先, 证明取定某个 $s \in \mathbb{N}$, $0 < \theta < 1$, 上述构造的 $\widetilde{\Omega}(t)$ 关于 Hölder 范数 $|\cdot|_{s+\theta}$ 收敛. 由 Sobolev 空间中椭圆算子的估计, 对任意的 $\sigma \in \wedge^\bullet T_X^*(X)$, 有

$$\begin{aligned}|\partial_H \sigma|_{s+\theta} &\leq C_1 |\sigma|_{s+1+\theta}, \\ |\bar{\partial}_H \sigma|_{s+\theta} &\leq C_2 |\sigma|_{s+1+\theta}, \\ |\partial_H^* \sigma|_{s+\theta} &\leq C_3 |\sigma|_{s+1+\theta}, \\ |\bar{\partial}_H^* \sigma|_{s+\theta} &\leq C_4 |\sigma|_{s+1+\theta}, \\ |G_{\bar{\partial}_H} \sigma|_{s+\theta} &\leq C_5 |\sigma|_{s-2+\theta}, \\ |G_{H,BC} \sigma|_{s+\theta} &\leq C_6 |\sigma|_{s-4+\theta},\end{aligned}$$

其中 $|\cdot|_{s+\theta}$ 为 Hölder 范数, C_i ($1 \leq i \leq 6$) 为仅依赖于 s 和 θ 且与 σ 选取无关的正常数. 为了本文的自包含, 以下给出上述不等式的证明. 首先, 证明存在仅依赖于 s 和 θ 的正常数 C_{21} , 使得 $|d_H \sigma|_{s+\theta} \leq C_{21} |\sigma|_{s+1+\theta}$. 事实上, 由于 $\wedge^\bullet T_X^*(X)$ 上 Hölder 范数的定义, X 紧致, 以及 d_H 为 1 阶微分算子, 所以由文献 [34, 第 414 页, 引理 2.9(2)], 结论成立. 具体地, 因为 X 紧致, 所以, 对任意的

$H \in H^3(X, \mathbb{R})$, 存在仅依赖于 s 和 θ 的正常数 C'_{22} , 使得 $|H|_{s'+\theta} \leq C'_{22}$ ($\forall 0 \leq s' \leq s$). 仍然因为 X 紧致, 又由 Hölder 范数的定义, 所以有 $|d\sigma|_{s+\theta} \leq |\sigma|_{s+1+\theta}$. 所以,

$$\begin{aligned} |d_H \sigma|_{s+\theta} &\leq |d\sigma|_{s+\theta} + |H \wedge \sigma|_{s+\theta} \\ &\leq |\sigma|_{s+1+\theta} + C_{20} |H|_{s+\theta} |\sigma|_{s+\theta} \\ &\leq |\sigma|_{s+1+\theta} + C_{20} C'_{22} |\sigma|_{s+1+\theta} \\ &=: C_{21} |\sigma|_{s+1+\theta}, \end{aligned}$$

其中第 2 个不等式成立是因为文献 [34, 第 414 页, 引理 2.9(2)] 以及 $\wedge^\bullet \bar{L}(X)$ 上 Hölder 范数的定义; 第 3 个不等式成立是因为对任意的 σ , $|\sigma|_{s+\theta} \leq |\sigma|_{s+1+\theta}$. 接着, 我们证明 $|\bar{\partial}_H \sigma|_{s+\theta} \leq C_2 |\sigma|_{s+1+\theta}$. 因为定义 3.1, 对任意的 $\sigma \in U_X^{-n+p}(X) \subset \wedge^\bullet T_X^*(X)$, σ 局部地可表示为 $\sigma := l\rho$, 其中 $l \in \wedge^p \bar{L}(X)$, $\rho \in U_X^{-n}(X)$. 且不妨假定 ρ 在局部坐标 $\{dx^{i_1} \wedge \cdots \wedge dx^{i_\alpha}\}$ 下的系数为 1 (若 $\sigma := (f_1 l)(f_2 \rho)$, 其中 f_1 和 f_2 为光滑函数, $f_1 l \in \wedge^p \bar{L}(X)$, $f_2 \rho \in U_X^{-n}(X)$, 则 $\sigma = (f_1 f_2 l)\rho$). 假定 $L : \{l_i\}_{i=1}^{2n}$ 和 $\bar{L} \cong L^* : \{l^i\}_{i=1}^{2n}$ 分别为 L 和 $\bar{L}$ 的一组局部基, 则局部地, $l := \frac{1}{p!} l_{i_1, \dots, i_p} l^{i_1} \cdots l^{i_p} \in \wedge^p \bar{L}(X)$. 且由于 $[L, L]_H \subset L$, 我们可以局部地假设 $[l_j, l_k]_H := c_{jk}^m l_m$, 其中 c_{jk}^m 为光滑函数, 满足 $c_{jk}^m = -c_{kj}^m$. 所以, 由定义 2.6, 有

$$\begin{aligned} (d_L l)(l_{i_1}, \dots, l_{i_{p+1}}) &:= \sum_{j=1}^{p+1} (-1)^{j-1} \pi_T(l_{i_j}) l(l_{i_1}, \dots, \hat{l}_{i_j}, \dots, l_{i_{p+1}}) \\ &\quad + \sum_{1 \leq j < k \leq p+1} (-1)^{j+k} c_{i_j i_k}^m l(l_m, l_{i_1}, \dots, \hat{l}_{i_j}, \dots, \hat{l}_{i_k}, \dots, l_{i_{p+1}}) \\ &= \sum_{j=1}^{p+1} (-1)^{j-1} \pi_T(l_{i_j}) l_{i_1, \dots, \hat{i}_j, \dots, i_{p+1}} \\ &\quad + \sum_{1 \leq j < k \leq p+1} (-1)^{j+k} c_{i_j i_k}^m l_{m, i_1, \dots, \hat{i}_j, \dots, \hat{i}_k, \dots, i_{p+1}}. \end{aligned}$$

任取 j 和 k , 若 $j < k$, 则

$$\begin{aligned} &(-1)^{j-1} \pi_T(l_{i_j}) l_{i_1, \dots, \hat{i}_j, \dots, i_k, \dots, i_{p+1}} \\ &= (-1)^{j-1} \pi_T(l_{i_k}) l_{i_1, \dots, \hat{i}_k, \dots, i_j, \dots, i_{p+1}} \quad (\text{交换 } i_j \text{ 和 } i_k) \\ &= (-1)^{j-1+k-j+1} \pi_T(l_{i_k}) l_{i_1, \dots, i_j, \dots, \hat{i}_k, \dots, i_{p+1}} \quad (\text{将第 } k \text{ 个位置的 } i_j \text{ 移动到 } \hat{i}_k \text{ 所在的第 } j \text{ 个位置}) \\ &= -(-1)^{k-1} \pi_T(l_{i_k}) l_{i_1, \dots, i_j, \dots, \hat{i}_k, \dots, i_{p+1}}. \end{aligned}$$

对于 $j > k$, 我们也有类似结果. 任取 j 、 k 和 r , 若 $j < k < r$, 则

$$\begin{aligned} &(-1)^{j+k} c_{i_j i_k}^m l_{m, i_1, \dots, \hat{i}_j, \dots, \hat{i}_k, \dots, i_r, \dots, i_{p+1}} \\ &= (-1)^{j+k} c_{i_j i_r}^m l_{m, i_1, \dots, \hat{i}_j, \dots, \hat{i}_r, \dots, i_k, \dots, i_{p+1}} \quad (\text{交换 } i_k \text{ 和 } i_r) \\ &= (-1)^{j+k+r-k+1} c_{i_j i_r}^m l_{m, i_1, \dots, \hat{i}_j, \dots, \hat{i}_r, \dots, i_k, \dots, i_{p+1}} \quad (\text{将第 } r \text{ 个位置的 } i_k \text{ 移动到 } \hat{i}_r \text{ 所在的第 } k \text{ 个位置}) \\ &= -(-1)^{j+r} c_{i_j i_r}^m l_{m, i_1, \dots, \hat{i}_j, \dots, \hat{i}_r, \dots, i_k, \dots, i_{p+1}}, \\ &(-1)^{j+k} c_{i_j i_k}^m l_{m, i_1, \dots, \hat{i}_j, \dots, \hat{i}_k, \dots, i_r, \dots, i_{p+1}} \\ &= (-1)^{j+k} c_{i_r i_k}^m l_{m, i_1, \dots, \hat{i}_r, \dots, \hat{i}_k, \dots, i_j, \dots, i_{p+1}} \quad (\text{交换 } i_j \text{ 和 } i_r) \end{aligned}$$

$$\begin{aligned}
&= (-1)^{j+k+r-j} c_{i_r i_k}^m l_{m, i_1, \dots, i_j, \dots, \hat{i}_k, \dots, \hat{i}_r, \dots, i_{p+1}} \quad (\text{将第 } r \text{ 个位置的 } i_j \text{ 移动到 } \hat{i}_r \text{ 所在的第 } j \text{ 个位置}) \\
&= (-1)^{k+r} c_{i_r i_k}^m l_{m, i_1, \dots, i_j, \dots, \hat{i}_k, \dots, \hat{i}_r, \dots, i_{p+1}} \\
&= -(-1)^{k+r} c_{i_k i_r}^m l_{m, i_1, \dots, i_k, \dots, \hat{i}_j, \dots, \hat{i}_r, \dots, i_{p+1}}.
\end{aligned}$$

对于 $j < r < k$, $r < j < k$, 我们也有类似结果. 所以, 上述 $(d_L l)(l_{i_1}, \dots, l_{i_{p+1}})$ 的表达式关于下指标 $i_1, \dots, i_{p+1}$ 反称. 所以,

$$d_L l := \frac{1}{p+1} (d_L l)_{i_1, \dots, i_{p+1}} l^{i_1} \dots l^{i_{p+1}},$$

其中

$$\begin{aligned}
(d_L l)_{i_1, \dots, i_{p+1}} &:= \sum_{j=1}^{p+1} (-1)^{j-1} \pi_T(l_{i_j}) l_{i_1, \dots, \hat{i}_j, \dots, i_{p+1}} \\
&\quad + \sum_{1 \leq j < k \leq p+1} (-1)^{j+k} c_{i_j i_k}^m l_{m, i_1, \dots, \hat{i}_j, \dots, \hat{i}_k, \dots, i_{p+1}},
\end{aligned}$$

即 d_L 为 1 阶微分算子, 且其 1 阶偏导数项 $\pi_T(l_{i_j}) \in T_X$ 的系数和 0 阶偏导数项 $c_{i_j i_k}^m$ 均为光滑函数. 类似于 $\wedge^\bullet T_X^*(X)$ 上 Hölder 范数的定义, $\wedge^\bullet \bar{L}(X)$ 上 Hölder 范数仍然定义为在局部坐标下其系数的 Hölder 范数在单位分解下的和, 且在不同坐标系下定义的范数等价. 所以, 类似于 d_H 相同的讨论, 由文献 [34, 第 414 页, 引理 2.9(2)], 有

$$|(d_L l)|_{s+\theta} \leq C_{23} M_{s+\theta} |l|_{s+1+\theta},$$

其中 C_{23} 为仅依赖于 s 和 θ 的正常数, $M_{s+\theta} > 0$ 为 d_L 的 Hölder 范数, 仅依赖于 s 和 θ . 又因为 X 紧致, 所以, $M_{s+\theta} < +\infty$, 且

$$|(d_L l)|_{s+\theta} \leq C_{23} M_{s+\theta} |l|_{s+\theta} =: C_{24} |l|_{s+\theta},$$

其中 C_{24} 为仅依赖于 s 和 θ 的正常数. 所以, $\bar{\partial}_H$ 为 1 阶微分算子且

$$\begin{aligned}
|\bar{\partial}_H \sigma|_{s+\theta} &\leq |(d_L l) \rho|_{s+\theta} + |\bar{l} \bar{\partial}_H \rho|_{s+\theta} \\
&= |(d_L l) \rho|_{s+\theta} + |l d_H \rho|_{s+\theta} \\
&= |(d_L l)|_{s+\theta} + |l(H \wedge \rho)|_{s+\theta} \\
&\leq C_{24} |l|_{s+1+\theta} + (s+1) C'_{22} |l \rho|_{s+\theta} \\
&=: C_2 |l \rho|_{s+1+\theta} =: C_2 |\sigma|_{s+1+\theta},
\end{aligned}$$

其中第 1 个等式成立是因为 $\partial_H \rho = 0 \in U_X^{-n-1}(X)$; 第 2 和 3 个等式成立是因为取线丛 $U_X^{-n}(X)$ 上的局部基 ρ 固定且假定 ρ 在局部坐标 $\{dx^{i_1} \wedge \dots \wedge dx^{i_n}\}$ 下的系数为 1, 从而 $d\rho = 0$. 再接着, 我们需要证明 $|\partial_H \sigma|_{s+\theta} \leq C_1 |\sigma|_{s+1+\theta}$. 因为假定 $d_H = \partial_H + \bar{\partial}_H$, 所以,

$$\begin{aligned}
|\partial_H \sigma|_{s+\theta} &\leq |d_H \sigma|_{s+\theta} + |\bar{\partial}_H \sigma|_{s+\theta} \\
&\leq C_1 |\sigma|_{s+1+\theta},
\end{aligned}$$

其中 C_1 为仅依赖于 s 和 θ 的正常数 (参见文献 [9, 引理 6]). 以下, 我们要证明 $|\partial_H^* \sigma|_{s+\theta} \leq C_3 |\sigma|_{s+1+\theta}$, $|\bar{\partial}_H^* \sigma|_{s+\theta} \leq C_4 |\sigma|_{s+1+\theta}$. 因为

$$\bar{\partial}_H^* := * \partial_H *^{-1} := a_1 \dots a_{2n} \partial_H a_{2n} \dots a_1,$$

$$\partial_H^* := * \bar{\partial}_H^* {}^{\perp} := a_1 \cdots a_{2n} \bar{\partial}_H a_{2n} \cdots a_1,$$

其中 $a_i \in C_+$, C_+ 为关于 G 特征值为 $+1$ 的特征子空间, 所以, 由于 ∂_H 和 $\bar{\partial}_H$ 为 1 阶微分算子, 因此, ∂_H^* 和 $\bar{\partial}_H^*$ 仍然为 1 阶微分算子. 又因为 X 紧致, 所以, 由与上述相同的证明, 有

$$\begin{aligned} |\partial_H^* \sigma|_{s+\theta} &:= |a_1 \cdots a_{2n} \bar{\partial}_H a_{2n} \cdots a_1 \sigma|_{s+\theta} \\ &\leq C_{31} |\bar{\partial}_H a_{2n} \cdots a_1 \sigma|_{s+\theta} \\ &\leq C_{31} C_2 |a_{2n} \cdots a_1 \sigma|_{s+1+\theta} \\ &\leq C_{31} C_2 C_{32} |\sigma|_{s+1+\theta} =: C_3 |\sigma|_{s+1+\theta}, \end{aligned}$$

其中 C_{31} 和 C_{32} 为仅依赖于 s 和 θ 的正常数, 第 1 和 3 个不等式成立是因为 $* := a_1 \cdots a_{2n}$ 为 0 次微分算子. 类似地, $|\bar{\partial}_H^* \sigma|_{s+\theta} \leq C_4 |\sigma|_{s+1+\theta}$ 结论成立. 最后, 我们需要证明 $|G_{\bar{\partial}_H} \sigma|_{s+\theta} \leq C_5 |\sigma|_{s-2+\theta}$, $|G_{H,BC} \sigma|_{s+\theta} \leq C_6 |\sigma|_{s-4+\theta}$. 由文献 [19, 第 12 页] 中讨论可知, $\Delta_{\bar{\partial}_H}$ 和 $\Delta_{H,BC}$ 分别为 2 和 4 阶强椭圆微分算子. 所以, 由文献 [34, 定理 7.4(2)], 结论成立. 由 $\widetilde{\Omega(t)}$ 的构造, 有

$$\begin{aligned} \widetilde{\Omega(t)} &= \Omega_0 - \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \\ &\quad - \bar{\partial}_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \\ &\quad + \partial_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right). \end{aligned} \quad (4.9)$$

受到文献 [36] 的启发, 定义如下算子:

$$\begin{aligned} \mathbb{F} &:= - \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \\ &\quad - \bar{\partial}_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \right) \\ &\quad + \partial_H(\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \right), \end{aligned}$$

则

$$\Omega_0 = (1 - \mathbb{F}) \widetilde{\Omega(t)}.$$

由上述对 ∂_H 、 ∂_H^* 、 $\bar{\partial}_H$ 、 $\bar{\partial}_H^*$ 、 $G_{\bar{\partial}_H}$ 和 $G_{H,BC}$ 等算子的估计, 以及 $\epsilon(t)$ 光滑且在 $t \rightarrow 0$ 时收敛到 0, 所以, 我们可以选取充分小的 $|\epsilon(t)|_{s+\theta}$, 使得

$$|\Omega_0|_{s+\theta} \geq (1 - C_{s+\theta}) |\widetilde{\Omega(t)}|_{s+\theta},$$

其中 $C_{s+\theta}$ 为仅依赖于 s 和 θ 且小于 1 的正常数. 所以, $\widetilde{\Omega(t)}$ 关于 Hölder 范数 $|\cdot|_{s+\theta}$ 收敛.

为了将 $\widetilde{\Omega(t)}$ 正则化并证明其关于 t 光滑, 我们希望 $\widetilde{\Omega(t)}$ 关于 Hölder 范数 $|\cdot|_{s+\theta}$ 的收敛半径不会随着 $s \rightarrow +\infty$ 而趋向于 0. 我们将采用类似文献 [21, 34] 中的方法. 因为 $\Omega_0 \in H_{\bar{\partial}_H}^0(X)$, 将广义

Laplace 算子 $\Delta_{\bar{\partial}_H}$ 作用到 (4.9) 两边, 可得

$$\begin{aligned}\Delta_{\bar{\partial}_H} \widetilde{\Omega(t)} &= -\Delta_{\bar{\partial}_H} \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \\ &\quad - \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \\ &\quad + \Delta_{\bar{\partial}_H} \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right).\end{aligned}$$

在 $\widetilde{\Omega(t)}$ 关于 Hölder 范数 $|\cdot|_{s+\theta}$ 的收敛半径内, 选取某个充分小的 $r > 0$. 对每一个 $l \in \mathbb{N}$, 选取一个光滑函数 $\eta^l(t)$, 使得

$$\eta^l(t) = \begin{cases} 1, & |t| \leq \left(\frac{1}{2} + \frac{1}{2^{l+1}}\right)r, \\ 0, & |t| \geq \left(\frac{1}{2} + \frac{1}{2^l}\right)r. \end{cases}$$

我们再选取 X 上覆盖 $\{U_\alpha\}_{\alpha=1}^N$ 的一个单位分解 $\{p_\alpha\}$, 即 $\{p_\alpha\}$ 满足对于任意的 $x \in X$, $\sum_{\alpha=1}^N p_\alpha(x) = 1$, 且 p_α 的支集 $\text{supp } p_\alpha \subset U_\alpha$. 令

$$p_\alpha^l(x, t) := p_\alpha(x) \eta^l(t).$$

我们先证明 $\eta^3 \widetilde{\Omega(t)}$ 为 $C^{s+1+\theta}$. 假定 U_α 上的光滑坐标为 $x^1, \dots, x^{2n}$, 又设 $\frac{1}{2}(t + \bar{t}) := x^{2n+1}$, $\frac{1}{2i}(t - \bar{t}) := x^{2n+2}$. 记 Δ_i^h ($i = 1, \dots, 2n+2$) 为文献 [34, 第 367 页 (1.12)] 所定义的差商, 有

$$\begin{aligned}\Delta_{\bar{\partial}_H} \Delta_i^h p_\alpha^3 \widetilde{\Omega(t)} &= (\Delta_{\bar{\partial}_H} \Delta_i^h p_\alpha^3 \widetilde{\Omega(t)} - \Delta_i^h \Delta_{\bar{\partial}_H} p_\alpha^3 \widetilde{\Omega(t)}) + (\Delta_i^h \Delta_{\bar{\partial}_H} p_\alpha^3 \widetilde{\Omega(t)} - \Delta_i^h p_\alpha^3 \Delta_{\bar{\partial}_H} \widetilde{\Omega(t)}) + \Delta_i^h p_\alpha^3 \Delta_{\bar{\partial}_H} \widetilde{\Omega(t)} \\ &= \Delta_i^h p_\alpha^3 \left(-\Delta_{\bar{\partial}_H} \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right. \\ &\quad \left. - \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right. \\ &\quad \left. + \Delta_{\bar{\partial}_H} \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right) \\ &\quad + (\Delta_{\bar{\partial}_H} \Delta_i^h p_\alpha^3 \widetilde{\Omega(t)} - \Delta_i^h \Delta_{\bar{\partial}_H} p_\alpha^3 \widetilde{\Omega(t)}) + (\Delta_i^h \Delta_{\bar{\partial}_H} p_\alpha^3 \widetilde{\Omega(t)} - \Delta_i^h p_\alpha^3 \Delta_{\bar{\partial}_H} \widetilde{\Omega(t)}) \\ &=: F^1.\end{aligned}$$

由于文献 [19, 第 7 页] 的讨论, $\Delta_{\bar{\partial}_H}$ 的主要部分为对角型的 2 阶强椭圆算子且满足文献 [34, 第 417 页, 定理 2.3] 的条件, 所以有以下估计:

$$\begin{aligned}|\Delta_i^h p_\alpha^3 \widetilde{\Omega(t)}|_{s+\theta} &\leq C_s (|\Delta_{\bar{\partial}_H} \Delta_i^h p_\alpha^3 \widetilde{\Omega(t)}|_{s-2+\theta} + |\Delta_i^h p_\alpha^3 \widetilde{\Omega(t)}|_0) \\ &= C_s (|F^1|_{s-2+\theta} + |\Delta_i^h p_\alpha^3 \widetilde{\Omega(t)}|_0),\end{aligned}\tag{4.10}$$

其中 C_s 为一个仅依赖于 s 和 θ 的正常数. 以下对 F^1 进行估计:

$$|F^1|_{s-2+\theta} \leq \left| \Delta_i^h p_\alpha^3 \Delta_{\bar{\partial}_H} \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right|_{s-2+\theta}$$

$$\begin{aligned}
 & + \left| \Delta_i^h p_\alpha^3 \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right|_{s-2+\theta} \\
 & + \left| \Delta_i^h p_\alpha^3 \Delta_{\bar{\partial}_H} \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right|_{s-2+\theta} \\
 & + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta},
 \end{aligned}$$

其中在上述 $\widetilde{\Omega(t)}$ 的低阶偏导数项中, 我们把求差商 Δ_i^h 也看作求 1 阶偏导数. 区别于仅含求偏导数的算子, 下面要对 Green 算子 $G_{H,BC}$ 做一些讨论. 对于 X 上光滑形式 u 和 w , 有

$$\Delta_{H,BC} u w - u \Delta_{H,BC} w = w \text{ 的至多 3 阶偏导数项,}$$

且上述 w 的至多 3 阶偏导数项中包含 u 的 1 到 4 阶偏导数项. 令 $w := G_{H,BC} v$ (v 为 X 上光滑形式), 则

$$\Delta_{H,BC} u G_{H,BC} v - u(v - \mathbb{H}_{H,BC} v) = G_{H,BC} v \text{ 的至多 3 阶偏导数项,}$$

对上式左作用 $G_{H,BC}$, 有

$$u G_{H,BC} v - \mathbb{H}_{H,BC} u G_{H,BC} v = G_{H,BC} u(v - \mathbb{H}_{H,BC} v) + G_{H,BC}(G_{H,BC} v \text{ 的至多 3 阶偏导数项}).$$

将 $u := p_\alpha^3, v := \bar{\partial}_H \sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} := \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)}$ 代入上式, 可得

$$\begin{aligned}
 & p_\alpha^3 G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} - \mathbb{H}_{H,BC}(p_\alpha^3 G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)}) \\
 & = G_{H,BC} p_\alpha^3 \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} - G_{H,BC} p_\alpha^3 \mathbb{H}_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \\
 & \quad + G_{H,BC}(G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项}) \\
 & = G_{H,BC} p_\alpha^3 \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} + G_{H,BC}(G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项}),
 \end{aligned}$$

其中第 2 个等式成立是因为由广义 $\partial_H \bar{\partial}_H$ - 引理, $\bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \in \text{Im} \partial_H \bar{\partial}_H$. 对上式两端同时作用 $\bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^*$, 可得

$$\begin{aligned}
 & \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* p_\alpha^3 G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} - \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* \mathbb{H}_{H,BC}(p_\alpha^3 G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)}) \\
 & = \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* p_\alpha^3 G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \\
 & = \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} p_\alpha^3 \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \\
 & \quad + \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC}(G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项}),
 \end{aligned}$$

其中第 1 个等式成立是因为 $\text{Ker} \Delta_{H,BC} = \text{Ker} \partial_H \cap \text{Ker} \bar{\partial}_H \cap \text{Ker}(\partial_H \bar{\partial}_H)^*$. 所以,

$$\begin{aligned}
 & p_\alpha^3 \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \\
 & = \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \epsilon(t)'' p_\alpha^3 \widetilde{\Omega(t)} + \widetilde{\Omega(t)} \text{ 的低阶偏导数项} \\
 & \quad + \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC}(G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项}),
 \end{aligned}$$

其中上述 $G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)}$ 的至多 3 阶偏导数项中包含 p_α^3 的 1 到 4 阶偏导数, 上述 $\widetilde{\Omega(t)}$ 的低阶偏导数项中包含 p_α^3 和 $\epsilon(t)''$ 的偏导数, 且我们把求差商 Δ_i^h 也看作求 1 阶偏导数. 类似地,

$$\Delta_{H,BC} \Delta_i^h w - \Delta_i^h \Delta_{H,BC} w = \text{不包含 } w \text{ 的 4 阶偏导数的差商的项.}$$

所以, 由上述相同的讨论, 有

$$\begin{aligned} & \Delta_i^h \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \\ &= \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \epsilon(t)'' \Delta_i^h \widetilde{\Omega(t)} + \widetilde{\Omega(t)} \text{ 的低阶偏导数项} \\ &+ \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (\text{不包含 } G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的 4 阶偏导数的差商的项}). \end{aligned}$$

所以, 利用上述对 ∂_H 、 ∂_H^* 、 $\bar{\partial}_H$ 、 $\bar{\partial}_H^*$ 、 $G_{\bar{\partial}_H}$ 和 $G_{H,BC}$ 等算子的估计 (参见文献 [34, 第 455 页, 引理 8.1、8.2(i) 和 8.2(ii)], $\eta^1(t) \cdot \eta^3(t) = \eta^3(t)$, 可知存在仅依赖 s 和 α 的正常数 K_2 、 K'_2 、 K''_2 、 K'''_2 、 K''''_2 、 L_2 和 L'_2 , 使得

$$\begin{aligned} & \left| \Delta_i^h p_\alpha^3 \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right|_{s-2+\theta} \\ & \leq |\Delta_i^h \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H (p_\alpha^1 \epsilon(t)'' p_\alpha^3 \widetilde{\Omega(t)})|_{s-2+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta} \\ & + |\Delta_i^h \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项})|_{s-2+\theta} \\ & \leq |\bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H (p_\alpha^1 \epsilon(t)'' \Delta_i^h p_\alpha^3 \widetilde{\Omega(t)})|_{s-2+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta} \\ & + |\Delta_i^h \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项})|_{s-2+\theta} \\ & + |\bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (\text{不包含 } G_{H,BC} \bar{\partial}_H \epsilon(t)'' p_\alpha^3 \widetilde{\Omega(t)} \text{ 的 4 阶偏导数的差商的项})|_{s-2+\theta} \\ & \leq K_2 |(p_\alpha^1 \epsilon(t)'' \Delta_i^h p_\alpha^3 \widetilde{\Omega(t)})|_{s+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta} \\ & + K'_2 |G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项}|_{s+\theta} \\ & + K''_2 |\text{不包含 } G_{H,BC} \bar{\partial}_H \epsilon(t)'' p_\alpha^3 \widetilde{\Omega(t)} \text{ 的 4 阶偏导数的差商的项}|_{s-1+\theta} \\ & \leq K_2 |(p_\alpha^1 \epsilon(t)'' \Delta_i^h p_\alpha^3 \widetilde{\Omega(t)})|_{s+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta} \\ & + K'''_2 |G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)}|_{s+3+\theta} + K''''_2 |G_{H,BC} \bar{\partial}_H \epsilon(t)'' p_\alpha^3 \widetilde{\Omega(t)}|_{s+3+\theta} \\ & \leq K_2 |(p_\alpha^1 \epsilon(t)'' \Delta_i^h p_\alpha^3 \widetilde{\Omega(t)})|_{s+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta} + K''''_2 |\epsilon(t)'' p_\alpha^3 \widetilde{\Omega(t)}|_{s+\theta} \\ & \leq L_2 |\eta^1 \epsilon(t)''|_0 |\Delta_i^h p_\alpha^3 \widetilde{\Omega(t)}|_{s+\theta} + L'_2 |\eta^1 \epsilon(t)''|_{s+\theta} |\eta^3 \widetilde{\Omega(t)}|_{s+\theta}, \end{aligned}$$

其中倒数第 2 个不等式成立是因为上述 $G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)}$ 的至多 3 阶偏导数项中包含 p_α^3 的 1 到 4 阶偏导数, 最后一个不等式成立是因为上述 $\widetilde{\Omega(t)}$ 的低阶偏导数项中包含 p_α^3 和 $\epsilon(t)''$ 的偏导数. 类似地, 令

$$\begin{aligned} \partial_H \sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} &=: \partial_H \epsilon(t)''' \widetilde{\Omega(t)}, \\ \Delta_{\bar{\partial}_H} \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{(k)!} \frac{(\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k}{k!} \widetilde{\Omega(t)} &=: \Delta_{\bar{\partial}_H} \epsilon(t)' \widetilde{\Omega(t)}, \end{aligned}$$

存在仅依赖 s 和 θ 的正常数 L_i 和 L'_i ($i = 1, 3, 4$), 使得

$$\begin{aligned} & \left| \Delta_i^h p_\alpha^3 \Delta_{\bar{\partial}_H} \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right|_{s-2+\theta} \\ & \leq L_1 |\eta^1 \epsilon(t)'|_0 |\Delta_i^h p_\alpha^3 \widetilde{\Omega(t)}|_{s+\theta} + L'_1 |\eta^1 \epsilon(t)'|_{s+\theta} |\eta^3 \widetilde{\Omega(t)}|_{s+\theta}, \\ & \left| \Delta_i^h p_\alpha^3 \Delta_{\bar{\partial}_H} \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right|_{s-2+\theta} \end{aligned}$$

$$\begin{aligned} &\leq L_3 |\eta^1 \epsilon(t)'''|_0 |\Delta_i^h p_\alpha^3 \widetilde{\Omega}(t)|_{s+\theta} + L'_3 |\eta^1 \epsilon(t)'''|_{s+\theta} |\eta^3 \widetilde{\Omega}(t)|_{s+\theta}, \\ &|\widetilde{\Omega}(t)| \text{ 的低阶偏导数项 }|_{s-2+\theta} \\ &\leq L'_4 (|\eta^1 \epsilon(t)'|_{s+\theta} + |\eta^1 \epsilon(t)''|_{s+\theta} + |\eta^1 \epsilon(t)'''|_{s+\theta}) |\eta^3 \widetilde{\Omega}(t)|_{s+\theta}. \end{aligned}$$

所以, 存在仅依赖 s 和 θ 的正常数 M_0 和 M_s , 使得

$$|F^1|_{s+\theta} \leq M_0 A(r) |\Delta_i^h p_\alpha^3 \widetilde{\Omega}(t)|_{s+\theta} + M_s (|\eta^1 \epsilon(t)'|_{s+\theta} + |\eta^1 \epsilon(t)''|_{s+\theta} + |\eta^1 \epsilon(t)'''|_{s+\theta}) |\eta^3 \widetilde{\Omega}(t)|_{s+\theta},$$

其中 $A(t)$ 为文献 [34, 第 452 页, (8.3)] 所定义的幂级数, 满足 $|\epsilon(t)'|_{s+\theta}, |\epsilon(t)''|_{s+\theta}, |\epsilon(t)'''|_{s+\theta} \ll A(t)$. 所以, 由 (4.10) 和文献 [34, 第 455 页, 引理 8.2(i) 和 8.3(ii)], 有

$$\begin{aligned} |\Delta_i^h p_\alpha^3 \widetilde{\Omega}(t)|_{s+\theta} &\leq C_s M_0 A(r) |\Delta_i^h p_\alpha^3 \widetilde{\Omega}(t)|_{s+\theta} + C_s M_s (|\eta^1 \epsilon(t)'|_{s+\theta} \\ &\quad + |\eta^1 \epsilon(t)''|_{s+\theta} + |\eta^1 \epsilon(t)'''|_{s+\theta}) |\eta^3 \widetilde{\Omega}(t)|_{s+\theta} + C_s |p_\alpha^3 \widetilde{\Omega}(t)|_1. \end{aligned}$$

我们适当缩小上述选取的充分小的 r , 使得

$$M_0 C_s A(r) < \frac{1}{2},$$

则得到

$$|\Delta_i^h p_\alpha^3 \widetilde{\Omega}(t)|_{s+\theta} \leq 2C_s M_s (|\eta^1 \epsilon(t)'|_{s+\theta} + |\eta^1 \epsilon(t)''|_{s+\theta} + |\eta^1 \epsilon(t)'''|_{s+\theta}) |\eta^3 \widetilde{\Omega}(t)|_{s+\theta} + 2C_s |p_\alpha^3 \widetilde{\Omega}(t)|_1.$$

因为 $\widetilde{\Omega}(t)$ 为 $C^{s+\theta}$, 上述公式的右端有界. 所以, 由文献 [34, 第 455 页, 引理 8.2(iii)], $p_\alpha^3 \widetilde{\Omega}(t)$ 为 $C^{s+1+\theta}$, 进而 $\eta^3 \widetilde{\Omega}(t)$ 为 $C^{s+1+\theta}$.

下面将证明 $\eta^5 \widetilde{\Omega}(t)$ 为 $C^{s+2+\theta}$. 记 D_j 为 $\frac{\partial}{\partial x^j}$ ($i = 1, \dots, 2n+2$), 则有

$$\begin{aligned} \Delta_{\bar{\partial}_H} \Delta_i^h D_j p_\alpha^5 \widetilde{\Omega}(t) &= (\Delta_{\bar{\partial}_H} \Delta_i^h D_j p_\alpha^5 \widetilde{\Omega}(t) - \Delta_i^h \Delta_{\bar{\partial}_H} D_j p_\alpha^5 \widetilde{\Omega}(t)) + (\Delta_i^h \Delta_{\bar{\partial}_H} D_j p_\alpha^5 \widetilde{\Omega}(t) - \Delta_i^h D_j \Delta_{\bar{\partial}_H} p_\alpha^5 \widetilde{\Omega}(t)) \\ &\quad + (\Delta_i^h D_j \Delta_{\bar{\partial}_H} p_\alpha^5 \widetilde{\Omega}(t) - \Delta_i^h D_j p_\alpha^5 \Delta_{\bar{\partial}_H} \widetilde{\Omega}(t)) + \Delta_i^h D_j p_\alpha^5 \Delta_{\bar{\partial}_H} \widetilde{\Omega}(t) \\ &= \Delta_i^h D_j p_\alpha^5 \left(-\Delta_{\bar{\partial}_H} \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega}(t) \right. \\ &\quad \left. - \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega}(t) \right) \right. \\ &\quad \left. + \Delta_{\bar{\partial}_H} \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega}(t) \right) \right) \\ &\quad + (\Delta_{\bar{\partial}_H} \Delta_i^h D_j p_\alpha^5 \widetilde{\Omega}(t) - \Delta_i^h \Delta_{\bar{\partial}_H} D_j p_\alpha^5 \widetilde{\Omega}(t)) + (\Delta_i^h \Delta_{\bar{\partial}_H} D_j p_\alpha^5 \widetilde{\Omega}(t) - \Delta_i^h D_j \Delta_{\bar{\partial}_H} p_\alpha^5 \widetilde{\Omega}(t)) \\ &\quad + (\Delta_i^h D_j \Delta_{\bar{\partial}_H} p_\alpha^5 \widetilde{\Omega}(t) - \Delta_i^h D_j p_\alpha^5 \Delta_{\bar{\partial}_H} \widetilde{\Omega}(t)) \\ &= \Delta_i^h D_j p_\alpha^5 \left(-\Delta_{\bar{\partial}_H} \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega}(t) \right. \\ &\quad \left. - \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega}(t) \right) \right. \\ &\quad \left. + \Delta_{\bar{\partial}_H} \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega}(t) \right) \right) \end{aligned}$$

$$+ \widetilde{\Omega(t)} \text{ 的低阶偏导数项} \\ =: F^2,$$

其中在上述 $\widetilde{\Omega(t)}$ 的低阶偏导数项中, 我们把求差商 Δ_i^h 也看作求 1 阶偏导数. 因为 $\Delta_{\bar{\partial}_H}$ 的主要部分为一个对角型强椭圆算子且满足文献 [34, 第 417 页, 定理 2.3] 的条件, 所以有以下估计:

$$|\Delta_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} \leq C_s (|\Delta_{\bar{\partial}_H} \Delta_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s-2+\theta} + |\Delta_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_0) \\ = C_s (|F^2|_{s-2+\theta} + |\Delta_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_0),$$

其中 C_s 为与 (4.10) 中相同的正常数. 以下对 F^2 进行估计:

$$|F^2|_{s-2+\theta} \leq \left| \Delta_i^h D_j p_\alpha^5 \Delta_{\bar{\partial}_H} \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right|_{s-2+\theta} \\ + \left| \Delta_i^h D_j p_\alpha^5 \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right|_{s-2+\theta} \\ + \left| \Delta_i^h D_j p_\alpha^5 \Delta_{\bar{\partial}_H} \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right|_{s-2+\theta} \\ + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta}.$$

因为对于 D_j 和光滑形式 w , 有

$$\Delta_{H,BC} D_j w - D_j \Delta_{H,BC} w = w \text{ 的至多 4 阶偏导数项},$$

所以, 由上述相同的讨论, 有

$$D_j \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \\ = \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \epsilon(t)'' D_j \widetilde{\Omega(t)} + \widetilde{\Omega(t)} \text{ 的低阶偏导数项} \\ + \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 4 阶偏导数项}).$$

利用文献 [34, 第 455 页, 引理 8.1、8.2(i) 和 8.2(ii)], $\eta^3(t) \cdot \eta^5(t) = \eta^5(t)$, 可知存在仅依赖 $s+1$ 和 θ 的正常数 R_2 、 R'_2 、 R''_2 、 R'''_2 、 R''''_2 、 L'_2 及上述相同的 K_2 和 L_2 , 使得

$$\left| \Delta_i^h D_j p_\alpha^5 \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k-1}}{(k-1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right|_{s-2+\theta} \\ \leq |\Delta_i^h D_j \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H (p_\alpha^3 \epsilon(t)'' p_\alpha^5 \widetilde{\Omega(t)})|_{s-2+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta} \\ + |\Delta_i^h D_j \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项})|_{s-2+\theta} \\ \leq |\Delta_i^h \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H (p_\alpha^3 \epsilon(t)'' D_j p_\alpha^5 \widetilde{\Omega(t)})|_{s-2+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta} \\ + |\Delta_i^h D_j \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项})|_{s-2+\theta} \\ + |\Delta_i^h \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (G_{H,BC} \bar{\partial}_H \epsilon(t)'' p_\alpha^5 \widetilde{\Omega(t)} \text{ 的至多 4 阶偏导数项})|_{s-2+\theta} \\ \leq |\bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \bar{\partial}_H (p_\alpha^3 \epsilon(t)'' \Delta_i^h D_j p_\alpha^5 \widetilde{\Omega(t)})|_{s-2+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项}|_{s-2+\theta} \\ + |\Delta_i^h D_j \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项})|_{s-2+\theta}$$

$$\begin{aligned}
 & + |\triangle_i^h \bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (G_{H,BC} \bar{\partial}_H \epsilon(t)'' p_\alpha^5 \widetilde{\Omega(t)}) \text{ 的至多 4 阶偏导数项} |_{s-2+\theta} \\
 & + |\bar{\partial}_H \bar{\partial}_H^* \bar{\partial}_H (\partial_H \bar{\partial}_H)^* G_{H,BC} (\text{不包含 } G_{H,BC} \bar{\partial}_H \epsilon(t)'' D_j p_\alpha^5 \widetilde{\Omega(t)} \text{ 的 4 阶偏导数的差商项}) |_{s-2+\theta} \\
 \leq & K_2 |p_\alpha^3 \epsilon(t)'' \triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项} |_{s-2+\theta} \\
 & + R_2 |G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)} \text{ 的至多 3 阶偏导数项} |_{s+1+\theta} \\
 & + R'_2 |G_{H,BC} \bar{\partial}_H \epsilon(t)'' p_\alpha^5 \widetilde{\Omega(t)} \text{ 的至多 4 阶偏导数项} |_{s+\theta} \\
 & + R''_2 |\text{不包含 } G_{H,BC} \bar{\partial}_H \epsilon(t)'' D_j p_\alpha^5 \widetilde{\Omega(t)} \text{ 的 4 阶偏导数的差商项} |_{s-1+\theta} \\
 \leq & K_2 |p_\alpha^3 \epsilon(t)'' \triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项} |_{s-2+\theta} + R_2''' |\epsilon(t)'' \widetilde{\Omega(t)}|_{s+1+\theta} \\
 & + R_2'''' |\epsilon(t)'' p_\alpha^5 \widetilde{\Omega(t)}|_{s+1+\theta} \\
 \leq & K_2 |p_\alpha^3 \epsilon(t)'' \triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} + |\widetilde{\Omega(t)} \text{ 的低阶偏导数项} |_{s-2+\theta} + R_2'''' |\epsilon(t)'' p_\alpha^5 \widetilde{\Omega(t)}|_{s+1+\theta} \\
 \leq & L_2 |\eta^3 \epsilon(t)'|_0 |\triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} + L_2'' |\eta^3 \epsilon(t)'|_0 |\eta^5 \widetilde{\Omega(t)}|_{s+1+\theta},
 \end{aligned}$$

其中倒数第 2 个不等式成立是因为上述 $G_{H,BC} \bar{\partial}_H \epsilon(t)'' \widetilde{\Omega(t)}$ 的至多 3 阶偏导数项中包含 p_α^5 的 1 到 4 阶偏导数, 最后一个不等式成立是因为上述 $\widetilde{\Omega(t)}$ 的低阶偏导数项中包含 p_α^3 和 $\epsilon(t)''$ 的偏导数. 类似地, 可知存在仅依赖 $s+1$ 和 θ 的正常数 L'_i 以及与上述相同的 L_i ($i=1,3,4$), 使得

$$\begin{aligned}
 & \left| \triangle_i^h D_j p_\alpha^5 \bar{\partial}_H \sum_{k=1}^{\infty} \frac{\epsilon(t)^k}{k!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right|_{s-2+\theta} \\
 & \leq L_1 |\eta^3 \epsilon(t)'|_0 |\triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} + L_1'' |\eta^3 \epsilon(t)'|_{s+1+\theta} |\eta^5 \widetilde{\Omega(t)}|_{s+1+\theta}, \\
 & \left| \triangle_i^h D_j p_\alpha^5 \bar{\partial}_H \partial_H (\partial_H \bar{\partial}_H)^* G_{H,BC} \partial_H \left(\sum_{k=0}^{\infty} \frac{\epsilon(t)^{k+1}}{(k+1)!} \frac{1}{k!} (\overline{\epsilon(t)}(1 - \epsilon(t)\overline{\epsilon(t)})^{-1})^k \widetilde{\Omega(t)} \right) \right|_{s-2+\theta} \\
 & \leq L_3 |\eta^3 \epsilon(t)'''|_0 |\triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} + L_3'' |\eta^3 \epsilon(t)'''|_{s+1+\theta} |\eta^5 \widetilde{\Omega(t)}|_{s+1+\theta}, \\
 & |\widetilde{\Omega(t)} \text{ 的低阶偏导数项} |_{s-2+\theta} \\
 & \leq L_4'' (|\eta^3 \epsilon(t)'|_{s+1+\theta} + |\eta^3 \epsilon(t)''|_{s+1+\theta} + |\eta^3 \epsilon(t)'''|_{s+1+\theta}) |\eta^5 \widetilde{\Omega(t)}|_{s+1+\theta},
 \end{aligned}$$

则存在仅依赖 s 和 θ 的正常数 M_{s+1} 以及与上述相同的 M_0 , 使得

$$\begin{aligned}
 |F^2|_{s+\theta} \leq & M_0 A(r) |\triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} + M_{s+1} (|\eta^3 \epsilon(t)'|_{s+1+\theta} + |\eta^3 \epsilon(t)''|_{s+1+\theta} \\
 & + |\eta^3 \epsilon(t)'''|_{s+1+\theta}) |\eta^5 \widetilde{\Omega(t)}|_{s+1+\theta},
 \end{aligned}$$

所以,

$$\begin{aligned}
 |\triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} \leq & C_s M_0 A(r) |\triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} + C_s M_{s+1} (|\eta^3 \epsilon(t)'|_{s+1+\theta} + |\eta^3 \epsilon(t)''|_{s+1+\theta} \\
 & + |\eta^3 \epsilon(t)'''|_{s+1+\theta}) |\eta^5 \widetilde{\Omega(t)}|_{s+1+\theta} + C_s |p_\alpha^5 \widetilde{\Omega(t)}|_2.
 \end{aligned}$$

因为我们选取的 r 满足

$$M_0 C_s A(r) < \frac{1}{2},$$

所以,

$$|\triangle_i^h D_j p_\alpha^5 \widetilde{\Omega(t)}|_{s+\theta} \leq 2C_s M_{s+1} (|\eta^3 \epsilon(t)'|_{s+1+\theta} + |\eta^3 \epsilon(t)''|_{s+1+\theta} + |\eta^3 \epsilon(t)'''|_{s+1+\theta}) |\eta^5 \widetilde{\Omega(t)}|_{s+1+\theta}$$

$$+ 2C_s |p_\alpha^5 \widetilde{\Omega}(t)|_2.$$

因为 $p_\alpha^3 \widetilde{\Omega}(t)$ 为 $C^{s+1+\theta}$, 所以上述公式的右端有界. 所以, 由文献 [34, 第 455 页, 引理 8.2(iii)], 有 $p_\alpha^5 \widetilde{\Omega}(t)$ 为 $C^{s+2+\theta}$, 进而 $\eta^5 \widetilde{\Omega}(t)$ 为 $C^{s+2+\theta}$. 用类似的方法, 我们可以证明, 对于任意的 $l \in \mathbb{N}$, $\eta^{2l+1} \widetilde{\Omega}(t)$ 为 $C^{s+l+\theta}$, 且其中可以选取固定的 r , 使得 r 仅依赖 s 和 θ 且与 l 无关. 因为在 $|t| \leq \frac{r}{2}$ 上, $\eta^{2l+1} = 1$, 所以, $\widetilde{\Omega}(t)$ 在 $|t| < \frac{r}{2}$ 上为 C^∞ . 最后, 由于 $\widetilde{\Omega}(t)$ 的定义 (4.1), 以及在 $|\epsilon(t)|_{s+\alpha}$ 充分小时, 算子 $e^{-\overline{\epsilon(t)}(1-\epsilon(t)\overline{\epsilon(t)})^{-1}}$ 、 $e^{-\epsilon(t)}$ 和 $e^{\epsilon(t)|\overline{\epsilon(t)}}$ 均可逆, 所以, 我们可以得出 $\Omega(t)$.

致谢 作者感谢刘克峰、饶胜、朱盛茂和夏玮等教授的鼓励与帮助, 也感谢审稿人的细心审稿.

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Local extensions of d_H -closed real forms of pure type on compact generalized Hermitian manifolds

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Abstract In this paper, we give a local extension of d_H -closed real forms of pure type on deformations of compact generalized Hermitian manifolds with the generalized $\partial_H\bar{\partial}_H$ -lemma being satisfied.

Keywords deformations of complex structures, Hodge theory, Hermitian and Kählerian manifolds

MSC(2020) 32G05, 58A14, 53C55

doi: 10.1360/SSM-2019-0209