一个新的边界层方程及其在形状控制中的应用

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摘要 本文给出固壁边界上 (即一个二维流形上) 的流体速度梯度和压力的二阶偏微分方程, 从而也给出边界上法向应力, 以及流体中运动物体所受的阻力和升力的计算公式. 本方法的创新在于边界上法向速度梯度不是通过在边界层内速度梯度的数值微分达到, 而是通过它与其他变量一起作为一组偏微分方程的解而得到, 证明边界层方程组的适定性问题, 并且给出解关于边界形状的 Gâteaux 导数所满足的偏微分方程. 本文将本方法应用于飞机外形的形状最优控制, 给出阻力泛函关于形状第一变分的可计算形式. 数值例子表明, 用本方法得到的阻力精度比通用程序得到要高.

关键词 边界层 阻力 形状优化控制 Navier-Stokes 方程

MSC (2010) 主题分类 65M30, 76D05, 78A10

1 引言

黏性流体在固壁边界附近流动时, 当 Reynolds 比较大时, 形成一个边界层, 在它里面产生很大的法向速度梯度, 即使没有产生湍流的情形下, 在边界层内仍然有很多小的旋涡, 消耗大量能量. 边界层的厚度 $\delta = \text{Re}^{-1/2}$ 非常小. 由于黏性的作用, 旋涡总是在流 - 固交界面上产生并向流体内部扩散, 它是与固壁的黏附条件相联系的. 一个简单的例子是, 一个无限大的平板上方有与平板平行的均匀无穷远来流 u_∞ , 设初始有个切向压强梯度, Navier-Stokes 方程^[1,2] 简化为

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}, \\ u &= 0, \quad t > 0, \quad y = 0, \quad u = u_\infty, \quad t = 0, \quad y > 0,\end{aligned}$$

它的解为

$$\begin{aligned}u(y, t) &= u_\infty \text{erf}(\eta) - \frac{1}{\nu} \frac{\partial p}{\partial x} \left[\left(\frac{1}{2} y^2 + \nu t \right) \text{erf}(\eta) + y \sqrt{\frac{\nu t}{\pi}} e^{-\eta^2} - \frac{1}{2} y^2 \right], \\ \text{erf}(\eta) &= -\frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\xi^2} d\xi, \quad \eta = \frac{1}{2} y (\nu t)^{-\frac{1}{2}},\end{aligned}$$

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涡量为

$$\omega = \frac{\partial u}{\partial y} = \frac{u_\infty}{\sqrt{\pi \nu t}} e^{-\eta^2} - \frac{1}{\nu} \frac{\partial p}{\partial x} \left[\sqrt{\frac{4\nu t}{\pi}} e^{-\eta^2} - y(1 - \operatorname{erf}(\eta)) \right],$$

这些结果说明了在很短的时间内, 壁面附近的流动状态.

在计算流动中的物体所受的阻力或流动中物体表面 $\mathfrak{S}$ 形状优化的计算时, 人们必须计算物体表面的法向速度梯度 $u_1 := \frac{\partial u}{\partial n}$ 和法向应力 σ_n , 这使得人们不得不使用精细的网格. 例如, 在计算飞机外部绕流时, 通常把几乎 80% 的网格都集中在边界层附近. 不但影响网格的质量和花费大量的人工, 而且速度梯度的数值微分精度不高^[3, 4].

本文利用半测地坐标系、微分几何方法和张量分析工具, 给出 $\mathbf{u}_1, p_0 = p|_{\mathfrak{S}}$ 满足的一组二维流形上的二阶偏微分方程, 证明边界层方程组的适定性问题, 并且给出解关于边界形状的 Gateaux 导数所满足的偏微分方程. 本文给出阻力泛函的计算公式, 并应用于飞机外形最优控制, 给出一些数值计算例子. 本文组织如下: 第 2 节, 建立在物体表面基础上的曲面的微分几何和半测地坐标系; 第 3 节, 边界层上在半测地坐标系下 Navier-Stokes 方程的速度 - 压力变形式和解 Taylor 展式 (关于贯裁变量); 第 4 节, 边界层方程; 第 5 节, 解关于形状的 Gateaux 导数方程; 第 6 节, 形状优化; 第 7 节, 阻力泛函计算公式, 形状控制的共轭梯度算法, 一维极小化的计算公式; 第 8 节, 计算例子.

2 建立在物体表面基础上的半测地坐标系

假设物体表面是一个二维流形, 它由若干二维曲面光滑拼接而成, 记为

$$\mathfrak{S} = \bigcup_{\alpha} \mathfrak{S}_{\alpha}.$$

设 $\omega \subset \mathbb{R}^2$ 是一个 Lipschitz 区域, $\gamma = \partial\omega$, 设 $\theta: \bar{\omega} \subset \mathbb{R}^2 \mapsto \mathfrak{S}_{\alpha} \in E^3$ 是一个光滑单射浸入, 使得对任何 $(x^1, x^2) \in \omega$, $\mathbf{e}_{\alpha} = \partial_{\alpha}\theta$ 是线性无关的,

$$\mathfrak{S} = \{\theta(x^1, x^2), \forall (x^1, x^2) \in \omega\}$$

是一个 $\mathbb{R}^3$ 中的曲面. 记 Ω_{δ} 厚度为 δ 的边界层, 底层是 $\mathfrak{S}$, 顶层记为

$$\tilde{\mathfrak{S}} = \{\theta(x^1, x^2) + \delta \mathbf{n}(x^1, x^2), \forall (x^1, x^2) \in \omega\},$$

它与底层的测地距离是 δ , 其中 $\mathbf{n}(x^1, x^2)$ 是 $\mathfrak{S}$ 在点 (x^1, x^2) 的单位外法线. 为简单, 设 Ω_{δ} 是一个闭的环形状,

$$\begin{aligned} \Omega_{\delta} &= \{\Theta(x, \xi) = \theta(x) + \xi \mathbf{n}; x = (x^1, x^2) \in \omega, 0 \leq \xi \leq \delta\}, \\ \mathbf{n}(x) &= \frac{\mathbf{e}_1(x) \times \mathbf{e}_2(x)}{|\mathbf{e}_1(x) \times \mathbf{e}_2(x)|}, \quad \forall x \in \omega, \\ \mathbf{e}_{\alpha}(x) &= \frac{\partial \theta}{\partial x^{\alpha}}(x), \end{aligned}$$

这里 $\mathbf{n}(x)$ 为物体表面的单位法向量, $\mathbf{e}_{\alpha}(x)$ 为 $\mathfrak{S}$ 上在 x 点的基向量, 那么, $x^1, x^2, x^3 = \xi$ 就是建立在以物体表面 $\mathfrak{S}$ 为基础上的 S -族坐标系, 即半测地坐标系 (见图 1). 且三维 Euclid 空间 E^3 和曲面 $\mathfrak{S}$ 的度量张量分别是

$$g_{ij} = \frac{\partial \Theta}{\partial x^i} \frac{\partial \Theta}{\partial x^j}, \quad a_{\alpha\beta}(x) = \mathbf{e}_{\alpha}(x) \mathbf{e}_{\beta}(x),$$

图 1 半测地坐标

并成立如下关系^[1,2]:

$$\begin{cases} g_{\alpha\beta}(x, \xi) = a_{\alpha\beta}(x) - 2\xi b_{\alpha\beta}(x) + \xi^2 c_{\alpha\beta}(x), \\ g_{\alpha 3}(x, \xi) = g_{3\alpha}(x, \xi) = 0, \quad g_{33}(x, \xi) = 1, \\ g(x, \xi) = \det(g_{ij}) = \theta^2 a(x), \quad a = \det(a_{\alpha\beta}), \\ g^{\alpha\beta}(x, \xi) = \theta^{-2}(a^{\alpha\beta}(x) - 2K\hat{b}^{\alpha\beta}(x)\xi + K^2\xi^2\hat{c}^{\alpha\beta}(x)) = \theta^{-2}(p(\xi)a^{\alpha\beta} + q(\xi)b^{\alpha\beta}), \\ g^{3\alpha}(x, \xi) = g^{\alpha 3}(x, \xi) = 0, \quad g^{33}(x, \xi) = 1, \\ p(\xi) = 1 - 4H\xi + (4H^2 - K)\xi^2, \quad q(\xi) = 2\xi - 2H\xi^2, \quad \theta = 1 - 2H\xi + K\xi^2, \end{cases} \quad (2.1)$$

其中 $x = (x^1, x^2)$ 为曲面上的 Gauss 坐标系, $b_{\alpha\beta}$ 和 $c_{\alpha\beta}$ 分别为曲面的第二和第三基本型, $a^{\alpha\beta}$ 为度量张量的逆变分量, $a^{\alpha\beta}a_{\beta\lambda} = \delta_{\lambda}^{\alpha}$, $b_{\alpha\beta}$ 和 $c_{\alpha\beta}$ 的逆变分量和逆矩阵分别是

$$b^{\alpha\beta} = a^{\alpha\lambda}a^{\beta\sigma}b_{\lambda\sigma}, \quad c^{\alpha\beta} = a^{\alpha\lambda}a^{\beta\sigma}c_{\lambda\sigma}, \quad \hat{b}^{\alpha\beta}b_{\beta\lambda} = \delta_{\lambda}^{\alpha}, \quad \hat{c}^{\alpha\beta}c_{\beta\lambda} = \delta_{\lambda}^{\alpha},$$

H 和 K 分别为曲面 $\mathfrak{S}$ 的平均和 Gauss 曲率, g^{ij} 为度量张量的逆变分量. 从而可以定义空间和曲面上的共轭标架 $\mathbf{R}^i = g^{ij}\mathbf{R}_j = g^{ij}\partial_j\mathbf{R}$, $\mathbf{e}^{\alpha} = a^{\alpha\beta}\mathbf{e}_{\beta}$. 以后总假定, 拉丁字母 $i, j, k, \dots$ 跑过 $1, 2, 3$; 而希腊字母跑过 $1, 2$; 上下指标相同表示求和; 在字母上冠以星号 $*$ 表示在曲面 $\mathfrak{S}$ 上的量, 如 ∇_{α}^* 表示在 $\mathfrak{S}$ 上的协变导数. 下面列出一些在空间和曲面上一些微分算子之间的关系. 设 $\varepsilon^{\alpha\beta}$ 和 $\varepsilon_{\alpha\beta}$ 是 $\mathfrak{S}$ 上的循环张量,

$$\varepsilon_{\alpha\beta} = \begin{cases} \sqrt{a}, \\ -\sqrt{a}, \\ 0, \end{cases} \quad \varepsilon^{\alpha\beta} = \begin{cases} \frac{1}{\sqrt{a}}, & (\alpha, \beta) : (1, 2) \text{ 的偶排列,} \\ -\frac{1}{\sqrt{a}}, & (\alpha, \beta) : (1, 2) \text{ 的奇排列,} \\ 0, & \text{其他,} \end{cases} \quad (2.2)$$

第一、第二和第三基本型之间有如下关系^[1]:

$$\begin{cases} Ka_{\alpha\beta} - 2Hb_{\alpha\beta} + c_{\alpha\beta} = 0, & a^{\alpha\beta} - 2H\hat{b}^{\alpha\beta} + K\hat{c}^{\alpha\beta} = 0, \\ 2Ha^{\alpha\beta} - b^{\alpha\beta} - K\hat{b}^{\alpha\beta} = 0, & (K - 4H^2)a^{\alpha\beta} + 2Hb^{\alpha\beta} + K^2\hat{c}^{\alpha\beta} = 0. \end{cases} \quad (2.3)$$

在 S - 坐标系下, E^3 中和 $\mathfrak{S}$ 上的 Christoffel 记号有如下关系^[1-3]:

$$\Gamma_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^{\gamma*} + \theta^{-1} R_{\alpha\beta}^\gamma, \quad \Gamma_{\beta 3}^\alpha = \theta^{-1} I_{\beta}^\alpha, \quad \Gamma_{\alpha\beta}^3 = J_{\alpha\beta}, \quad \Gamma_{\beta 3}^3 = \Gamma_{3\beta}^3 = \Gamma_{33}^\alpha = \Gamma_{33}^3 = 0, \quad (2.4)$$

其中

$$\begin{cases} I_{\beta}^\alpha = -b_{\beta}^\alpha + K\xi\delta_{\beta}^\alpha, & J_{\alpha\beta} = b_{\alpha\beta} - \xi c_{\alpha\beta}, \\ R_{\beta\sigma}^\alpha = -\nabla_{\sigma}^* b_{\beta}^\alpha \xi + K\hat{b}_{\mu}^{\alpha} \nabla_{\sigma}^* b_{\beta}^{\mu} \xi^2 = -\nabla_{\sigma}^* b_{\beta}^{\alpha} \xi + (2H\delta_{\mu}^{\alpha} - b_{\mu}^{\alpha}) \nabla_{\sigma}^* b_{\beta}^{\mu} \xi^2. \end{cases} \quad (2.5)$$

关于协变导数^[3,5]

$$\begin{aligned} \nabla_i u^j &= \frac{\partial u^j}{\partial x^i} + \Gamma_{ik}^j u^k, \quad \forall u \in H^1(\Omega)^3, \quad \nabla_{\alpha}^* u^{\beta} = \frac{\partial u^{\beta}}{\partial x^{\alpha}} + \Gamma_{\alpha\lambda}^{\beta*} u^{\lambda}, \quad \forall u \text{ 在 } T\mathfrak{S} \text{ 上}, \\ \operatorname{div}(u) &= \nabla_i u^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} u^i), \quad \operatorname{div} u = \nabla_{\alpha}^* u^{\alpha} = \frac{1}{\sqrt{a}} \frac{\partial}{\partial x^{\alpha}} (\sqrt{a} u^{\alpha}), \\ \operatorname{rot}(u) &= \nabla \times u = \varepsilon^{ijk} \nabla_j u_k e_i, \quad \operatorname{rot}^*(u) = \varepsilon^{\alpha\beta} \nabla_{\alpha}^* u_{\beta} n, \end{aligned} \quad (2.6)$$

这里 $u_i = g_{ij} u^j$ 是 u 的协变分量. 在 $\mathbb{R}^3$ 中和 $\mathfrak{S}$ 上的变形张量分别是

$$\begin{aligned} e_{ij}(u) &= \frac{1}{2} (\nabla_i u_j + \nabla_j u_i) = \frac{1}{2} (\delta_i^k g_{jl} + \delta_j^k g_{il}) \nabla_k u^l, \\ e_{\alpha\beta}^*(u) &= \frac{1}{2} (\nabla_{\alpha}^* u_{\beta} + \nabla_{\beta}^* u_{\alpha}) = \frac{1}{2} (\delta_{\alpha}^{\lambda} a_{\beta\sigma} + \delta_{\beta}^{\lambda} a_{\alpha\sigma}) \nabla_{\lambda}^* u^{\sigma}. \end{aligned}$$

显然, $\operatorname{div}(u) = g^{ij} e_{ij}(u)$, $\operatorname{div}^*(u) = a^{\alpha\beta} e_{\alpha\beta}^*(u)$. 在 S - 坐标系下, 关于协变导数之间的关系^[3,5] 为

$$\begin{cases} \nabla_{\alpha} u^{\beta} = \nabla_{\alpha}^* u^{\beta} + \theta^{-1} (I_{\alpha}^{\beta} u^3 + R_{\alpha\lambda}^{\beta} u^{\lambda}), & \nabla_{\alpha} u^3 = \frac{\partial u^3}{\partial x^{\alpha}} + J_{\alpha\beta} u^{\beta}, \\ \nabla_3 u^{\beta} = \frac{\partial u^{\beta}}{\partial \xi} + \theta^{-1} I_{\alpha}^{\beta} u^{\alpha}, & \nabla_3 u^3 = \frac{\partial u^3}{\partial \xi}, \\ \operatorname{div} u = \operatorname{div}^*(u) + \frac{\partial u^3}{\partial \xi} + \theta^{-1} [-2H u^3 + (2K u^3 - 2u^{\lambda} \nabla_{\lambda} H) \xi + u^{\lambda} \nabla_{\lambda} K \xi^2], \end{cases} \quad (2.7)$$

变形张量之间关系为

$$\begin{cases} e_{\alpha\beta}(u) = \frac{1}{2} (\nabla_{\alpha} u_{\beta} + \nabla_{\beta} u_{\alpha}) = \gamma_{\alpha\beta}(u) + \frac{1}{2} \gamma_{\alpha\beta}^1(u) \xi + \frac{1}{2} \gamma_{\alpha\beta}^2(u) \xi^2, \\ e_{\alpha 3}(u) = \frac{1}{2} \left(g_{\alpha\beta} \frac{\partial u^{\beta}}{\partial \xi} + \nabla_{\alpha}^* u^3 \right) = \gamma_{\alpha 3}(u) + \frac{1}{2} \gamma_{\alpha 3}^1(u) \xi + \frac{1}{2} \gamma_{\alpha 3}^2(u) \xi^2, \\ e_{33}(u) = \nabla_3 u^3 = \frac{\partial u^3}{\partial \xi}, \end{cases} \quad (2.8)$$

即变形张量在半测地坐标系下是贯截变量的一个二次多项式,

$$\begin{aligned} \gamma_{\alpha\beta}(u) &= e_{\alpha\beta}^*(u) - b_{\alpha\beta} u^3, \quad \gamma^{\alpha\beta}(u) = a^{\alpha\nu} a^{\beta\sigma} \gamma_{\nu\sigma}(u), \\ \gamma_{\alpha\beta}^1(u) &= -(b_{\alpha\lambda} \nabla_{\beta}^* u^{\lambda} + b_{\beta\lambda} \nabla_{\alpha}^* u^{\lambda}) + c_{\alpha\beta} u^3 - \nabla_{\lambda}^* b_{\alpha\beta} u^{\lambda}, \\ \gamma_{\alpha\beta}^2(u) &= \frac{1}{2} (c_{\alpha\lambda} \nabla_{\beta}^* u^{\lambda} + c_{\beta\lambda} \nabla_{\alpha}^* u^{\lambda} + \nabla_{\lambda}^* c_{\alpha\beta} u^{\lambda}), \\ \gamma_{\alpha 3}(u) &= \gamma_{3\alpha} = \frac{1}{2} \left(a_{\alpha\beta} \frac{\partial u^{\beta}}{\partial \xi} + \nabla_{\alpha}^* u^3 \right), \end{aligned} \quad (2.9)$$

$$\begin{aligned}\gamma_{\alpha 3}^1(u) &= -b_{\alpha\beta} \frac{\partial u^\beta}{\partial \xi}, \quad \gamma_{\alpha 3}^2(u) = \frac{1}{2} c_{\alpha\beta} \frac{\partial u^\beta}{\partial \xi}, \\ \gamma_{33}(u) &= \frac{\partial u^3}{\partial \xi}, \quad \gamma_{33}^1(u) = \gamma_{33}^2(u) = 0.\end{aligned}$$

现在设想, $\mathfrak{S}$ 发生一个变形, 即在 $\mathfrak{S}$ 上任一点受到一个位移 $\boldsymbol{\eta}$, 于是 $\mathfrak{S}$ 变为新的曲面 $\mathfrak{S}_\eta$. 下面给出变形后的曲面的几何, 它的度量张量, 第一、第二和第三基本型: $a_{\alpha\beta}(\eta)$, $a^{\alpha\beta}(\eta)$, $b_{\alpha\beta}(\eta)$, $\hat{b}^{\alpha\beta}(\eta)$, $c_{\alpha\beta}(\eta)$, $\hat{c}^{\alpha\beta}(\eta)$, $H(\eta)$, $K(\eta)$ 和单位法向量 $\mathbf{n}(\eta)$, 成立如下定理.

定理 1 ^[3,5] $\mathfrak{S}_\eta$ 上几何量与 $\mathfrak{S}$ 相应的几何量有如下关系:

$$\begin{cases} a_{\alpha\beta}(\eta) - a_{\alpha\beta} = 2\gamma_{\alpha\beta}(\eta) + o(\|\delta a_{\alpha\beta}\|), \\ b_{\alpha\beta}(\eta) - b_{\alpha\beta} = \rho_{\alpha\beta}(\eta) + o(\|\delta b_{\alpha\beta}\|), \\ \sqrt{a(\eta)} - \sqrt{a} = \frac{1}{2}\gamma_0(\eta)\sqrt{a}, \\ a = (1 + \gamma_0(\eta))a + o(|\eta|^2), \\ \mathbf{n}(\eta) - \mathbf{n} = \frac{1}{2}\gamma_0(\eta)\mathbf{n}, \end{cases} \quad (2.10)$$

其中法向量归一化后, $|\mathbf{n}(\eta)| = 1$, 所以, $\mathbf{n}(\eta) = \mathbf{n}$,

$$\begin{aligned}\rho_{\alpha\beta}(\eta) &= \frac{1}{2}[\nabla_\alpha^* \nabla_\beta^* + \nabla_\beta^* \nabla_\alpha^*]\eta^3 + b_{\alpha\sigma} \nabla_\beta^* \eta^\sigma + b_{\beta\sigma} \nabla_\alpha^* \eta^\sigma - c_{\alpha\beta} \eta^3 + \nabla_\sigma^* b_{\alpha\beta} \eta^\sigma, \\ \gamma_{\alpha\beta}(\eta) &= \frac{1}{2}(a_{\alpha\lambda} \nabla_\beta^* \eta^\lambda + a_{\beta\lambda} \nabla_\alpha^* \eta^\lambda) - b_{\alpha\beta} \eta^3, \\ \gamma_0(\eta) &= a^{\alpha\beta} \gamma_{\alpha\beta}(\eta) = \operatorname{div}^* \eta - 2H\eta^3.\end{aligned} \quad (2.11)$$

定理 2 在曲面 $\mathfrak{S}_\eta$ 上还成立如下公式:

$$\begin{aligned}a^{\alpha\beta}(\eta) - a^{\alpha\beta} &= 2a^{\alpha\beta}\gamma_0(\eta) + o(\|\eta\|), \\ b^{\alpha\beta}(\eta) &= b^{\alpha\beta} + \mathbf{b}^{\alpha\beta}(\eta), \quad \mathbf{b}^{\alpha\beta}(\eta) = 4\gamma_0(\eta)b^{\alpha\beta} + \rho^{\alpha\beta}(\eta), \\ a(\eta) &= (1 + \gamma_0(\eta))a, \quad b(\eta) = b(1 + \rho_b(\eta)), \\ K(\eta) &= K + \hat{K}(\eta), \quad \hat{K}(\eta) = K(\rho_b(\eta) - \gamma_0(\eta)), \\ H(\eta) &= H + \hat{H}(\eta), \quad \hat{H}(\eta) = 4H\gamma_0(\eta) + \rho_0(\eta), \\ c_{\alpha\beta}(\eta) &= c_{\alpha\beta} + \hat{c}_{\alpha\beta}(\eta), \\ \hat{c}_{\alpha\beta}(\eta) &= 2H\rho_{\alpha\beta}(\eta) + 2\hat{H}(\eta)b_{\alpha\beta} - 2K\gamma_{\alpha\beta}(\eta) - \hat{K}(\eta)a_{\alpha\beta}, \\ c^{\alpha\beta}(\eta) &= c^{\alpha\beta} + \hat{c}^{\alpha\beta}(\eta), \\ \hat{c}^{\alpha\beta}(\eta) &= 2H\hat{b}^{\alpha\beta}(\eta) + 2\hat{H}(\eta)b^{\alpha\beta} - 2K\gamma_{\alpha\beta}(\eta)a^{\alpha\beta} - \hat{K}(\eta)a^{\alpha\beta}, \\ b_\beta^\alpha(\eta) &= b_\beta^\alpha + \hat{b}_\beta^\alpha(\eta), \quad c_\beta^\alpha(\eta) = c_\beta^\alpha + \hat{c}_\beta^\alpha(\eta), \\ \hat{b}_\beta^\alpha(\eta) &= 2\gamma_0(\eta)b_\beta^\alpha + a^{\alpha\lambda}\rho_{\lambda\beta}(\eta), \\ \hat{b}_\beta^\alpha(\eta) &= 2\gamma_0(\eta)c_\beta^\alpha + a^{\alpha\lambda}\hat{c}_{\lambda\beta}(\eta),\end{aligned} \quad (2.12)$$

其中

$$\rho_b(\eta) = b^{\alpha\beta}\rho_{\alpha\beta}(\eta), \quad \rho_0(\eta) = a^{\alpha\beta}\rho_{\alpha\beta}(\eta), \quad b = \det(b_{\alpha\beta}), \quad b(\eta) = \det(b_{\alpha\beta}(\eta)), \quad (2.13)$$

而 $c_\beta^\lambda = a^{\lambda\sigma} c_{\sigma\beta}$ 是二阶混合张量.

证明 应用 (2.10) 和张量运算, 立即可得 $\varepsilon^{\alpha\beta}(\eta) = \frac{\varepsilon^{\alpha\beta}}{\sqrt{1+\gamma_0(\eta)}}$, 故有

$$\begin{aligned}
 a^{\alpha\beta}(\eta) &= \varepsilon^{\alpha\lambda}(\eta) \varepsilon^{\beta\sigma}(\eta) a_{\lambda\sigma}(\eta) = \varepsilon^{\alpha\lambda} \varepsilon^{\beta\sigma} a_{\lambda\sigma} \left(1 + \frac{2\gamma_0(\eta)}{1+\gamma_0(\eta)} \right) \\
 &= a^{\alpha\beta} + a^{\alpha\beta} \cdot 2\gamma_0(\eta) + o(\|\eta\|), \\
 b(\eta) &= b_{22}(\eta)b_{11}(\eta) - (b_{12}(\eta))^2 \\
 &= b + b_{22}\rho_{11}(\eta) + b_{11}\rho_{22}(\eta) + 2b_{12}\rho_{12}(\eta) = b + bb^{\alpha\beta}\rho_{\alpha\beta}(\eta), \\
 K(\eta) &= \frac{b(\eta)}{a(\eta)} = \frac{b}{a} \frac{1+b^{\alpha\beta}\rho_{\alpha\beta}(\eta)}{1+\gamma_0(\eta)} = K(1+\rho_b(\eta))(1-\gamma_0(\eta)) + o(|\eta|^2) \\
 &= K + K(\rho_b(\eta) - \gamma_0(\eta)) + o(\|\eta\|), \\
 H(\eta) - H &= a^{\alpha\beta}(\eta)b_{\alpha\beta}(\eta) - a^{\alpha\beta}b_{\alpha\beta} \\
 &= a^{\alpha\beta}\rho_{\alpha\beta}(\eta) + 2a^{\alpha\beta}b_{\alpha\beta}\gamma_0(\eta) + 2a^{\alpha\beta}\rho_{\alpha\beta}(\eta)\gamma_0(\eta) + o(\|\eta\|) \\
 &= \rho_0(\eta) + 4H\gamma_0(\eta) = o(\|\eta\|).
 \end{aligned} \tag{2.14}$$

用同样的方法, 以及公式

$$\begin{aligned}
 Ka_{\alpha\beta} - 2Hb_{\alpha\beta} + c_{\alpha\beta} &= 0, \quad Ka^{\alpha\beta} - 2Hb^{\alpha\beta} + c^{\alpha\beta} = 0, \\
 c_\beta^\alpha(\eta) &= a^{\alpha\lambda}(\eta)c_{\lambda\beta}(\eta), \quad b_\beta^\alpha(\eta) = a^{\alpha\lambda}(\eta)b_{\lambda\beta}(\eta)
 \end{aligned}$$

可以推出其他公式. □

3 边界层内 Navier-Stokes 方程原始变量的变分方法

为简单, 我们假设边界层 Ω_δ 是一个闭环, 内表面即底层是物体表面 $\mathfrak{S}$, 外表面即顶层是 $\tilde{\mathfrak{S}}$, 定常不可压缩 Navier-Stokes 方程的不变形式为

$$\begin{cases} -2\mu\nabla_j e^{ij}(u) + g^{ij}\nabla_j p + u^j\nabla_j u^i = f^i, & \text{在 } \Omega_\delta \text{ 内,} \\ \nabla_j u^j = 0, \\ \sigma(u, p)n|_{\tilde{\mathfrak{S}}}^+ = \sigma(u, p)n|_{\tilde{\mathfrak{S}}}^-, \quad \mathbf{u}|_{\tilde{\mathfrak{S}}}^+ = \mathbf{u}|_{\tilde{\mathfrak{S}}}^-, & \text{在顶层 } \tilde{\mathfrak{S}}, \\ u|_{\mathfrak{S}} = 0, & \text{在底层物体表面上 } \mathfrak{S}, \\ \text{在其他方向上满足周期性边界条件.} \end{cases} \tag{3.1}$$

令

$$\begin{aligned}
 \mathbf{V}(\Omega_\delta) &= \{\mathbf{u} \in \mathbf{H}^1(\Omega_\delta), \mathbf{u}|_{\mathfrak{S}} = \mathbf{u}|_{\xi=0} = 0, \text{在 } x^\alpha \text{ 方向满足周期性边界条件}\}, \\
 \mathbf{M}(\Omega_\delta) &= \{q \in L^2(\Omega_\delta), \text{在 } x^\alpha \text{ 方向满足周期性边界条件}\}.
 \end{aligned}$$

对应于 (3.1) 的变分问题是

$$\begin{cases} \text{求 } \mathbf{u} \in \mathbf{V}(\Omega_\delta), p \in \mathbf{M}(\Omega_\delta), \text{ 使得} \\ a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}; \mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{V}(\Omega_\delta), \\ (\operatorname{div} \mathbf{u}, q) = 0, \quad q \in L^2(\Omega_\delta), \end{cases} \tag{3.2}$$

利用 (2.11), 有

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) = \int_{\Omega_\delta} 2\mu e^{ij}(\mathbf{u}) e_{ij}(\mathbf{v}) \sqrt{g} dx d\xi = \int_{\Omega_\delta} 2\mu g^{ik} g^{jm} e_{km}(\mathbf{u}) e_{ij}(\mathbf{v}) \theta \sqrt{a} dx d\xi, \\ b(u; u, v) = \int_{\Omega_\delta} u^j \nabla_j u^i g_{ik} v^k \theta \sqrt{a} dx d\xi, \\ \langle \mathbf{F}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle + \langle \sigma \cdot \mathbf{n}, \mathbf{v} \rangle|_{\mathfrak{S}} = \langle \mathbf{f}, \mathbf{v} \rangle + \langle \mathbf{h}, \mathbf{v} \rangle|_{\xi=\delta}, \\ h^i := \sigma^{ij}(\mathbf{u}, p) n_j = (2\mu e^{ij}(\mathbf{u}) - g^{ij} p) n_j, \quad \mathfrak{S} \text{ 上的法向应力.} \end{cases} \quad (3.3)$$

下面采用建立在 $\mathfrak{S}$ 基础上的 S - 坐标系 (x^α, ξ) . 令黏性张量 $A^{ijkm} = 2\mu\theta g^{ik} g^{jm}$, 那么, 由 (2.1) 可知,

$$\begin{aligned} A^{\alpha 3 \beta 3} &= 2\mu\theta g^{\alpha 3} g^{\beta 3} = 0, \quad A^{3\alpha 3\beta} = 2\mu\theta g^{3\alpha} g^{3\beta} = 0, \\ A^{\alpha 3 3 \beta} &= A^{3\alpha \beta 3} = A^{\alpha \beta \lambda 3} = A^{\alpha \beta 3 \lambda} = A^{\alpha 3 \beta \lambda} = A^{3\alpha \beta \lambda} = 0, \\ A^{\alpha \beta 3 3} &= A^{33 \alpha \beta} = 2\mu\theta g^{\alpha \beta} g^{33} = 2\mu\theta^{-1}(p(\xi)a^{\alpha\beta} + q(\xi)b^{\alpha\beta}), \\ A^{\alpha \beta \lambda \sigma} &= 2\mu\theta^{-3}(p(\xi)a^{\alpha\beta} + q(\xi)b^{\alpha\beta})(p(\xi)a^{\lambda\sigma} + q(\xi)b^{\lambda\sigma}) \\ &= 2\mu\theta^{-3}[p^2(\xi)a^{\alpha\beta}a^{\lambda\sigma} + q^2(\xi)b^{\alpha\beta}b^{\lambda\sigma} + p(\xi)q(\xi)(a^{\alpha\beta}b^{\lambda\sigma} + a^{\lambda\sigma}b^{\alpha\beta})]. \end{aligned}$$

由 (2.1),

$$\begin{aligned} p^2(\xi) &= 1 - 8H\xi + 2(12H^2 - K)\xi^2 - 8H(4H^2 - K)\xi^3 + (4H^2 - K)^2\xi^4, \\ q^2(\xi) &= 4\xi^2 - 8H\xi^3 + 4H^2\xi^4, \\ p(\xi)q(\xi) &= 2\xi - 10H\xi^2 + 2(8H^2 - K)\xi^3 - 2H(4H^2 - K)\xi^4, \end{aligned}$$

黏性张量是贯截变量 ξ 的有理多项式. 利用 (2.1), 对很小的 ξ 有 Taylor 展式

$$\begin{aligned} \theta^{-1} &= 1 + 2H\xi + (4H^2 - K)\xi^2 + (8H^3 - 4HK)\xi^3 + \cdots, \\ \theta^{-2} &= 1 + 4H\xi + (12H^2 - 2K)\xi^2 + \cdots, \\ \theta^{-3} &= 1 + 6H\xi + (24H^2 - 3K)\xi^2 + \cdots. \end{aligned} \quad (3.4)$$

于是经过简单计算, 可得如下命题.

命题 3 黏性张量有下面关于贯截变量 ξ 的 Taylor 展式:

$$\begin{aligned} A^{\alpha \beta \lambda \sigma} &= a_0^{\alpha \beta \lambda \sigma} + a_1^{\alpha \beta \lambda \sigma} \xi + a_2^{\alpha \beta \lambda \sigma} \xi^2 + \cdots, \\ A^{3333} &= 2\mu(1 - 2H\xi + K\xi^2), \\ A^{33\alpha\beta} &= A^{\alpha\beta 33} = 2\mu(a^{\alpha\beta} + 2(b^{\alpha\beta} - Hb^{\alpha\beta})\xi + 2(Hb^{\alpha\beta} - Ka^{\alpha\beta})\xi^2 + \cdots), \\ A^{3\alpha 3\beta} &= A^{\alpha 3 \beta 3} = A^{33\alpha\beta} = A^{\alpha\beta 33} = A^{3\alpha\beta 3} = A^{\alpha 3 \beta 3} = 0, \\ A^{\alpha 3 3 \beta} &= A^{3\alpha \beta 3} = A^{\alpha \beta \lambda 3} = A^{\alpha \beta 3 \lambda} = A^{\alpha 3 \beta \lambda} = A^{3\alpha \beta \lambda} = 0, \\ a_0^{\alpha \beta \lambda \sigma} &= 2\mu a^{\alpha\beta} a^{\lambda\sigma}, \quad a_1^{\alpha \beta \lambda \sigma} = 4\mu(a^{\alpha\beta} b^{\lambda\sigma} + a^{\lambda\sigma} b^{\alpha\beta} - H a^{\alpha\beta} a^{\lambda\sigma}), \\ a_2^{\alpha \beta \lambda \sigma} &= 2\mu[-5K a^{\alpha\beta} a^{\lambda\sigma} + 2H(a^{\alpha\beta} b^{\lambda\sigma} + a^{\lambda\sigma} b^{\alpha\beta}) + 4b^{\alpha\beta} b^{\lambda\sigma}]. \end{aligned} \quad (3.5)$$

考察二次形式

$$\begin{aligned}
 A(\mathbf{u}, \mathbf{v}) &:= A^{ijklm} e_{ik}(\mathbf{u}) e_{jm}(\mathbf{v}) \\
 &= A^{\alpha\beta\lambda\sigma} e_{\alpha\lambda}(\mathbf{u}) e_{\beta\sigma}(\mathbf{v}) + A^{\alpha\beta\lambda 3} e_{\alpha\lambda}(\mathbf{u}) e_{\beta 3}(\mathbf{v}) + A^{\alpha\beta 3\lambda} e_{\alpha 3}(\mathbf{u}) e_{\beta\lambda}(\mathbf{v}) \\
 &\quad + A^{\alpha 3\beta\lambda} e_{\alpha\beta}(\mathbf{u}) e_{3\lambda}(\mathbf{v}) + A^{3\alpha\beta\lambda} e_{3\beta}(\mathbf{u}) e_{\alpha\lambda}(\mathbf{v}) + A^{\alpha\beta 33} e_{\alpha 3}(\mathbf{u}) e_{\beta 3}(\mathbf{v}) \\
 &\quad + A^{33\alpha\beta} e_{3\alpha}(\mathbf{u}) e_{3\beta}(\mathbf{v}) + A^{\alpha 3\beta 3} e_{\alpha\beta}(\mathbf{u}) e_{33}(\mathbf{v}) + A^{3\alpha 3\beta} e_{33}(\mathbf{u}) e_{\alpha\beta}(\mathbf{v}) \\
 &\quad + A^{\alpha 33\beta} e_{\alpha 3}(\mathbf{u}) e_{3\beta}(\mathbf{v}) + A^{3\alpha 3\beta} e_{3\beta}(\mathbf{u}) e_{\alpha 3}(\mathbf{v}) + A^{3333} e_{33}(\mathbf{u}) e_{33}(\mathbf{v}) \\
 &= A_m(\mathbf{u}, \mathbf{v}) + A_b(\mathbf{u}, \mathbf{v}), \\
 A_m(\mathbf{u}, \mathbf{v}) &:= A^{\alpha\beta\lambda\sigma} e_{\alpha\lambda}(\mathbf{u}) e_{\beta\sigma}(\mathbf{v}), \\
 A_b(\mathbf{u}, \mathbf{v}) &:= 2\mu\theta g^{\alpha\beta} e_{3\alpha}(\mathbf{u}) e_{3\beta}(\mathbf{v}) + 2\mu\theta e_{33}(\mathbf{u}) e_{33}(\mathbf{v}),
 \end{aligned} \tag{3.6}$$

其中 $A_m(\mathbf{u}, \mathbf{v})$ 称为膜双线性形式, 而 $A_b(\mathbf{u}, \mathbf{v})$ 称为弯曲双线性形式. 首先考查弯曲双线性形式. 实际上,

$$e_{3\alpha}(\mathbf{u}) = \frac{1}{2}(g_{\alpha\lambda} \nabla_3 u^\lambda + \nabla_\alpha u^3) = \frac{1}{2} \left(g_{\alpha\lambda} \left(\frac{\partial u^\lambda}{\partial \xi} + \theta^{-1} I_\nu^\lambda u^\nu \right) + \nabla_\alpha^* u^3 + J_{\alpha\nu} u^\nu \right),$$

但是容易证明 (参见文献 [3, 5])

$$g_{\alpha\lambda} \theta^{-1} I_\nu^\lambda + J_{\alpha\nu} = 0, \tag{3.7}$$

于是, 有

$$\begin{aligned}
 e_{3\alpha}(\mathbf{u}) &= \frac{1}{2} \left(g_{\alpha\lambda} \frac{\partial u^\lambda}{\partial \xi} + \nabla_\alpha^* u^3 \right), \\
 g^{\alpha\beta} e_{3\alpha}(\mathbf{u}) e_{3\beta}(\mathbf{v}) &= \frac{1}{4} g^{\alpha\beta} \left(g_{\alpha\lambda} g_{\beta\sigma} \frac{\partial u^\lambda}{\partial \xi} \frac{\partial v^\sigma}{\partial \xi} + g_{\alpha\lambda} \frac{\partial u^\lambda}{\partial \xi} \nabla_\beta^* v^3 + g_{\beta\sigma} \frac{\partial v^\sigma}{\partial \xi} \nabla_\alpha^* u^3 + \nabla_\alpha^* u^3 \nabla_\beta^* v^3 \right) \\
 &= \frac{1}{4} \left[g_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} \frac{\partial v^\sigma}{\partial \xi} + \frac{\partial u^\beta}{\partial \xi} \nabla_\beta^* v^3 + \frac{\partial v^\beta}{\partial \xi} \nabla_\beta^* u^3 + g^{\alpha\beta} \nabla_\alpha^* u^3 \nabla_\beta^* v^3 \right],
 \end{aligned}$$

这里用到了 $g^{\alpha\beta} g_{\beta\lambda} = \delta_\lambda^\alpha$. 于是, 有

$$2\mu\theta g^{\alpha\beta} e_{3\alpha}(\mathbf{u}) e_{3\beta}(\mathbf{v}) = \frac{1}{2}\mu \left[\theta \left(g_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} \frac{\partial v^\sigma}{\partial \xi} + \frac{\partial u^\beta}{\partial \xi} \nabla_\beta^* v^3 + \frac{\partial v^\beta}{\partial \xi} \nabla_\beta^* u^3 \right) + \theta g^{\alpha\beta} \nabla_\alpha^* u^3 \nabla_\beta^* v^3 \right],$$

应用 (3.4) 和 (2.1),

$$\begin{aligned}
 \theta g^{\alpha\beta} &= \theta^{-1} (p(\xi) a^{\alpha\beta} + q(\xi) b^{\alpha\beta}) \\
 &= (1 - 2H\xi - 2K\xi^2) a^{\alpha\beta} + (2\xi + 2H\xi^2) b^{\alpha\beta} + o(|\xi|^3) \\
 &= a^{\alpha\beta} + 2(b^{\alpha\beta} - H a^{\alpha\beta}) \xi + 2(H b^{\alpha\beta} - K a^{\alpha\beta}) \xi^2 + o(|\xi|^3), \\
 \theta g_{\lambda\sigma} &= a_{\lambda\sigma} - 2(H a_{\lambda\sigma} + b_{\lambda\sigma}) \xi + 6H b_{\lambda\sigma} \xi^2 + o(|\xi|^3),
 \end{aligned}$$

这里用到 (2.3) 的第一式, 代入 (3.6),

$$A_b(\mathbf{u}, \mathbf{v}) = I_0(u, v) + I_1(u, v)\xi + I_2(u, v)\xi^2 + \cdots,$$

$$\begin{aligned}
I_0(\mathbf{u}, \mathbf{v}) &= \frac{1}{2}\mu \left(a_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} \frac{\partial v^\sigma}{\partial \xi} + \frac{\partial u^\beta}{\partial \xi} \nabla_\beta^* v^3 + \frac{\partial v^\beta}{\partial \xi} \nabla_\beta^* u^3 + a^{\alpha\beta} \nabla_\alpha^* u^3 \nabla_\beta^* v^3 \right) + 2\mu \frac{\partial u^3}{\partial \xi} \frac{\partial v^3}{\partial \xi}, \\
I_1(\mathbf{u}, \mathbf{v}) &= \frac{1}{2}\mu \left(-2(Ha_{\lambda\sigma} + b_{\lambda\sigma}) \frac{\partial u^\lambda}{\partial \xi} \frac{\partial v^\sigma}{\partial \xi} - 2H \left(\frac{\partial u^\beta}{\partial \xi} \nabla_\beta^* v^3 + \frac{\partial v^\beta}{\partial \xi} \nabla_\beta^* u^3 \right) \right. \\
&\quad \left. + 2(b^{\alpha\beta} - Ha^{\alpha\beta}) \nabla_\alpha^* u^3 \nabla_\beta^* v^3 \right) - 4\mu H \frac{\partial u^3}{\partial \xi} \frac{\partial v^3}{\partial \xi}, \\
I_2(\mathbf{u}, \mathbf{v}) &= \frac{1}{2}\mu \left(6Hb_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} \frac{\partial v^\sigma}{\partial \xi} + K \left(\frac{\partial u^\beta}{\partial \xi} \nabla_\beta^* v^3 + \frac{\partial v^\beta}{\partial \xi} \nabla_\beta^* u^3 \right) \right. \\
&\quad \left. + 2(Hb^{\alpha\beta} - Ka^{\alpha\beta}) \nabla_\alpha^* u^3 \nabla_\beta^* v^3 \right) + 2\mu K \frac{\partial u^3}{\partial \xi} \frac{\partial v^3}{\partial \xi}.
\end{aligned} \tag{3.8}$$

命题 4 由 (3.3) 所定义的双线性形式 $a(\cdot, \cdot)$ 可以表达为膜双线性形式和弯曲双线性形式之和,

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) = a_m(\mathbf{u}, \mathbf{v}) + a_b(\mathbf{u}, \mathbf{v}), \\ a_m(\mathbf{u}, \mathbf{v}) = \int_{\Omega_\delta} A_m(\mathbf{u}, \mathbf{v}) \sqrt{a} dx d\xi \\ \quad = \int_{\Omega_\delta} [J_0(\mathbf{u}, \mathbf{v}) + J_1(\mathbf{u}, \mathbf{v})\xi + J_2(\mathbf{u}, \mathbf{v})\xi^2 + J_R(\mathbf{u}, \mathbf{v}, \xi)] \sqrt{a} dx d\xi, \\ a_b(\mathbf{u}, \mathbf{v}) = \int_{\Omega_\delta} A_b(\mathbf{u}, \mathbf{v}) \sqrt{a} dx d\xi \\ \quad = \int_{\Omega_{ij}} [I_0(\mathbf{u}, \mathbf{v}) + I_1(\mathbf{u}, \mathbf{v})\xi + I_2(\mathbf{u}, \mathbf{v})\xi^2 + I_R(\mathbf{u}, \mathbf{v}, \xi)] \sqrt{a} dx d\xi, \end{cases} \tag{3.9}$$

其中

$$\begin{cases} J_0(\mathbf{u}, \mathbf{v}) := a_0^{\alpha\beta\lambda\sigma} \gamma_{\alpha\lambda}(\mathbf{u}) \gamma_{\beta\sigma}(\mathbf{v}), \\ J_1(\mathbf{u}, \mathbf{v}) = a_0^{\alpha\beta\lambda\sigma} (\gamma_{\alpha\lambda}^1(\mathbf{u}) \gamma_{\beta\sigma}(\mathbf{v}) + \gamma_{\alpha\lambda}(\mathbf{u}) \gamma_{\beta\sigma}^1(\mathbf{v})) + a_1^{\alpha\beta\lambda\sigma} \gamma_{\alpha\lambda}(\mathbf{u}) \gamma_{\beta\sigma}(\mathbf{v}), \\ J_2(\mathbf{u}, \mathbf{v}) := a_0^{\alpha\beta\lambda\sigma} (\gamma_{\alpha\lambda}^2(\mathbf{u}) \gamma_{\beta\sigma}(\mathbf{v}) + \gamma_{\alpha\lambda}(\mathbf{u}) \gamma_{\beta\sigma}^2(\mathbf{v}) + \gamma_{\beta\sigma}^1(\mathbf{u}) \gamma_{\beta\sigma}^1(\mathbf{v})) \\ \quad + a_2^{\alpha\beta\lambda\sigma} (\gamma_{\alpha\lambda}(\mathbf{u}) \gamma_{\beta\sigma}(\mathbf{v})) + a_1^{\alpha\beta\lambda\sigma} (\gamma_{\alpha\lambda}^1(\mathbf{u}) \gamma_{\beta\sigma}(\mathbf{v}) + \gamma_{\alpha\lambda}(\mathbf{u}) \gamma_{\beta\sigma}^1(\mathbf{v})), \end{cases} \tag{3.10}$$

而 I_α 由 (3.8) 所定义的. 这里 J_R 和 I_R ($R = 0, 1, 2$) 分别为对应的 Taylor 余项.

证明 实际上, 由 (2.8) 和 (3.5) 容易得到膜双线性形式,

$$\begin{aligned}
a_m(\mathbf{u}, \mathbf{v}) &= \int_{\Omega_\delta} A_m(\mathbf{u}, \mathbf{v}) \sqrt{a} dx d\xi \\
&= \int_{\Omega_\delta} A^{\alpha\beta\lambda\sigma} e_{\alpha\lambda}(\mathbf{u}) e_{\beta\sigma}(\mathbf{v}) \sqrt{a} dx d\xi \\
&= (a_0^{\alpha\beta\lambda\sigma} + a_1^{\alpha\beta\lambda\sigma} \xi + a_2^{\alpha\beta\lambda\sigma} \xi^2 + A_R^{\alpha\beta\lambda\sigma}) (\gamma_{\alpha\lambda}(u) + \gamma_{\alpha\lambda}^1(u)\xi + \gamma_{\alpha\lambda}^2(u)\xi^2) \\
&\quad \times (\gamma_{\beta\sigma}(v) + \gamma_{\beta\sigma}^1(v)\xi + \gamma_{\beta\sigma}^2(v)\xi^2),
\end{aligned}$$

简单计算可推出 (3.9) 和 (3.10). $a_b(\mathbf{u}, \mathbf{v})$ 直接由命题 3 得到. 证毕. $\square$

下面考虑非线性项.

命题 5 三线性形式 $b(\mathbf{u}, \mathbf{u}, \mathbf{v})$ 可以表示为膜三线性形式和弯曲三线性形式之和,

$$\begin{aligned}
 b(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= \int_{\Omega_\delta} (\mathbf{u} \nabla) \mathbf{u} \mathbf{v} \theta \sqrt{a} dx d\xi = b_m(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b_b(\mathbf{u}, \mathbf{u}, \mathbf{v}), \\
 b_m(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= \int_{\Omega_\delta} \varphi_m(\mathbf{u}, \mathbf{u}, \mathbf{v}) \sqrt{a} dx d\xi \\
 &= \int_{\Omega_\delta} \theta [(g_{\lambda\sigma} \nabla_\beta^* u^\lambda u^\beta - 2J_{\beta\sigma} u^3 u^\beta + (-\nabla_\sigma^* b_{\beta\gamma} \xi + b_\sigma^\mu \nabla_\mu^* b_{\beta\gamma} \xi^2) u^\beta u^\gamma) v^\sigma \\
 &\quad + (u^\beta \nabla_\beta^* u^3 + J_{\beta\gamma} u^\beta u^\gamma) v^3] \sqrt{a} dx d\xi, \\
 b_b(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= \int_{\Omega_\delta} \varphi_b(\mathbf{u}, \mathbf{u}, \mathbf{v}) \sqrt{a} dx d\xi = \int_{\Omega_\delta} \theta \left[g_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} u^3 v^\sigma + u^3 \frac{\partial u^3}{\partial \xi} v^3 \right] \sqrt{a} dx d\xi,
 \end{aligned} \tag{3.11}$$

它们关于贯截变量的 Taylor 展式为

$$\begin{aligned}
 \varphi_m(\mathbf{u}, \mathbf{u}, \mathbf{v}) &:= \sum_{k=0}^2 \varphi_m^k(\mathbf{u}, \mathbf{u}, \mathbf{v}) \xi^k + o(|\xi|^3), \\
 \varphi_m^0(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= (a_{\lambda\sigma} \nabla_\beta^* u^\lambda u^\beta - 2b_{\beta\sigma} u^3 u^\beta) v^\sigma + (u^\beta \nabla_\beta^* u^3 + b_{\beta\gamma} u^\beta u^\gamma) v^3, \\
 \varphi_m^1(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= (-2(Ha_{\lambda\sigma} + b_{\lambda\sigma}) \nabla_\beta^* u^\lambda u^\beta + 2(2Hb_{\beta\sigma} + c_{\beta\sigma}) u^3 u^\beta - \nabla_\sigma^* b_{\beta\lambda} u^\beta u^\lambda) v^\sigma \\
 &\quad - ((2Hb_{\beta\lambda} + c_{\beta\lambda}) u^\beta u^\lambda + 2Hu^\beta \nabla_\beta^* u^3) v^3, \\
 \varphi_m^2(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= (6Hb_{\lambda\sigma} \nabla_\beta^* u^\lambda u^\beta + (b_\sigma^\mu + 2H\delta_\sigma^\mu) \nabla_\mu^* b_{\lambda\beta} u^\lambda u^\beta - 2(2Hc_{\beta\sigma} + Kb_{\beta\sigma}) u^3 u^\beta) v^\sigma \\
 &\quad + (Ku^\beta \nabla_\beta^* u^3 + (Kb_{\beta\sigma} + 2Hc_{\beta\sigma}) u^\beta u^\sigma) v^3,
 \end{aligned} \tag{3.12}$$

和

$$\begin{aligned}
 \varphi_b(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= \varphi_b^0(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \varphi_b^1(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \varphi_b^2(\mathbf{u}, \mathbf{u}, \mathbf{v}) + o(|\xi|^3), \\
 \varphi_b^0(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= a_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} u^3 v^\sigma + u^3 \frac{\partial u^3}{\partial \xi} v^3, \\
 \varphi_b^1(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= -2(Ha_{\lambda\sigma} + b_{\lambda\sigma}) \frac{\partial u^\lambda}{\partial \xi} u^3 v^\sigma - 2Hu^3 \frac{\partial u^3}{\partial \xi} v^3, \\
 \varphi_b^2(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= 6Hb_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} u^3 v^\sigma + Ku^3 \frac{\partial u^3}{\partial \xi} v^3.
 \end{aligned} \tag{3.13}$$

证明 实际上, 三线性形式的被积表达式为

$$\varphi(\mathbf{u}, \mathbf{u}, \mathbf{v}) := g_{ik} u^j \nabla_j u^i v^k \theta = \theta [g_{\lambda\sigma} (u^\beta \nabla_\beta u^\lambda + u^3 \nabla_3 u^\lambda) v^\sigma + (u^\beta \nabla_\beta u^3 + u^3 \nabla_3 u^3) v^3],$$

应用公式 (2.7),

$$\begin{aligned}
 \varphi(\mathbf{u}, \mathbf{u}, \mathbf{v}) &:= \theta \left[g_{\lambda\sigma} (\nabla_\beta^* u^\lambda + \theta^{-1} (I_\beta^\lambda u^3 + R_{\beta\gamma}^\lambda u^\gamma) u^\beta) v^\sigma \right. \\
 &\quad \left. + g_{\lambda\sigma} \left(\frac{\partial u^\lambda}{\partial \xi} + \theta^{-1} I_\beta^\lambda u^\beta \right) u^3 v^\sigma + u^\beta (\nabla_\beta^* u^3 + J_{\beta\lambda} u^\lambda) v^3 + u^3 \frac{\partial u^3}{\partial \xi} v^3 \right].
 \end{aligned}$$

因为文献 [3, 5],

$$g_{\lambda\sigma} I_\beta^\lambda = -\theta J_{\beta\sigma}, \quad g_{\lambda\sigma} R_{\beta\gamma}^\lambda = \theta (-\nabla_\sigma^* b_{\beta\gamma} \xi + b_\sigma^\mu \nabla_\mu^* b_{\beta\gamma} \xi^2),$$

$$\begin{aligned}\varphi(\mathbf{u}, \mathbf{u}, \mathbf{v}) &:= \theta[(g_{\lambda\sigma} \overset{*}{\nabla}_\beta u^\lambda u^\beta - 2J_{\beta\sigma} u^3 u^\beta + (-\overset{*}{\nabla}_\sigma b_{\beta\gamma} \xi + b_\sigma^\mu \overset{*}{\nabla}_\mu b_{\beta\gamma} \xi^2) u^\beta u^\gamma) v^\sigma \\ &\quad + (u^\beta \overset{*}{\nabla}_\beta u^3 + J_{\beta\lambda} u^\beta u^\lambda) v^3] + \theta \left(g_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} u^3 v^\sigma + u^3 \frac{\partial u^3}{\partial \xi} v^3 \right) \\ &= \varphi_m(\mathbf{u}, \mathbf{v}) + \varphi_b(\mathbf{u}, \mathbf{v}),\end{aligned}$$

这里

$$\begin{aligned}\varphi_m(\mathbf{u}, \mathbf{u}, \mathbf{v}) &:= \theta[(g_{\lambda\sigma} \overset{*}{\nabla}_\beta u^\lambda u^\beta - 2J_{\beta\sigma} u^3 u^\beta + (-\overset{*}{\nabla}_\sigma b_{\beta\gamma} \xi \\ &\quad + b_\sigma^\mu \overset{*}{\nabla}_\mu b_{\beta\gamma} \xi^2) u^\beta u^\gamma) v^\sigma + (u^\beta \overset{*}{\nabla}_\beta u^3 + J_{\beta\gamma} u^\beta u^\gamma) v^3], \\ \varphi_b(\mathbf{u}, \mathbf{v}) &= \theta \left(g_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} u^3 v^\sigma + u^3 \frac{\partial u^3}{\partial \xi} v^3 \right).\end{aligned}\quad (3.14)$$

由于

$$\theta = 1 - 2H\xi + K\xi^2, \quad g_{\lambda\sigma} = a_{\lambda\sigma} - 2b_{\lambda\sigma}\xi + c_{\lambda\sigma}\xi^2, \quad J_{\lambda\sigma} = b_{\lambda\sigma} - c_{\lambda\sigma}\xi,$$

经过计算,

$$\begin{aligned}\varphi_m(\mathbf{u}, \mathbf{u}, \mathbf{v}) &:= \sum_{k=0}^2 \varphi_m^k(\mathbf{u}, \mathbf{u}, \mathbf{v}) \xi^k + o(|\xi|^3), \\ \varphi_m^0(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= (a_{\lambda\sigma} \overset{*}{\nabla}_\beta u^\lambda u^\beta - 2b_{\beta\sigma} u^3 u^\beta) v^\sigma + (u^\beta \overset{*}{\nabla}_\beta u^3 + b_{\beta\gamma} u^\beta u^\gamma) v^3, \\ \varphi_m^1(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= (-2(Ha_{\lambda\sigma} + b_{\lambda\sigma}) \overset{*}{\nabla}_\beta u^\lambda u^\beta + 2(2Hb_{\beta\sigma} + c_{\beta\sigma}) u^3 u^\beta \\ &\quad - \overset{*}{\nabla}_\sigma b_{\beta\lambda} u^\beta u^\lambda) v^\sigma - ((2Hb_{\beta\lambda} + c_{\beta\lambda}) u^\beta u^\lambda + 2Hu^\beta \overset{*}{\nabla}_\beta u^3) v^3, \\ \varphi_m^2(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= ((Ka_{\lambda\sigma} + 4Hb_{\lambda\sigma} + c_{\lambda\sigma}) \overset{*}{\nabla}_\beta u^\lambda u^\beta + (b_\sigma^\mu + 2H\delta_\sigma^\mu) \overset{*}{\nabla}_\mu b_{\lambda\beta} u^\lambda u^\beta \\ &\quad + 2(2Hc_{\beta\sigma} + Kb_{\beta\sigma})) v^\sigma + (Ku^\beta \overset{*}{\nabla}_\beta u^3 + (Kb_{\beta\sigma} + 2Hc_{\beta\sigma}) u^\beta u^\sigma) v^3, \\ \varphi_m^2(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= (6Hb_{\lambda\sigma} \overset{*}{\nabla}_\beta u^\lambda u^\beta + (b_\sigma^\mu + 2H\delta_\sigma^\mu) \overset{*}{\nabla}_\mu b_{\lambda\beta} u^\lambda u^\beta \\ &\quad - 2(2Hc_{\beta\sigma} + Kb_{\beta\sigma}) u^\beta u^3) v^\sigma + (Ku^\beta \overset{*}{\nabla}_\beta u^3 + (Kb_{\beta\sigma} + 2Hc_{\beta\sigma}) u^\beta u^\sigma) v^3,\end{aligned}$$

以及

$$\begin{aligned}\varphi_b(\mathbf{u}, \mathbf{v}) &= \varphi_b^0(\mathbf{u}, \mathbf{v}) + \varphi_b^1(\mathbf{u}, \mathbf{v}) + \varphi_b^2(\mathbf{u}, \mathbf{v}) + o(|\xi|^3), \\ \varphi_b^0(\mathbf{u}, \mathbf{v}) &= a_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} u^3 v^\sigma + u^3 \frac{\partial u^3}{\partial \xi} v^3, \\ \varphi_b^1(\mathbf{u}, \mathbf{v}) &= -2(Ha_{\lambda\sigma} + b_{\lambda\sigma}) \frac{\partial u^\lambda}{\partial \xi} u^3 v^\sigma - 2Hu^3 \frac{\partial u^3}{\partial \xi} v^3, \\ \varphi_b^2(\mathbf{u}, \mathbf{v}) &= 6Hb_{\lambda\sigma} \frac{\partial u^\lambda}{\partial \xi} u^3 v^\sigma + Ku^3 \frac{\partial u^3}{\partial \xi} v^3,\end{aligned}$$

这就证明了 (3.12) 和 (3.13). 证毕. $\square$

下面给出双线性形式

$$\mathcal{D}(p, v) := (p, \operatorname{div} \mathbf{v}) = \int_{\Omega_\delta} p \operatorname{div} \mathbf{v} \theta \sqrt{a} dx d\xi,$$

在 S - 坐标系下的形式. 从 (3.7),

$$\begin{cases} \operatorname{div} \mathbf{u} = \frac{\partial u^3}{\partial \xi} + \operatorname{div}^* \mathbf{u} + \theta^{-1}[-2Hu^3 + (2Ku^3 - 2u^\alpha \nabla_\alpha^* H)\xi + u^\alpha \nabla_\alpha^* K\xi^2], \\ \theta \operatorname{div} \mathbf{u} = \theta \frac{\partial u^3}{\partial \xi} + \gamma_0(u) + (-2H \operatorname{div}^* \mathbf{u} + 2Ku^3 - 2u^\alpha \nabla_\alpha^* H)\xi + (K \operatorname{div}^* \mathbf{u} + u^\alpha \nabla_\alpha^* K)\xi^2, \\ \gamma_0(\mathbf{u}) = \operatorname{div}^* \mathbf{u} - 2Hu^3, \end{cases} \quad (3.15)$$

所以, 有下面的命题.

命题 6

$$(p, \operatorname{div} \mathbf{v}) = \int_{\Omega_\delta} D_m(p, \mathbf{v}) \sqrt{a} dx d\xi + \int_{\Omega_\delta} D_b(p, \mathbf{v}) \sqrt{a} dx d\xi = \mathcal{D}_m(p, \mathbf{v}) + \mathcal{D}_b(p, \mathbf{v}), \quad (3.16)$$

其中

$$\begin{cases} D_m(p, \mathbf{v}) = \int_{\Omega_\delta} p[\gamma_0(\mathbf{v}) + (-2H \operatorname{div}^* \mathbf{v} + 2Kv^3 - 2v^\alpha \nabla_\alpha^* H)\xi \\ \quad + (K \operatorname{div}^* \mathbf{v} + v^\alpha \nabla_\alpha^* K)\xi^2] \sqrt{a} dx d\xi, \\ D_b(p, \mathbf{v}) = \int_{\Omega_\delta} \left(\theta p \frac{\partial v^3}{\partial \xi} \right) \sqrt{a} dx d\xi. \end{cases} \quad (3.17)$$

综合以上所述, 对于在 $\mathfrak{S}$ 与 $\mathfrak{S}_\delta$ 之间的流层内的变分问题 (3.2) 可以表述为

$$\begin{cases} \text{求 } \mathbf{u} \in \mathbf{V}(\Omega_\delta), p \in M(\Omega_\delta), \text{ 使得} \\ a_m(\mathbf{u}, \mathbf{v}) + b_m(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \mathcal{D}_m(p, \mathbf{v}) + a_b(\mathbf{u}, \mathbf{v}) \\ \quad + b_b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \mathcal{D}_b(p, \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{V}(\Omega_\delta), \\ \mathcal{D}_m(q, \mathbf{u}) + \mathcal{D}_b(q, \mathbf{u}) = 0, \quad q \in M(\Omega_\delta), \end{cases} \quad (3.18)$$

其中

$$\langle \mathbf{F}, \mathbf{v} \rangle = (\mathbf{f}, \mathbf{v}) + \langle \sigma \cdot \mathbf{n}, \mathbf{v} \rangle|_{\mathfrak{S}}, \quad (3.19)$$

$\mathbf{f}$ 为体积力. 通常体积力为重力, 可以省略.

下面计算边界层顶层 $\mathfrak{S} = \{\xi = \delta\}$ 上的法向应力. 由于 $\mathbf{n}|_{\xi=0} = (0, 0, 1)$, 由定理 1 知,

$$\begin{aligned} \delta \mathbf{n} &= (0, 0, \delta), \quad \gamma_0(\delta \mathbf{n}) = \operatorname{div}^*(0) - 2H\delta = -2H\delta, \\ \mathbf{n}(\delta) &= \left(1 + \frac{1}{2}\gamma_0(\delta \mathbf{n})\right) \mathbf{n} = (1 - H)\mathbf{n} \approx \mathbf{n} \quad (\text{归一化后}), \\ \sqrt{a(\delta \mathbf{n})} &= \left(1 + \frac{1}{2}\gamma_0(\delta \mathbf{n})\right) \sqrt{a} = (1 - H\delta)\sqrt{a} \quad (\delta \text{ 很小使得 } H\delta < 1), \\ g_{ij}(\delta) &= \{a_{\alpha\beta} - 2b_{\alpha\beta}\delta + c_{\alpha\beta}\delta^2, g_{3\alpha}(\delta) = 0, g_{33}(\delta) = 1\}, \\ \sigma(u, p) \mathbf{n} \mathbf{v}|_{\mathfrak{S}} &= (2\mu e^{ij}(\mathbf{u}) - g^{ij}p) n_j g_{ik} v^k|_{\xi=\delta} = (2\mu e^{i3}(\mathbf{u}) - g^{i3}p) g_{ik} v^k \\ &= [2\mu g^{il} g_{ik} e_{l3}(\mathbf{u}) v^k - p v^3] = (2\mu e_{k3}(\mathbf{u}) v^k - p v^3) \\ &= [2\mu e_{3\alpha}(\mathbf{u}) v^\alpha + 2\mu e_{33}(\mathbf{u}) v^3 - p v^3] \end{aligned} \quad (3.20)$$

$$= \mu \left(g_{\alpha\beta}(\delta) \frac{\partial u^\alpha}{\partial \xi} + \nabla_\beta^* u^3 \right) v^\beta + \left(2\mu \frac{\partial u^3}{\partial \xi} - p \right) v^3,$$

其中 $g_{\alpha\beta}(\delta) = a_{\alpha\beta} - 2b_{\alpha\beta}\delta + c_{\alpha\beta}\delta^2$. 于是我们得到如下结果.

命题 7 在 $\tilde{\mathfrak{S}}$ 上的法向应力沿任何方向 $\mathbf{v}$ 的做功率为

$$\left\{ \begin{aligned} \langle \sigma \cdot \mathbf{n}, \mathbf{v} \rangle|_{\tilde{\mathfrak{S}}} &= \int_{\tilde{\mathfrak{S}}} \sigma^{ij}(\mathbf{u}, p) n_j g_{ik} v^k \sqrt{a(\delta)} dx \\ &= \int_{\xi=\delta} \left[\mu \left(g_{\alpha\beta}(\delta) \frac{\partial u^\alpha}{\partial \xi} + \nabla_\beta^* u^3 \right) v^\beta + \left(2\mu \frac{\partial u^3}{\partial \xi} - p \right) v^3 \right] (1 - H\delta) \sqrt{a} dx \\ &= \langle \mathbf{h}, \mathbf{v} \rangle|_{\xi=\delta}, \\ h_\beta(\mathbf{u}) &= \mu \left(g_{\alpha\beta}(\delta) \frac{\partial u^\alpha}{\partial \xi} + \nabla_\beta^* u^3 \right) (1 - H\delta), \\ h^3(\mathbf{u}, p) &= \left(2\mu \frac{\partial u^3}{\partial \xi} - p \right) (1 - H\delta), \\ g_{\alpha\beta}(\delta) &= a_{\alpha\beta} - 2b_{\alpha\beta}\delta + c_{\alpha\beta}\delta^2, \end{aligned} \right. \quad (3.21)$$

其中速度和压力是在顶层的外侧计算的. 由连接条件, $\mathbf{h}|_- = \mathbf{h}|_+$, 即在边界层顶层的内外侧, 法向应力是连续的.

4 边界层方程

假设在边界层内, Navier-Stokes 方程的解和外力可以展成关于贯截变量 ξ 的 Taylor 级数

$$\left\{ \begin{aligned} \mathbf{u}(x, \xi) &= \mathbf{u}_1(x)\xi + \mathbf{u}_2(x)\xi^2 + \cdots, \\ p(x, \xi) &= p_0(x) + p_1(x)\xi + p_2(x)\xi^2 + \cdots, \\ \mathbf{F}(x, \xi) &= \mathbf{F}_0(x) + \mathbf{F}_1(x)\xi + \mathbf{F}_2(x)\xi^2 + \cdots, \end{aligned} \right. \quad (4.1)$$

检验函数同样也展成 Taylor 级数

$$\mathbf{v} = \mathbf{v}_1(x)\xi + \mathbf{v}_2(x)\xi^2 + \cdots, \quad q = q_0(x) + q_1(x)\xi + q_2(x)\xi^2 + \cdots.$$

为了简单, 考虑到边界条件, 记

$$U = \{\mathbf{u}_1, \mathbf{u}_2\}, \quad V = \{\mathbf{v}_1, \mathbf{v}_2\}, \quad P = \{p_0, p_1, p_2\}, \quad Q = \{q_0, q_1, q_2\},$$

那么, 由命题 5 和 6, 膜双线性和三线性的级数形式为

$$\begin{aligned} a_m(\mathbf{u}, \mathbf{v}) &= a_m^0(U, V)\delta + a_m^1(U, V)\frac{\delta^2}{2} + a_m^2(U, V)\frac{\delta^3}{3} + \cdots, \\ b_m(\mathbf{u}, \mathbf{u}, \mathbf{v}) &= b_m^3(u_1, u_1, v_1)\frac{\delta^4}{4} + \cdots, \end{aligned} \quad (4.2)$$

其中

$$a_m^0(U, V) := \int_{\mathfrak{S}} J_0(\mathbf{u}_0, \mathbf{v}_0) \sqrt{a} dx d\xi,$$

$$\begin{aligned}
 a_m^1(U, V) &:= \int_{\mathfrak{S}} (J_0(\mathbf{u}_1, \mathbf{v}_0) + J_0(\mathbf{u}_0, \mathbf{v}_1) + J_1(\mathbf{u}_0, \mathbf{v}_0)) \sqrt{a} dx d\xi, \\
 a_m^2(U, V) &:= \int_{\mathfrak{S}} (J_0(\mathbf{u}_2, \mathbf{v}_0) + J_0(\mathbf{u}_0, \mathbf{v}_2) + J_0(\mathbf{u}_1, \mathbf{v}_1) \\
 &\quad + J_1(\mathbf{u}_1, \mathbf{v}_0) + J_1(\mathbf{u}_0, \mathbf{v}_1) + J_2(\mathbf{u}_0, \mathbf{v}_0)) \sqrt{a} dx d\xi, \\
 b_m(u_1, u_1, v_1) &= \int_{\mathfrak{S}} \varphi_m^0(u_1, u_1, v_1) \sqrt{a} dx \\
 &= \int_{\mathfrak{S}} [a_{\lambda\sigma} u_1^\beta \overset{*}{\nabla}_\beta u_1^\lambda - 2b_{\beta\sigma} u_1^3 u_1^\beta] v_1^\sigma \sqrt{a} dx,
 \end{aligned} \tag{4.3}$$

关于压力项, 由命题 6,

$$\begin{aligned}
 (p, \operatorname{div} \mathbf{v}) &= \mathcal{D}_0(P, V) \delta + \mathcal{D}_1(P, V) \frac{\delta^2}{2} + \mathcal{D}_2(P, V) \frac{\delta^3}{3} + \mathcal{D}_b^R(p, \mathbf{v}), \\
 \mathcal{D}_0(P, V) &= \int_D p_0 v_1^3 \sqrt{a} dx, \\
 \mathcal{D}_1(P, V) &= \int_D [p_0 \overset{*}{\operatorname{div}} \mathbf{v}_1 + 2p_0 v_2^3 + (p_1 - 4Hp_0) v_1^3] \sqrt{a} dx, \\
 \mathcal{D}_2(P, V) &= \int_D [p_0 \overset{*}{\operatorname{div}} \mathbf{v}_2 + (p_1 - 2Hp_0) \overset{*}{\operatorname{div}} \mathbf{v}_1 + (2p_1 - 6Hp_0) v_2^3 \\
 &\quad + (3Kp_0 - 4Hp_1 + p_2) v_1^3 - 2p_0 \overset{*}{\nabla}_\alpha H v_1^\alpha] \sqrt{a} dx.
 \end{aligned} \tag{4.4}$$

连续性方程同样由命题 6 给出

$$\begin{aligned}
 (\operatorname{div} \mathbf{u}, q) &= \int_{\Omega_\delta} q \left[\theta \frac{\partial u^3}{\partial \xi} + \gamma_0(\mathbf{u}) + (-2H \overset{*}{\operatorname{div}} \mathbf{u} + 2Ku^3 \right. \\
 &\quad \left. - 2u_1^\alpha \overset{*}{\nabla}_\alpha H) \xi + (K \overset{*}{\operatorname{div}} \mathbf{u} + u_1^\alpha \overset{*}{\nabla}_\alpha K) \xi^2 \right] \sqrt{a} dx d\xi \\
 &= \int_D \left\{ q_0(u_1^3 + \gamma_0(\mathbf{u}_0)) \delta + [q_1(u_1^3 + \gamma_0(\mathbf{u}_0)) + q_0(\gamma_0(u_1) + 2u_2^3 - 2Hu_1^3)] \frac{\delta^2}{2} \right. \\
 &\quad + [q_2(u_1^3 + \gamma_0(\mathbf{u}_0)) + q_1(\gamma_0(\mathbf{u}_1) + 2u_2^3 - 2Hu_1^3) \\
 &\quad + q_0(\gamma_0(\mathbf{u}_2) + (-2H \overset{*}{\operatorname{div}} \mathbf{u}_1 + 2Ku_1^3 - 2u_1^\alpha \overset{*}{\nabla}_\alpha H) \\
 &\quad \left. + Ku_1^3 - 4Hu_2^3)] \frac{\delta^3}{3} + \dots \right\} \sqrt{a} dx \\
 &= 0, \quad \forall Q(L^2(D))^3.
 \end{aligned} \tag{4.5}$$

毫无疑问, 由于 q_0, q_1 和 q_2 的任意性和相互独立, 可以推出

$$\begin{aligned}
 u_1^3 + \gamma_0(\mathbf{u}_0) &= u_1^3 = 0, \\
 \gamma_0(\mathbf{u}_1) + 2u_2^3 - 2Hu_1^3 &= \gamma_0(\mathbf{u}_1) + 2u_2^3 = 0, \\
 \gamma_0(\mathbf{u}_2) - 2H \overset{*}{\operatorname{div}} \mathbf{u}_1 + 3Ku_1^3 - 2u_1^\alpha \overset{*}{\nabla}_\alpha H - 4Hu_2^3 \\
 &= \gamma_0(\mathbf{u}_2) - 2H \overset{*}{\operatorname{div}} \mathbf{u}_1 - 2u_1^\alpha \overset{*}{\nabla}_\alpha H - 4Hu_2^3 = 0.
 \end{aligned}$$

因为 $\gamma_0(\mathbf{u}) = \overset{*}{\operatorname{div}} \mathbf{u} - 2Hu^3$, 所以, 我们有下面命题.

命题 8 由连续性方程推出

$$\begin{cases} u_1^3 = 0, \\ u_2^3 = -\frac{1}{2}\gamma_0(\mathbf{u}_1) = -\frac{1}{2} \operatorname{div}^* \mathbf{u}_1, \\ \operatorname{div}^* \mathbf{u}_2 + H \operatorname{div}^* \mathbf{u}_1 - 2u_1^\alpha \nabla_\alpha^* H = 0. \end{cases} \quad (4.6)$$

弯曲算子的级数形式 从命题 4 和 5 可得

$$\begin{aligned} a_b(\mathbf{u}, \mathbf{v}) &= \int_{\Omega_\delta} A_b(\mathbf{u}, \mathbf{v}) \sqrt{a} dx d\xi \\ &= \int_{\mathfrak{S}} \int_0^\delta [I_0(u, v) + I_1(u, v)\xi + I_2(u, v)\xi^2 + I_R] dx \sqrt{a} dx \\ &= \int_D \left[\Lambda_0(U, V)\delta + \Lambda_1(U, V)\frac{\delta^2}{2} + \Lambda_2(U, V)\frac{\delta^3}{3} + \Lambda_R \right] \sqrt{a} dx \\ &= a_b^0(U, V)\delta + a_b^1(U, V)\frac{\delta^2}{2} + a_b^2(U, V)\frac{\delta^3}{3} + a_b^R(\mathbf{u}, \mathbf{v}), \end{aligned} \quad (4.7)$$

其中

$$\begin{aligned} \Lambda_0(U, V) &:= 2\mu u_1^3 v_1^3 + \frac{1}{2}\mu a_{\alpha\beta} u_1^\alpha v_1^\beta, \\ \Lambda_1(U, V) &:= \frac{1}{2}\mu(a_{\alpha\beta}(u_1^\alpha v_2^\beta + u_2^\alpha v_1^\beta) + u_1^\beta \nabla_\beta^* v_1^3 + v_1^\beta \nabla_\beta^* u_1^3) \\ &\quad + 2\mu(u_1^3 v_2^3 + u_2^3 v_1^3) + \frac{1}{2}\mu(-2(Ha_{\lambda\sigma} + b_{\lambda\sigma})u_1^\lambda v_1^\sigma) - 4\mu H u_1^3 v_1^3, \\ \Lambda_2(U, V) &:= 2\mu a_{\alpha\beta} u_2^\alpha v_2^\beta + 8\mu u_2^3 v_2^3 + \frac{1}{2}\mu[2u_2^\beta \nabla_\beta^* v_1^3 + u_1^\beta \nabla_\beta^* v_2^3 + 2v_2^\beta \nabla_\beta^* u_1^3 \\ &\quad + v_1^\beta \nabla_\beta^* u_2^3 + a^{\alpha\beta} \nabla_\alpha^* u_1^3 \nabla_\beta^* v_1^3] + 3\mu H b_{\alpha\beta} u_1^\alpha v_1^\beta + 2\mu K u_1^3 v_1^3 \\ &\quad + \frac{1}{2}\mu[-2(Ha_{\alpha\beta} + b_{\alpha\beta})(2u_1^\alpha v_2^\beta + 2u_2^\alpha v_1^\beta) - 2H(u_1^\beta \nabla_\beta^* v_1^3 + v_1^\beta \nabla_\beta^* u_1^3)] \\ &\quad - 8\mu H(u_2^3 v_1^3 + u_1^3 v_2^3). \end{aligned}$$

利用连续性方程 (4.6), 那么,

$$\begin{aligned} \Lambda_0(U, V) &:= \frac{1}{2}\mu a_{\alpha\beta} u_1^\alpha v_1^\beta, \\ \Lambda_1(U, V) &:= \frac{1}{2}\mu a_{\alpha\beta} u_1^\alpha v_2^\beta + \frac{\mu}{2}(a_{\alpha\beta} u_2^\alpha - 2(Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha)v_1^\beta + 2\mu u_2^3 v_1^3 + \frac{\mu}{2}u_1^\beta \nabla_\beta^* v_1^3, \\ \Lambda_2(U, V) &:= 2\mu(a_{\alpha\beta} u_2^\alpha - (Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha)v_2^\beta + \frac{\mu}{2}[\nabla_\beta^* u_2^3 + 6Hb_{\alpha\beta} u_1^\alpha \\ &\quad - 4(Ha_{\alpha\beta} + b_{\alpha\beta})u_2^\alpha]v_1^\beta + 8\mu u_2^3(v_2^3 - H v_1^3) + \frac{\mu}{2}u_1^\beta \nabla_\beta^* v_2^3 + \mu(u_2^\beta - H u_1^\beta) \nabla_\beta^* v_1^3. \end{aligned} \quad (4.8)$$

为简单, 可以令

$$\begin{aligned} a_b(U, V) &= a_b^0(U, V)\delta + a_b^1(U, V)\frac{\delta^2}{2} + a_b^2(U, V)\frac{\delta^3}{3} \\ &= (\pi_\beta^2(U), v_2^\beta) + (\pi_\beta^3(U), v_2^3) + (\pi_\beta^1(U), v_1^\beta) + (\pi_\beta^3(U), v_1^3) \\ &\quad + (\tilde{\pi}_2^\beta, \nabla_\beta^* v_2^3) + (\tilde{\pi}_1^\beta, \nabla_\beta^* v_1^3) \end{aligned}$$

$$\begin{aligned}
&= (\pi_\beta^2(U), v_2^\beta) + (\pi_3^2(U) - \nabla_\beta^* \tilde{\pi}_2^\beta, v_2^3) + (\pi_\beta^1(U), v_1^\beta) + (\pi_3^1(U) - \nabla_\beta^* \tilde{\pi}_1^\beta, v_1^3) \\
&= (\pi_\beta^2(U), v_2^\beta) + (\tilde{\pi}_3^2(U), v_2^3) + (\pi_\beta^1(U), v_1^\beta) + (\tilde{\pi}_3^1(U), v_1^3),
\end{aligned} \tag{4.9}$$

和

$$\begin{aligned}
\pi_\beta^2(U) &= \frac{\mu}{2} \left[a_{\alpha\beta} u_1^\alpha \frac{\delta^2}{2} + 4(a_{\alpha\beta} u_2^\alpha - (H a_{\alpha\beta} + b_{\alpha\beta}) u_1^\alpha) \frac{\delta^3}{3} \right], \\
\pi_3^2(U) &= 8\mu u_2^3 \frac{\delta^3}{3}, \quad \pi_3^1(U) = 2\mu \left(\frac{\delta^2}{2} - 4H \frac{\delta^3}{3} \right) u_2^3, \\
\pi_\beta^1(U) &= \frac{\mu}{2} \left[\left(a_{\alpha\beta} \frac{\delta^2}{2} - 4(H a_{\alpha\beta} + b_{\alpha\beta}) \frac{\delta^3}{3} \right) u_2^\alpha + \left(a_{\alpha\beta} \delta - 2(H a_{\alpha\beta} + b_{\alpha\beta}) \frac{\delta^2}{2} \right. \right. \\
&\quad \left. \left. + 6H b_{\alpha\beta} \frac{\delta^3}{3} \right) u_1^\alpha + \nabla_\beta^* u_1^3 \frac{\delta^2}{2} + \nabla_\beta^* u_2^3 \frac{\delta^3}{3} \right], \\
\tilde{\pi}_2^\beta(U) &= \frac{\mu}{2} u_1^\beta \frac{\delta^3}{3}, \quad \tilde{\pi}_1^\beta(U) = \frac{\mu}{2} \left[u_1^\beta \frac{\delta^2}{2} + 2(u_2^\beta - H u_1^\beta) \frac{\delta^3}{3} \right], \\
\tilde{\pi}_3^2(U) &= \pi_3^2(U) - \nabla_\beta^* \tilde{\pi}_2^\beta = -\frac{9}{2} \mu \operatorname{div} \mathbf{u}_1 \frac{\delta^3}{3}, \\
\tilde{\pi}_3^1(U) &= \pi_3^1(U) - \nabla_\beta^* \tilde{\pi}_1^\beta = \mu \left[\left(4H \frac{\delta^3}{3} - \frac{3}{2} \frac{\delta^2}{2} \right) \operatorname{div} \mathbf{u}_1 + (-\operatorname{div} \mathbf{u}_2 + \operatorname{div}(H \mathbf{u}_1)) \frac{\delta^3}{3} \right] \\
&\stackrel{\text{(应用 (4.6) 第三式)}}{=} \mu \left[\left(7H \frac{\delta^3}{3} - \frac{3}{2} \frac{\delta^2}{2} \right) \operatorname{div} \mathbf{u}_1 - 3 \operatorname{div}(H \mathbf{u}_1) \frac{\delta^3}{3} \right].
\end{aligned} \tag{4.10}$$

下面考查非线性项. 利用命题 5 和 8, 我们发现

$$\begin{aligned}
b_m(U, U, V) &= \int_{\mathfrak{S}} [(a_{\lambda\sigma} \nabla_\beta^* u_1^\lambda u_1^\beta - 2b_{\beta\sigma} u_1^3 u_1^\beta) v_1^\sigma + (u_1^\beta \nabla_\beta^* u_1^3 + b_{\beta\sigma} u_1^\beta u_1^\sigma) v_1^3] \frac{\delta^4}{4} \sqrt{a} dx + o(\delta^5) \\
&= \int_{\mathfrak{S}} [a_{\lambda\sigma} \nabla_\beta^* u_1^\lambda u_1^\beta v_1^\sigma + b_{\beta\sigma} u_1^\beta u_1^\sigma v_1^3] \frac{\delta^4}{4} \sqrt{a} dx + o(\delta^5), \\
b_b(U, U, V) &= \int_{\mathfrak{S}} \left[(a_{\lambda\sigma} u_1^\lambda u_1^3 v_1^\sigma + u_1^3 u_1^3 v_1^3) \frac{\delta^3}{3} + (a_{\lambda\sigma} (2u_2^\lambda u_1^3 v_1^\sigma + u_1^\lambda u_2^3 v_1^\sigma + u_1^\lambda u_1^3 v_2^\sigma) + u_2^3 u_1^3 v_1^3 \right. \\
&\quad \left. + 2u_1^3 u_2^3 v_1^3 + u_1^3 u_1^3 v_2^3 - 2(H a_{\lambda\sigma} + b_{\lambda\sigma}) u_1^\lambda u_1^3 v_1^\sigma - 2H u_1^3 u_1^3 v_1^3) \frac{\delta^4}{4} \right] \sqrt{a} dx + o(\delta^5) \\
&= \int_{\mathfrak{S}} \left(a_{\lambda\sigma} u_1^\lambda u_2^3 v_1^\sigma \frac{\delta^4}{4} \right) \sqrt{a} dx + o(\delta^5).
\end{aligned}$$

记三线性形式

$$b(U, U, U) = \int_D [a_{\lambda\sigma} (\nabla_\beta^* u_1^\lambda u_1^\beta + u_1^\lambda u_2^3) v_1^\sigma + b_{\beta\sigma} u_1^\beta u_1^\sigma v_1^3] \frac{\delta^4}{4} \sqrt{a} dx, \tag{4.11}$$

记三线性算子

$$\begin{aligned}
b(U, U, V) &= \mathcal{B}_0(U, V), \\
\mathcal{B}_0(U, V) &= [(a_{\alpha\beta} (\nabla_\lambda^* u_1^\alpha u_1^\lambda + u_1^\alpha u_2^3), v_1^\beta) + (b_{\alpha\beta} u_1^\alpha u_1^\beta, v_1^3)] \frac{\delta^4}{4} \\
&= \left[\left(a_{\alpha\beta} \left(\nabla_\lambda^* u_1^\alpha u_1^\lambda - \frac{1}{2} u_1^\alpha \operatorname{div} \mathbf{u}_1 \right), v_1^\beta \right) + (b_{\alpha\beta} u_1^\alpha u_1^\beta, v_1^3) \right] \frac{\delta^4}{4},
\end{aligned} \tag{4.12}$$

其中 $L^2(D)$ 内积

$$(f, g) = \int_D fg \sqrt{a} dx.$$

现在回到 Navier-Stokes 方程 (3.2), 将 (3.9), (3.11) 和 (3.16) 代入后得

$$\begin{cases} a_m(\mathbf{u}, \mathbf{v}) - \mathcal{D}_m(p, \mathbf{v}) + a_b(\mathbf{u}, \mathbf{v}) - \mathcal{D}_b(p, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle, \\ \mathcal{D}_m(q, \mathbf{u}) + \mathcal{D}_b(q, \mathbf{u}) = 0, \end{cases} \quad (4.13)$$

其中

$$\begin{aligned} a_m(U, V) &:= a_m^0(U, V)\delta + a_m^1(U, V)\frac{\delta^2}{2} + a_m^2(U, V)\frac{\delta^3}{3}, \\ \mathcal{D}_m(p, \mathbf{v}) &= \mathcal{D}_m^0(P, V)\delta + \mathcal{D}_m^1(P, V)\frac{\delta^2}{2} + \mathcal{D}_m^2(P, V)\frac{\delta^3}{3}, \\ a_b(U, V) &:= a_b^0(U, V)\delta + a_b^1(U, V)\frac{\delta^2}{2} + a_b^2(U, V)\frac{\delta^3}{3}, \\ \mathcal{D}_b(p, \mathbf{v}) &= \mathcal{D}_b^0(P, V)\delta + \mathcal{D}_b^1(P, V)\frac{\delta^2}{2} + \mathcal{D}_b^2(P, V)\frac{\delta^3}{3}, \\ \langle \mathbf{F}, \mathbf{v} \rangle &= \langle \mathbf{f}, \mathbf{v} \rangle + \langle \mathbf{h}(\mathbf{u}, p), \mathbf{v} \rangle, \end{aligned}$$

以及

$$V(\mathfrak{S}) = \{\mathbf{v} \mid \mathbf{v} \in H^1(\mathfrak{S}) \times H^1(\mathfrak{S}) \times H^1(\mathfrak{S}), \text{周期性边界条件}\}.$$

必须指出, $\mathbf{F}$ 中的法向应力 $\mathbf{h}(\mathbf{u}, p)$ 是在边界层顶层上外侧上的值. 那么, 考查 Navier-Stokes 方程 (4.13) 的近似方程, 有下面的引理.

引理 9 假设 Navier-Stokes 方程的解及外力可以展成 Taylor 级数 (4.1), 那么, $U = \{\mathbf{u}_1, \mathbf{u}_2\}$ 和 $P = \{p_0, p_1, p_2\}$ 近似地满足下列截断 Navier-Stokes 方程的变分问题:

$$\begin{cases} \text{求 } U = \{\mathbf{u}_1, \mathbf{u}_2\} \in V(\mathfrak{S}) \times V(\mathfrak{S}), \quad P = \{p_0, p_1, p_2\} \in L^2(\mathfrak{S}) \times L^2(\mathfrak{S}) \times L^2(\mathfrak{S}), \text{使得} \\ (\mathcal{L}_0(U, V) - \mathcal{D}_0(P, V))\delta + (\mathcal{L}_1(U, V) - \mathcal{D}_1(P, V))\frac{\delta^2}{2} \\ + (\mathcal{L}_2(U, V) - \mathcal{D}_2(P, V))\frac{\delta^3}{3} + \mathcal{B}_0(U, V) = \langle \mathbf{F}, \mathbf{V} \rangle, \\ \mathcal{D}_0(Q, U)\delta + \mathcal{D}_1(Q, U)\frac{\delta^2}{2} + \mathcal{D}_2(Q, U)\frac{\delta^3}{3} = 0, \end{cases} \quad (4.14)$$

或且, 等价地,

$$\begin{cases} \text{求 } U = \{\mathbf{u}_1, \mathbf{u}_2\} \in V(\mathfrak{S}) \times V(\mathfrak{S}), \quad P = \{p_0, p_1, p_2\} \in L^2(\mathfrak{S}) \times L^2(\mathfrak{S}) \times L^2(\mathfrak{S}), \text{使得} \\ (a_0^{\alpha\beta\lambda\sigma}\gamma_{\alpha\lambda}(\mathbf{u}_1), \gamma_{\beta\sigma}(\mathbf{v}_1))\frac{\delta^3}{3} + (L_k^\lambda(U, \delta), v_\lambda^k) + (B_k(U), v_1^k) - (D_k^\lambda(p, \delta), v_\lambda^k) = (h_k^\lambda, v_\lambda^k), \\ \forall V \in V(\mathfrak{S}) \times V(\mathfrak{S}), \\ \left(q_0, u_1^3\delta + (\operatorname{div}^* \mathbf{u}_1 + 2u_2^3 - 4Hu_1^3)\frac{\delta^2}{2} + (\operatorname{div}^* \mathbf{u}_2 - 2H \operatorname{div}^* \mathbf{u}_1 - 6Hu_2^3 + 3u_1^3 - 2u_1^\alpha \nabla_\alpha^* H)\frac{\delta^3}{3} \right) \\ + \left(q_1, u_1^3\frac{\delta^2}{2} + (\operatorname{div}^* \mathbf{u}_1 + 2u_2^3 - 4Hu_1^3)\frac{\delta^3}{3} \right) + \left(q_2, u_1^3\frac{\delta^3}{3} \right) = 0, \\ \forall Q \in L^2(\mathfrak{S}) \times L^2(\mathfrak{S}) \times L^2(\mathfrak{S}), \end{cases} \quad (4.15)$$

其中

$$\begin{cases} \mathcal{L}_0(U, V) := \mu(a_{\alpha\beta}u_1^\alpha, v_1^\beta), \\ \mathcal{L}_1(U, V) := \frac{\mu}{2}[(a_{\alpha\beta}u_1^\alpha, v_2^\beta) + (a_{\alpha\beta}u_2^\alpha - 2(Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha, v_1^\beta)] - \frac{3}{2}\mu(\operatorname{div}^* \mathbf{u}_1, v_1^3), \\ \mathcal{L}_2(U, V) := 2\mu(\gamma^{\beta\sigma}(\mathbf{u}_1), \gamma_{\beta\sigma}(\mathbf{v}_1)) + 2\mu(a_{\alpha\beta}u_2^\alpha - (Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha, v_1^\beta) \\ \quad - \frac{\mu}{4}([\nabla_\beta^* \operatorname{div}^* \mathbf{u}_1 - 12Hb_{\alpha\beta}u_1^\alpha + 8(Ha_{\alpha\beta} + b_{\alpha\beta})u_2^\alpha], v_1^\beta) \\ \quad - \frac{9}{2}\mu(\operatorname{div}^* \mathbf{u}_1, v_2^3) + \mu(4H \operatorname{div}^* \mathbf{u}_1 + \operatorname{div}^*(H\mathbf{u}_1) - \operatorname{div}^* \mathbf{u}_2, v_1^3), \end{cases} \quad (4.16)$$

$$\begin{cases} \mathcal{B}_0(U, V) := (B_k(U), v_1^k), \\ B_\beta(\mathbf{u}_1) = a_{\alpha\beta}(\nabla_\lambda^* u_1^\alpha u_1^\lambda - u_1^\alpha \operatorname{div}^* \mathbf{u}_1) \frac{\delta^4}{4}, \quad B_3(\mathbf{u}_1) = (b_{\alpha\beta}u_1^\alpha u_1^\beta) \frac{\delta^4}{4}, \end{cases} \quad (4.17)$$

$$\begin{cases} \mathcal{D}_0(P, V) = \int_D p_0 v_1^3 \sqrt{a} dx, \\ \mathcal{D}_1(P, V) = \int_D [p_0 \operatorname{div}^* \mathbf{v}_1 + 2p_0 v_2^3 + (p_1 - 4Hp_0)v_1^3] \sqrt{a} dx, \\ \mathcal{D}_2(P, V) = \int_D [p_0 \operatorname{div}^* \mathbf{v}_2 + (p_1 - 2Hp_0) \operatorname{div}^* \mathbf{v}_1 + (2p_1 - 6Hp_0)v_2^3 \\ \quad + (3Kp_0 - 4Hp_1 + p_2)v_1^3 - 2p_0 \nabla_\alpha^* H v_1^\alpha] \sqrt{a} dx, \end{cases} \quad (4.18)$$

$$\begin{cases} L_\beta^2(U) = \frac{\mu}{2} \left[a_{\alpha\beta} u_1^\alpha \frac{\delta^2}{2} + 4(a_{\alpha\beta} u_2^\alpha - (Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha) \frac{\delta^3}{3} \right], \\ L_3^2(U) = -\frac{9}{2}\mu \operatorname{div}^* \mathbf{u}_1 \frac{\delta^3}{3}, \\ L_\beta^1(U) = \frac{\mu}{4} \left[4a_{\alpha\beta} u_1^\alpha \delta + 2(a_{\alpha\beta} u_2^\alpha - 2(Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha) \frac{\delta^2}{2} \right. \\ \quad \left. - (\nabla_\beta^* \operatorname{div}^* \mathbf{u}_1 - 12Hb_{\alpha\beta}u_1^\alpha + 8(Ha_{\alpha\beta} + b_{\alpha\beta})u_2^\alpha) \frac{\delta^3}{3} \right], \\ L_3^1(U) = \mu \left(-\frac{3}{2} \operatorname{div}^* \mathbf{u}_1 \frac{\delta^2}{2} + (4H \operatorname{div}^* \mathbf{u}_1 + \operatorname{div}^*(H\mathbf{u}_1) - \operatorname{div}^* \mathbf{u}_2) \frac{\delta^3}{3} \right), \end{cases} \quad (4.19)$$

$$\begin{cases} D_3^1(P, \delta) = p_0 \delta + (p_1 - 4Hp_0) \frac{\delta^2}{2} + (3Kp_0 - 4Hp_1 + p_2) \frac{\delta^3}{3}, \\ D_3^2(P, \delta) = 2p_0 \frac{\delta^2}{2} + (2p_1 - 6Hp_0) \frac{\delta^3}{3}, \\ D_\beta^2(P, \delta) = -\nabla_\beta^* p_0 \frac{\delta^3}{3}, \\ D_\beta^1(P, \delta) = -\nabla_\beta^* \left(p_0 \frac{\delta^2}{2} + (p_1 - 2Hp_0) \frac{\delta^3}{3} \right) - 2p_0 \nabla_\beta^* H \frac{\delta^3}{3}. \end{cases} \quad (4.20)$$

证明 Navier-Stokes 方程的近似到三阶的截断变分问题

$$\begin{cases} \text{求 } U = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\} \in V(\mathfrak{I}_i) \times V(\mathfrak{I}_i) \times V(\mathfrak{I}_i), \\ P = \{p_0, p_1, p_2\} \in L^2(\mathfrak{I}_i) \times L^2(\mathfrak{I}_i) \times L^2(\mathfrak{I}_i), \text{ 使得} \\ a_m(U, V) - \mathcal{D}_m(P, V) + a_b(U, V) - \mathcal{D}_b(P, V) + \mathcal{B}_0(U, V) = \langle \mathbf{F}, V \rangle, \quad \forall V \in V(\mathfrak{I}_i)^3, \\ \mathcal{D}_m(Q, U) + \mathcal{D}_b(Q, U) = 0, \quad \forall Q \in L^2(\mathfrak{I}_i)^3. \end{cases} \quad (4.21)$$

注意 (3.10), (4.8) 和 $\mathbf{u}_0 = 0, \mathbf{v}_0 = 0, u_1^3 = 0, u_2^3 = -\frac{1}{2} \operatorname{div}^* \mathbf{u}_1$,

$$\begin{aligned}\mathcal{L}_0(U, V) &:= \int_{\mathfrak{S}} [J_0(\mathbf{u}_0, \mathbf{v}_0) + \Lambda_0(U, V)] \sqrt{a} dx = \int_{\mathfrak{S}} \mu a_{\alpha\beta} u_1^\alpha v_1^\beta \sqrt{a} dx = \mu(a_{\alpha\beta} u_1^\alpha, v_1^\beta), \\ \mathcal{L}_1(U, V) &:= \int_{\mathfrak{S}} [J_0(\mathbf{u}_1, \mathbf{v}_0) + J_0(\mathbf{u}_0, \mathbf{v}_1) + J_1(\mathbf{u}_0, \mathbf{v}_0) + \Lambda_1(U, V)] \sqrt{a} dx \\ &= \frac{\mu}{2} [(a_{\alpha\beta} u_1^\alpha, v_2^\beta) + (a_{\alpha\beta} u_2^\alpha - 2(Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha, v_1^\beta)] - \frac{3}{2} \mu(\operatorname{div}^* \mathbf{u}_1, v_1^3), \\ \mathcal{L}_2(U, V) &:= \int_{\mathfrak{S}} [J_0(\mathbf{u}_2, \mathbf{v}_0) + J_0(\mathbf{u}_0, \mathbf{v}_2) + J_0(\mathbf{u}_1, \mathbf{v}_1) \\ &\quad + J_1(\mathbf{u}_1, \mathbf{v}_0) + J_1(\mathbf{u}_0, \mathbf{v}_1) + J_2(\mathbf{u}_0, \mathbf{v}_0) + \Lambda_2(U, V)] \sqrt{a} dx \\ &= \int_{\mathfrak{S}} [J_0(\mathbf{u}_1, \mathbf{v}_1) + \Lambda_2(U, V)] \sqrt{a} dx,\end{aligned}\tag{4.22}$$

应用 Gauss 定理和边界条件,

$$\begin{aligned}\int_{\mathfrak{S}} [\Lambda_2(U, V)] \sqrt{a} dx &= 2\mu(a_{\alpha\beta} u_2^\alpha - (Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha, v_2^\beta) \\ &\quad + \frac{\mu}{2} \left(\left[-\frac{1}{2} \nabla_\beta^* \operatorname{div}^* \mathbf{u}_1 + 6Hb_{\alpha\beta} u_1^\alpha - 4(Ha_{\alpha\beta} + b_{\alpha\beta})u_2^\alpha \right], v_1^\beta \right) \\ &\quad - \frac{9}{2} \mu(\operatorname{div}^* \mathbf{u}_1, v_2^3) + \mu(4H \operatorname{div}^* \mathbf{u}_1 + \operatorname{div}^*(H\mathbf{u}_1) - \operatorname{div}^* \mathbf{u}_2, v_1^3).\end{aligned}$$

同时,

$$\int_{\mathfrak{S}} [J_0(\mathbf{u}_1, \mathbf{v}_1)] \sqrt{a} dx = (a_0^{\alpha\beta\lambda\sigma} \gamma_{\alpha\lambda}(\mathbf{u}_1), \gamma_{\beta\sigma}(\mathbf{v}_1)) = 2\mu(\gamma^{\beta\sigma}(\mathbf{u}_1), \gamma_{\beta\sigma}(\mathbf{v}_1)).$$

最后, 有

$$\left\{ \begin{aligned}\mathcal{L}_0(U, V) &:= \mu(a_{\alpha\beta} u_1^\alpha, v_1^\beta), \\ \mathcal{L}_1(U, V) &:= \frac{\mu}{2} [(a_{\alpha\beta} u_1^\alpha, v_2^\beta) + (a_{\alpha\beta} u_2^\alpha - 2(Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha, v_1^\beta)] - \frac{3}{2} \mu(\operatorname{div}^* \mathbf{u}_1, v_1^3), \\ \mathcal{L}_2(U, V) &:= 2\mu(\gamma^{\beta\sigma}(\mathbf{u}_1), \gamma_{\beta\sigma}(\mathbf{v}_1)) + 2\mu(a_{\alpha\beta} u_2^\alpha - (Ha_{\alpha\beta} + b_{\alpha\beta})u_1^\alpha, v_2^\beta) \\ &\quad - \frac{\mu}{4} ([\nabla_\beta^* \operatorname{div}^* \mathbf{u}_1 - 12Hb_{\alpha\beta} u_1^\alpha + 8(Ha_{\alpha\beta} + b_{\alpha\beta})u_2^\alpha], v_1^\beta) \\ &\quad - \frac{9}{2} \mu(\operatorname{div}^* \mathbf{u}_1, v_2^3) + \mu(4H \operatorname{div}^* \mathbf{u}_1 + \operatorname{div}^*(H\mathbf{u}_1) - \operatorname{div}^* \mathbf{u}_2, v_1^3).\end{aligned}\right.\tag{4.23}$$

由 (4.18), 压力项

$$\begin{aligned}\mathcal{D}_0(P, V)\delta + \mathcal{D}_1(P, V)\frac{\delta^2}{2} + \mathcal{D}_2(P, V)\frac{\delta^3}{3} \\ &= \left(p_0\delta + (p_1 - 4Hp_0)\frac{\delta^2}{2} + (3Kp_0 - 4Hp_1 + p_2)\frac{\delta^3}{3}, v_1^3 \right) \\ &\quad + \left(2p_0\frac{\delta^2}{2} + (2p_1 - 6Hp_0)\frac{\delta^3}{3}, v_2^3 \right) + \left(p_0\frac{\delta^2}{2} + (p_1 - 2Hp_0)\frac{\delta^3}{3}, \operatorname{div}^* \mathbf{v}_1 \right) \\ &\quad + \left(p_0\frac{\delta^3}{3}, \operatorname{div}^* \mathbf{v}_2 \right) - \left(2p_0 \nabla_\alpha^* H \frac{\delta^3}{3}, v_1^\alpha \right) \\ &= \left(p_0\delta + (p_1 - 4Hp_0)\frac{\delta^2}{2} + (3Kp_0 - 4Hp_1 + p_2)\frac{\delta^3}{3}, v_1^3 \right)\end{aligned}$$

$$\begin{aligned}
& + \left(2p_0 \frac{\delta^2}{2} + (2p_1 - 6Hp_0) \frac{\delta^3}{3}, v_2^3 \right) - \left(\nabla_\beta^* \left(p_0 \frac{\delta^2}{2} + (p_1 - 2Hp_0) \frac{\delta^3}{3} \right), v_1^\beta \right) \\
& - \left(\nabla_\beta^* p_0 \frac{\delta^3}{3}, v_2^\beta \right) - \left(2p_0 \nabla_\alpha^* H \frac{\delta^3}{3}, v_1^\alpha \right),
\end{aligned} \quad (4.24)$$

因而, 有

$$\mathcal{D}_0(P, V)\delta + \mathcal{D}_1(P, V)\frac{\delta^2}{2} + \mathcal{D}_2(P, V)\frac{\delta^3}{3} = (D_k^\lambda(P, \delta), v_\lambda^k), \quad (4.25)$$

其中 $D_k^\lambda(P, \delta)$ 由 (4.20) 所定义. 从 (4.23), 经过计算得

$$(\mathcal{L}_0(U, V))\delta + (\mathcal{L}_1(U, V))\frac{\delta^2}{2} + (\mathcal{L}_2(U, V))\frac{\delta^3}{3} = 2\mu(\gamma^{\beta\sigma}(\mathbf{u}_1), \gamma_{\beta\sigma}(\mathbf{v}_1))\frac{\delta^3}{3} + (L_k^\lambda(U), v_\lambda^k), \quad (4.26)$$

其中 $L_k^\lambda(U)$ 由 (4.19) 所定义. 于是, 综合以上各式, 变分问题 (4.15). 引理得证. $\square$

定理 10 (4.14) 中的解 $(\mathbf{u}_2, p_1)$ 可以表示为

$$\begin{cases} u_2^\alpha = K_\beta^\alpha u_1^\beta - \frac{1}{2\mu} a^{\alpha\beta} \nabla_\beta^* p_0 + F_2^\alpha, \\ p_1 \frac{\delta^3}{3} = \left(3H \frac{\delta^3}{3} - \frac{\delta^2}{2} \right) p_0 - \frac{9}{4} \mu \operatorname{div} \mathbf{u}_1 \frac{\delta^3}{3} - \frac{1}{2} h_3^2, \\ u_1^3 = 0, \end{cases} \quad (4.27)$$

其中

$$K_\beta^\alpha = \left(H - \frac{3}{8\delta} \right) \delta_\beta^\alpha + b_\beta^\alpha, \quad F_2^\alpha = \frac{2}{\mu} \left(\frac{\delta^3}{3} \right)^{-1} a^{\alpha\beta} h_\beta^2. \quad (4.28)$$

证明 由于检验函数 V 和 Q 的任意性和线性无关, 可令 $v_2 \neq 0, q_2 \neq 0, v_i = 0, q_i = 0, i = 1, 3, \dots$, 那么, 从 (4.15) 推出

$$(L_k^2(U) - D_k^2(P), v_2^k) = (h_k^2, v_2^k),$$

即

$$L_k^2(U) - D_k^2(P) = h_k^2, \quad k = 1, 2, 3.$$

利用 (4.19) 和 (4.20),

$$\begin{cases} \frac{\mu}{2} \left[a_{\alpha\beta} u_1^\alpha \frac{\delta^2}{2} + 4(a_{\alpha\beta} u_2^\alpha - (H a_{\alpha\beta} + b_{\alpha\beta}) u_1^\alpha) \frac{\delta^3}{3} \right] + \nabla_\beta^* p_0 \frac{\delta^3}{3} = h_\beta^2, \\ -\frac{9}{2} \mu \operatorname{div} \mathbf{u}_1 \frac{\delta^3}{3} - 2p_0 \frac{\delta^2}{2} - (2p_1 - 6Hp_0) \frac{\delta^3}{3} = h_3^2. \end{cases}$$

从而, 有

$$\begin{cases} u_2^\alpha = K_\beta^\alpha u_1^\beta - \frac{1}{2\mu} a^{\alpha\beta} \nabla_\beta^* p_0 + F_2^\alpha, \\ p_1 \frac{\delta^3}{3} = \left(3H \frac{\delta^3}{3} - \frac{\delta^2}{2} \right) p_0 - \frac{9}{4} \mu \operatorname{div} \mathbf{u}_1 \frac{\delta^3}{3} - \frac{1}{2} h_3^2, \end{cases} \quad (4.29)$$

其中

$$K_{\beta}^{\alpha} = \left(H - \frac{3}{8\delta}\right) \delta_{\beta}^{\alpha} + b_{\beta}^{\alpha}, \quad F_2^{\alpha} = \frac{1}{2\mu \frac{\delta^3}{3}} a^{\alpha\beta} h_{\beta}^2. \quad (4.30)$$

证毕. $\square$

定理 11 p_0 是下列边值问题的解,

$$\begin{cases} -\Delta^* p_0 + M_1(\mathbf{u}_1) = -2\mu \nabla_{\alpha}^* F_2^{\alpha}, \\ M_1(\mathbf{u}_1) = 2\mu \left[\left(2H - \frac{3}{8\delta}\right) \operatorname{div} \mathbf{u}_1 + u_1^{\alpha} \nabla_{\alpha}^* H + b_{\beta}^{\alpha} \nabla_{\alpha}^* u_1^{\beta} \right], \\ p_0 \text{ 使得周期性边界条件或无边界.} \end{cases} \quad (4.31)$$

证明 对 (4.28) 的第一式作用算子 ∇_{α}^* 和 $\Delta^* = a^{\alpha\beta} \nabla_{\alpha}^* \nabla_{\beta}^*$, 那么, 有

$$\nabla_{\alpha}^* u_2^{\alpha} = \nabla_{\alpha}^* (K_{\beta}^{\alpha} u_1^{\beta}) - \frac{1}{2\mu} \Delta^* p_0 + \nabla_{\alpha}^* F_2^{\alpha}.$$

(4.6) 的第三式

$$\operatorname{div} \mathbf{u}_2 + H \operatorname{div} \mathbf{u}_1 - 2u_1^{\alpha} \nabla_{\alpha}^* H = 0$$

和从 (4.28) 第一式可得

$$\begin{aligned} \nabla_{\alpha}^* (K_{\beta}^{\alpha} u_1^{\beta}) &= K_{\beta}^{\alpha} \nabla_{\alpha}^* (u_1^{\beta}) + \nabla_{\alpha}^* K_{\beta}^{\alpha} u_1^{\beta} = \left(H - \frac{3}{8\delta}\right) \nabla_{\alpha}^* u_1^{\alpha} + b_{\beta}^{\alpha} \nabla_{\alpha}^* u_1^{\beta} + \nabla_{\alpha}^* K_{\beta}^{\alpha} u_1^{\beta}, \\ \nabla_{\alpha}^* K_{\beta}^{\alpha} &= \nabla_{\alpha}^* \left(H - \frac{3}{8\delta}\right) \delta_{\beta}^{\alpha} + \nabla_{\alpha}^* b_{\beta}^{\alpha} = \nabla_{\beta}^* H + \nabla_{\beta}^* b_{\alpha}^{\alpha} = 3 \nabla_{\beta}^* H, \\ \nabla_{\alpha}^* (K_{\beta}^{\alpha} u_1^{\beta}) &= \left(H - \frac{3}{8\delta}\right) \operatorname{div} \mathbf{u}_1 + b_{\beta}^{\alpha} \nabla_{\alpha}^* u_1^{\beta} + 3u_1^{\beta} \nabla_{\beta}^* H. \end{aligned}$$

综合以上各式得

$$H \operatorname{div} \mathbf{u}_1 - 2u_1^{\alpha} \nabla_{\alpha}^* H + \left(H - \frac{3}{8\delta}\right) \operatorname{div} \mathbf{u}_1 + b_{\beta}^{\alpha} \nabla_{\alpha}^* u_1^{\beta} + 3u_1^{\beta} \nabla_{\beta}^* H - \frac{1}{2\mu} \Delta^* p_0 + \nabla_{\alpha}^* F_2^{\alpha} = 0.$$

这就推出 (4.28). 证毕. $\square$

利用定理 10 和 (2.3) 中的公式 $Ka_{\alpha\beta} - 2Hb_{\alpha\beta} + C_{\alpha\beta}$, 经过运算, (4.19) 和 (4.20) 可以表示为

$$\begin{aligned} L_{\beta}^1(U) &= \mu \left[\frac{29}{32} a_{\alpha\beta} \delta + 2((K - H^2)a_{\alpha\beta} + Hb_{\alpha\beta}) \frac{\delta^3}{3} \right] u_1^{\alpha} - \frac{\mu}{4} \nabla_{\beta}^* \operatorname{div} \mathbf{u}_1 \frac{\delta^3}{3} \\ &\quad + \left(\left(H \frac{\delta^3}{3} - \frac{1}{4} \frac{\delta^2}{2} \right) \delta_{\beta}^{\alpha} + b_{\beta}^{\alpha} \frac{\delta^3}{3} \right) \nabla_{\alpha}^* p_0 + \mu \left(\left(\frac{1}{2} \frac{\delta^2}{2} - 2H \frac{\delta^3}{3} \right) a_{\alpha\beta} - \frac{1}{2} b_{\alpha\beta} \frac{\delta^3}{3} \right) F_2^{\alpha}, \\ L_3^1(U) &= \mu \left(-\frac{3}{2} \operatorname{div} \mathbf{u}_1 \frac{\delta^2}{2} + (6H \operatorname{div} \mathbf{u}_1 - u_1^{\alpha} \nabla_{\alpha}^* H) \frac{\delta^3}{3} \right), \\ D_{\beta}^1(P) &= -H \nabla_{\beta}^* p_0 \frac{\delta^3}{3} - 3 \nabla_{\alpha}^* H p_0 \frac{\delta^3}{3} + \frac{9}{4} \mu \nabla_{\beta}^* \operatorname{div} \mathbf{u}_1 \frac{\delta^3}{3} - \frac{1}{2} \nabla_{\beta}^* h_3^2, \\ D_3^1(P) &= p_2 \frac{\delta^3}{3} + \frac{1}{4} \left(\delta + 12H \frac{\delta^2}{2} + 12(K - 4H^2) \frac{\delta^3}{3} \right) p_0 \\ &\quad - \frac{9}{4} \mu \left(\frac{\delta^2}{2} - 4H \frac{\delta^3}{3} \right) \operatorname{div} \mathbf{u}_1 - \left(\frac{3}{4} \delta^{-1} - 2H \right) h_3^2, \end{aligned} \quad (4.32)$$

从而, 有

$$\begin{aligned}
 (L_k^1(U) - D_k^1(P, U), v_\lambda^1) &= (l_\beta^1(\mathbf{u}_1) - d_\beta^1, v_1^\beta) + (l_3^1(\mathbf{u}_1) - d_3^1(p_0), v_1^3) + \left(p_2 \frac{\delta^3}{3}, v_1^3\right) \\
 &\quad + \left(\left(\frac{3}{4}\delta^{-1} - 2H\right)h_3^2, v_1^3\right) - \left(\frac{1}{2} \nabla_\beta^* h_3^2, v_1^\beta\right), \\
 l_\beta^1(\mathbf{u}_1) &= q_{\alpha\beta} u_1^\alpha - \frac{10}{4} \mu \nabla_\beta^* \operatorname{div} \mathbf{u}_1 \frac{\delta^3}{3}, \\
 l_3^1(\mathbf{u}_1) &= \mu \left(\frac{3}{4} \frac{\delta^2}{2} - 2H \frac{\delta^3}{3}\right) \operatorname{div} \mathbf{u}_1 - \mu \frac{\delta^3}{3} \operatorname{div}(H \mathbf{u}_1) \\
 &= \mu \left(\frac{3}{4} \frac{\delta^2}{2} - 3H \frac{\delta^3}{3}\right) \operatorname{div} \mathbf{u}_1 - \mu \frac{\delta^3}{3} \nabla_\alpha^* H u_1^\alpha, \\
 d_\beta^1(p_0) &= H \frac{\delta^3}{3} \nabla_\beta^* p_0 - 3 \nabla_\beta^* H p_0 \frac{\delta^3}{3}, \\
 d_3^1(p_0) &= d_0 p_0, \quad d_0 := \frac{1}{4} \left(\delta + 12H \frac{\delta^2}{2} + 12(K - 4H^2) \frac{\delta^3}{3}\right), \\
 q_{\alpha\beta} &= \mu \left(\frac{29}{32} a_{\alpha\beta} \delta - 2(H^2 - K) a_{\alpha\beta} + H b_{\alpha\beta} \frac{\delta^3}{3}\right).
 \end{aligned} \tag{4.33}$$

于是, 从 (4.15) 和 (4.31) 立即可以得到如下定理.

定理 12 (4.14) 中的解 $(\mathbf{u}_1, p_2)$ 满足下列变分问题:

$$\begin{cases} \text{求 } (\mathbf{u}_1, p_2) \in \mathbf{V}(D) \times L^2(D), \forall \mathbf{v}_1 \in \mathbf{V}(D), \text{ 使得} \\ (a_0^{\alpha\beta\lambda\sigma} \gamma_{\alpha\lambda}(\mathbf{u}_1), \gamma_{\beta\sigma}(\mathbf{v}_1)) \frac{\delta^3}{3} + (\mathbf{l}^1(\mathbf{u}_1), \mathbf{v}_1) - (\mathbf{d}^1(p_0), \mathbf{v}_1) \\ + b(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}_1) + (p_2, v_1^3) \frac{\delta^3}{3} = (\mathbf{F}^{(1)}, \mathbf{v}_1), \\ q_1, u_1^3 \frac{\delta^2}{2} + (\operatorname{div} \mathbf{u}_1 + 2u_2^3 - 4H u_1^3) \frac{\delta^3}{3} = 0, \end{cases} \tag{4.34}$$

其中 $\mathbf{l}$ 和 $\mathbf{d}$ 为 (4.34) 所定义,

$$\begin{cases} b(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}_1) = \mathcal{B}_0(U, V) := (B_k(U), v_1^k), \\ B_\beta(\mathbf{u}_1) = a_{\alpha\beta} (\nabla_\lambda^* u_1^\alpha u_1^\lambda - u_1^\alpha \operatorname{div} \mathbf{u}_1) \frac{\delta^4}{4}, \quad B_3(\mathbf{u}_1) = (b_{\alpha\beta} u_1^\alpha u_1^\beta) \frac{\delta^4}{4}, \\ F_\beta^{(1)} = h_\beta^1 + \frac{1}{2} \nabla_\beta^* h_3^2, \quad F_3^{(1)} = h_3^1 - \left(\frac{3}{4}\delta^{-1} - 2H\right)h_3^2, \end{cases} \tag{4.35}$$

而相应 $\mathbf{u}_1$ 和 p_2 所满足的边值问题是

$$\begin{cases} -\frac{\delta^3}{3} \mu \left(\Delta u_1^\alpha + \frac{7}{2} a^{\alpha\lambda} \nabla_\lambda^* \operatorname{div} \mathbf{u}_1 + K u_1^\alpha \right) + q_\beta^\alpha u_1^\beta + (u_1^\beta \nabla_\beta^* u_1^\alpha - u_1^\alpha \operatorname{div} \mathbf{u}_1) \frac{\delta^4}{4} \\ - H \frac{\delta^3}{3} a^{\alpha\beta} \nabla_\beta^* p_0 + 3a^{\alpha\gamma} \nabla_\beta^* H p_0 \frac{\delta^3}{3} = a^{\alpha\beta} F_\beta^{(1)}, \\ p_2 \frac{\delta^3}{3} = 2\mu \beta_0(\mathbf{u}_1) \frac{\delta^3}{3} - \mu \left(\frac{3}{4} \frac{\delta^2}{2} - 2H \frac{\delta^3}{3}\right) \operatorname{div} \mathbf{u}_1 + \mu \operatorname{div}(H \mathbf{u}_1) \frac{\delta^3}{3} \\ + d_0 p_0 + b_{\alpha\beta} u_1^\alpha u_1^\beta \frac{\delta^4}{4} + F_3^{(1)}, \\ \text{周期性边界条件, 且假设体积力忽略不计,} \end{cases} \tag{4.36}$$

其中

$$\begin{aligned} d_0 &:= \frac{1}{4} \left(\delta + 12H \frac{\delta^2}{2} + 12(K - 4H^2) \frac{\delta^3}{3} \right), \\ q_{\alpha\beta} &= \mu \left(\frac{29}{32} a_{\alpha\beta} \delta - 2((H^2 - K)a_{\alpha\beta} + Hb_{\alpha\beta}) \frac{\delta^3}{3} \right). \end{aligned} \quad (4.37)$$

证明 实际上, 从 (4.27) 和 (4.34) 立即可以推出 (4.35). 余下的只须证明 (4.37). 那么, 注意到

$$\begin{cases} \gamma_{\alpha\lambda}(\mathbf{u}_1) = e_{\alpha\lambda}^*(\mathbf{u}_1) - b_{\alpha\lambda} u_1^3 = e_{\alpha\lambda}^*(\mathbf{u}_1) \quad (\text{因为 } u_1^3 = 0), \\ a_0^{\alpha\beta\lambda\sigma} = 2\mu a^{\alpha\beta} a^{\lambda\sigma}, \\ a(\mathbf{u}_1, \mathbf{v}_1) := (a_0^{\alpha\beta\lambda\sigma} \gamma_{\alpha\lambda}(\mathbf{u}_1), \gamma_{\beta\sigma}(\mathbf{v}_1)) \frac{\delta^3}{3} = 2\mu \frac{\delta^3}{3} (a^{\alpha\beta} a^{\lambda\sigma} e_{\alpha\lambda}^*(\mathbf{u}_1), e_{\beta\sigma}^*(\mathbf{v}_1) - b_{\beta\sigma} v_1^3). \end{cases} \quad (4.38)$$

因为

$$a^{\alpha\beta} a^{\lambda\sigma} b_{\beta\sigma} = b^{\alpha\lambda}, \quad b^{\alpha\lambda} e_{\alpha\lambda}^*(\mathbf{u}_1) := \beta_0(\mathbf{u}_1),$$

所以, 考虑到指标对称性和度量张量协变导数为零, 有

$$\begin{cases} a(\mathbf{u}_1, \mathbf{v}_1) = 2\mu(a^{\alpha\beta\lambda\sigma} e_{\alpha\lambda}^*(\mathbf{u}_1), e_{\alpha\lambda}^*(\mathbf{v}_1)) - 2\mu(\beta_0(\mathbf{u}_1), v_1^3), \\ e_{\beta\sigma}^*(\mathbf{v}_1) = \frac{1}{2}(a_{\beta\nu} \nabla_\sigma^* v_1^\nu, a_{\sigma\nu} \nabla_\beta^* v_1^\nu), \\ (2\mu(a^{\alpha\beta} a^{\lambda\sigma} e_{\alpha\lambda}^*(\mathbf{u}_1)), a_{\sigma\nu} \nabla_\beta^* v_1^\nu) = -2\mu(\nabla_\beta^* (a^{\alpha\beta} a^{\lambda\sigma} e_{\alpha\lambda}^*(\mathbf{u}_1)), a_{\sigma\nu} v_1^\nu) \\ \quad = -2\mu(\nabla_\beta^* e^{\beta\sigma}(\mathbf{u}_1), a_{\sigma\nu} v_1^\nu). \end{cases} \quad (4.39)$$

下面证明如下一个通用的公式 (二维流形上的变形张量散度公式):

$$\begin{aligned} \nabla_\beta^* e^{\alpha\beta}(\mathbf{u}) &= \frac{1}{2}(\Delta^* u^\alpha + a^{\alpha\lambda} \nabla_\lambda^* (\operatorname{div} \mathbf{u}) + K u^\alpha), \\ (e^{\alpha\beta}(\mathbf{u}), e_{\alpha\beta}^*(\mathbf{v})) &= \frac{1}{2}[(\nabla^* \mathbf{u}, \nabla^* \mathbf{v}) + (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) - (K \mathbf{u}, \mathbf{v})], \end{aligned} \quad (4.40)$$

其中 K 是固壁边界曲面的 Gauss 曲率,

$$(\nabla^* \mathbf{u}, \nabla^* \mathbf{v}) = (a^{\alpha\beta} a_{\lambda\sigma} \nabla_\alpha^* u^\lambda, \nabla_\beta^* v^\sigma), \quad (K \mathbf{u}, \mathbf{v}) = (K a_{\alpha\beta} u^\alpha, v^\beta).$$

(4.40) 的证明. 实际上,

$$\nabla_\beta^* e^{\alpha\beta}(\mathbf{u}_1) = \frac{1}{2} \nabla_\beta^* (\nabla^\alpha u_1^\beta + \nabla^\beta u_1^\alpha) = \frac{1}{2} a^{\alpha\lambda} \nabla_\beta^* \nabla_\lambda^* u_1^\beta + \frac{1}{2} \nabla_\beta^* \nabla^\beta u_1^\alpha,$$

然而,

$$\nabla_\beta^* \nabla^\beta u_1^\alpha = \Delta^* u_1^\alpha,$$

利用文献 [3, 6–8] 中的 Ricci 公式、Ricci 张量和 Riemann 张量的关系, 曲面上 Ricci 张量的表达式

$$\begin{cases} \nabla_\beta^* \nabla_\lambda^* u_1^\sigma - \nabla_\lambda^* \nabla_\beta^* u_1^\sigma = R_{\nu\beta\lambda}^\sigma u_1^\nu, \\ R_{\sigma\lambda} = R_{\sigma\nu\lambda}^\nu, \quad R_{\alpha\beta} = K a_{\alpha\beta}, \end{cases}$$

那么,

$$\begin{aligned} a^{\alpha\lambda} \nabla_{\beta}^* \nabla_{\lambda}^* u_1^{\beta} &= a^{\alpha\lambda} (\nabla_{\lambda}^* \nabla_{\beta}^* u_1^{\beta} + R_{\sigma\beta\lambda}^{\beta} u_1^{\sigma}) = a^{\alpha\lambda} \nabla_{\lambda}^* (\operatorname{div} \mathbf{u}_1) + a^{\alpha\lambda} R_{\sigma\lambda} u_1^{\sigma} \\ &= a^{\alpha\lambda} \nabla_{\lambda}^* (\operatorname{div} \mathbf{u}_1) + a^{\alpha\lambda} K a_{\sigma\lambda} u_1^{\sigma} = a^{\alpha\lambda} \nabla_{\lambda}^* (\operatorname{div} \mathbf{u}_1) + K u_1^{\alpha}. \end{aligned}$$

这就推出 (4.40) 的第一式. 应用流形上的 Gauss 散度积分定理^[3,5] 即得 (4.40) 的第二式. 证毕. $\square$

将 (4.40) 代入 (4.39) 得

$$\begin{cases} 2\mu(a^{\alpha\beta\lambda\sigma} e_{\alpha\lambda}^*(\mathbf{u}_1), e_{\alpha\lambda}^*(\mathbf{v}_1)) = -\mu(\Delta u_1^{\alpha} + a^{\alpha\lambda} \nabla_{\lambda}^* (\operatorname{div} \mathbf{u}_1) + K u_1^{\alpha}, a_{\alpha\beta} v_1^{\beta}) \\ \quad = \mu[(\nabla \mathbf{u}_1, \nabla \mathbf{v}_1) + (\operatorname{div} \mathbf{u}_1, \operatorname{div} \mathbf{v}_1) - (a_{\alpha\beta} K u_1^{\alpha}, v_1^{\beta})], \\ a(\mathbf{u}_1, \mathbf{v}_1) = (a_0^{\alpha\beta\lambda\sigma} \gamma_{\alpha\lambda}(\mathbf{u}_1), \gamma_{\beta\sigma}(\mathbf{v}_1)) \\ \quad = 2\mu(a^{\alpha\beta} a^{\lambda\sigma} e_{\alpha\lambda}^*(\mathbf{u}_1), e_{\alpha\lambda}^*(\mathbf{v}_1)) - 2\mu(\beta_0(\mathbf{u}_1), v_1^3) \\ \quad = -\mu(\Delta u_1^{\alpha} + a^{\alpha\lambda} \nabla_{\lambda}^* (\operatorname{div} \mathbf{u}_1) + K u_1^{\alpha}, a_{\alpha\beta} v_1^{\beta}) - 2\mu(\beta_0(\mathbf{u}_1), v_1^3), \\ \frac{\delta^3}{3} [-\mu((\Delta u_1^{\alpha} + a^{\alpha\lambda} \nabla_{\lambda}^* (\operatorname{div} \mathbf{u}_1) + K u_1^{\alpha}), a_{\alpha\beta} v_1^{\beta}) - 2\mu(\beta_0(\mathbf{u}_1), v_1^3)] \\ \quad + (l^1(\mathbf{u}), \mathbf{v}_1) - (d^1(p_0), \mathbf{v}_1) + (p_2, v_1^3) \frac{\delta^3}{3} + b(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}_1) = (\mathbf{F}^{(1)}, \mathbf{v}_1), \end{cases} \quad (4.41)$$

即

$$\begin{cases} -\frac{\delta^3}{3} \mu(\Delta u_1^{\alpha} + a^{\alpha\lambda} \nabla_{\lambda}^* (\operatorname{div} \mathbf{u}_1) + K u_1^{\alpha}) + a^{\alpha\beta} l_{\beta}^1(\mathbf{u}_1) - a^{\alpha\beta} d_{\beta}^1(p_0) + a^{\alpha\beta} \mathcal{B}_{\beta}(\mathbf{u}_1) = a^{\alpha\beta} F_{\beta}^{(1)}, \\ p_2 \frac{\delta^3}{3} - 2\mu(\beta_0(\mathbf{u}_1)) \frac{\delta^3}{3} + l_3^1(\mathbf{u}_1) - d_3^1(p_0) + \mathcal{B}_3(\mathbf{u}_1) = F_3^{(1)}. \end{cases}$$

利用 (4.33), 上式可以表示为

$$\begin{cases} -\frac{\delta^3}{3} \mu \left(\Delta u_1^{\alpha} + \frac{7}{2} a^{\alpha\lambda} \nabla_{\lambda}^* \operatorname{div} \mathbf{u}_1 + K u_1^{\alpha} \right) + q_{\beta}^{\alpha} u_1^{\beta} + (u_1^{\beta} \nabla_{\beta}^* u_1^{\alpha} - u_1^{\alpha} \operatorname{div} \mathbf{u}_1) \frac{\delta^4}{4} \\ \quad - H \frac{\delta^3}{3} a^{\alpha\beta} \nabla_{\beta}^* p_0 + 3a^{\alpha\gamma} \nabla_{\beta}^* H p_0 \frac{\delta^3}{3} = a^{\alpha\beta} F_{\beta}^{(1)}, \\ p_2 \frac{\delta^3}{3} = 2\mu\beta_0(\mathbf{u}_1) \frac{\delta^3}{3} - \mu \left(\frac{3}{4} \frac{\delta^2}{2} - 2H \frac{\delta^3}{3} \right) \operatorname{div} \mathbf{u}_1 + \mu \operatorname{div} (H \mathbf{u}_1) \frac{\delta^3}{3} \\ \quad + \frac{1}{4} \left(\delta + 12H \frac{\delta^2}{2} + 10(K - 4H^2) \frac{\delta^3}{3} \right) p_0 + b_{\alpha\beta} u_1^{\alpha} u_1^{\beta} \frac{\delta^4}{4} + F_3^{(1)}, \\ \mathbf{u}_1 \text{ 在 } \mathfrak{S} \text{ 边界上满足周期性边界条件, 或无边界.} \end{cases}$$

这就是 (4.36) 和 (4.37). 证明结束. $\square$

综合以上结果, 得到边界层方程

$$\begin{cases} \text{求 } (\mathbf{u}_1, p_0) \in V(D) \times L_p^2(D), \text{ 使得} \\ -\frac{\delta^3}{3} \mu \left(\Delta u_1^{\alpha} + \frac{7}{2} a^{\alpha\lambda} \nabla_{\lambda}^* \operatorname{div} \mathbf{u}_1 + K u_1^{\alpha} \right) + q_{\beta}^{\alpha} u_1^{\beta} + B^{\alpha}(\mathbf{u}_1, \mathbf{u}_1) + d^{\alpha}(p_0) = a^{\alpha\beta} F_{\beta}^{(1)}, \\ -\frac{\delta^3}{3} \Delta p_0 + \frac{\delta^3}{3} M_1(\mathbf{u}_1) = f_p, \\ u_1^3 = 0, \quad \text{周期性边界条件,} \end{cases} \quad (4.42)$$

而 $\mathbf{u}_2, p_1$ 和 p_2 由下列公式得到

$$\begin{cases} u_2^\alpha = K_\beta^\alpha u_1^\beta - \frac{1}{2\mu} a^{\alpha\beta} \nabla_\beta^* p_0 + F_2^\alpha, & u_2^3 = -\frac{1}{2} \operatorname{div}^* \mathbf{u}_1, \\ p_1 = \left(3H - \frac{3}{2}\delta^{-1}\right) p_0 - \frac{9}{4} \mu \operatorname{div}^* \mathbf{u}_1 - \frac{3}{2} \delta^{-1} h_3^2, \\ p_2 \frac{\delta^3}{3} = 2\mu \beta_0(\mathbf{u}_1) \frac{\delta^3}{3} - \mu \left(\frac{3}{4} \frac{\delta^2}{2} - 2H \frac{\delta^3}{3}\right) \operatorname{div}^* \mathbf{u}_1 + \mu \operatorname{div}^*(H \mathbf{u}_1) \frac{\delta^3}{3} \\ \quad + d_0 p_0 + b_{\alpha\beta} u_1^\alpha u_1^\beta \frac{\delta^4}{4} + F_3^{(1)}, \end{cases} \quad (4.43)$$

其中

$$\begin{cases} q_{\alpha\beta} = \mu \left(\frac{29}{32} a_{\alpha\beta} \delta - 2((H^2 - K) a_{\alpha\beta} + H b_{\alpha\beta}) \frac{\delta^3}{3} \right), \\ K_\beta^\alpha = \left(H - \frac{3}{8\delta} \right) \delta_\beta^\alpha + b_\beta^\alpha, \\ d_0 = \frac{1}{4} \left(\delta + 12H \frac{\delta^2}{2} + 10(K - 4H^2) \frac{\delta^3}{3} \right), \\ d^\alpha(p_0) := -H \frac{\delta^3}{3} a^{\alpha\beta} \nabla_\beta^* p_0 + 3a^{\alpha\beta} \nabla_\beta^* H p_0 \frac{\delta^3}{3}, \\ B^\alpha(\mathbf{u}_1, \mathbf{u}_1) := (u_1^\beta \nabla_\beta^* u_1^\alpha - u_1^\alpha \operatorname{div}^* \mathbf{u}_1) \frac{\delta^4}{4}, \\ M_1(\mathbf{u}_1) = 2\mu \left[\left(2H - \frac{3}{8\delta} \right) \operatorname{div}^* \mathbf{u}_1 + u_1^\alpha \nabla_\alpha^* H + b_\beta^\alpha \nabla_\alpha^* u_1^\beta \right], \\ F_\beta^{(1)} = h_\beta^1 + \frac{1}{2} \nabla_\beta^* h_3^2, \quad F_3^{(1)} = h_3^1 - \left(\frac{3}{4} \delta^{-1} - 2H \right) h_3^2, \\ F_2^\alpha = \frac{1}{2\mu} \left(\frac{\delta^3}{3} \right)^{-1} a^{\alpha\beta} h_\beta^2, \quad f_p = -2\mu \frac{\delta^3}{3} \nabla_\beta^* F_2^\beta = -a^{\alpha\beta} \nabla_\beta^* h_\alpha^2. \end{cases} \quad (4.44)$$

显然, 应用 (4.42), 那么 (4.43) 对应的变分问题是

$$\begin{cases} \text{求 } U := (\mathbf{u}_1, p_0) \in \mathbf{Y}(D) = \mathbf{V}(D) \times L_p^2(D), \text{ 使得} \\ A(U, V) + b(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}) + \frac{\delta^3}{3} (M_1(u_1), q) + (a_{\alpha\beta} d^\alpha(p_0), v^\beta) = (\mathbf{F}^{(1)}, \mathbf{v}) + (f_p, q), \\ \forall V = (\mathbf{v}, q) \in \mathbf{Y}(D), \end{cases} \quad (4.45)$$

这里

$$\begin{aligned} A(U, V) &= 2\mu \frac{\delta^3}{3} (a^{\alpha\beta} a^{\lambda\sigma} e_{\alpha\lambda}^*(\mathbf{u}_1), e_{\beta\sigma}^*(\mathbf{v})) + \frac{\delta^3}{3} (\nabla^* p_0, \nabla^* q) + \frac{5}{2} \frac{\delta^3}{3} (\operatorname{div}^* \mathbf{u}_1, \operatorname{div}^* \mathbf{v}) + (q_{\alpha\beta} u_1^\alpha, v^\beta), \\ (a_{\alpha\beta} d^\alpha(p_0), v^\beta) &= \frac{\delta^3}{3} (-H \nabla_\beta^* p_0 + 3 \nabla_\beta^* H p_0, v^\beta), \\ \frac{\delta^3}{3} (M_1(u_1), q) &= -\frac{\mu}{16} \delta^2 (\operatorname{div}^* \mathbf{u}_1, q) + \frac{\delta^3}{32} \mu (H \operatorname{div}^* \mathbf{u}_1 + u_1^\alpha \nabla_\alpha^* H + b_\beta^\alpha \nabla_\alpha^* u_1^\beta, q). \end{aligned}$$

定理 13 假设流形 $\mathfrak{S}$ 充分光滑, 使得 $a_{\alpha\beta}, a^{\alpha\beta}, H, K \in C^1(\overline{D})$,

$$\sum_{\alpha, \beta} \{ \|a_{\alpha\beta}\|_{1,D} + \|a^{\alpha\beta}\|_{1,D} \} \leq C_\delta, \quad \{ \|H\|_{1,D}, \|K\|_{1,K}, \|a\|_{1,D} \} \leq C_\delta,$$

这里 C_δ 是一个绝对常数, 在不同的地方, 可能表示不同的意义. 那么, 只要 δ 充分小, 那么, 下面 (4.46) 第一式的强制性不等式成立, 并且如果方程右端满足

$$\|F^{(1)}\|_{-1,D} \leq \frac{C_0^2 \delta^6}{4M},$$

那么, (4.45) 存在唯一解 $U = (\mathbf{u}_1, p_0)$, 它满足 (4.46) 的第二式:

$$\begin{cases} A(U, U) + (a_{\alpha\beta} d^\alpha(p_0), u_1^\beta) + \frac{\delta^3}{3} (M_1(u_1), q) \geq C_0 \delta^3 \|\mathbf{u}_1\|_{1,D}^2 + \frac{7}{16} \mu \lambda \delta \|\mathbf{u}_1\|_{0,D}^2 + \frac{1}{2} C_1 \|p_0\|_{1,D}^2, \\ \|\mathbf{u}_1\|_{1,D} \leq \frac{C_0 \delta^3}{2M} - \sqrt{\left(\frac{C_0 \delta^3}{2M}\right)^2 - \frac{\|F^{(1)}\|_{-1,D}}{M}}, \\ \|p_0\|_{1,D} \leq \frac{2}{C_1 \delta^3} \|f_p\|_{-1,D}, \end{cases} \quad (4.46)$$

其中常数 C 只依赖于 F, f_p 的常数.

证明 (1) 由于 $a^{\alpha\beta}$ 是正定的, 所以, 存在正常数 λ 使得

$$a^{\alpha\beta} \xi_\alpha \xi_\beta \geq \lambda \sum_{\alpha=1}^2 |\xi|^2, \quad \forall \xi \in \mathbb{R}^2.$$

因此,

$$a^{\alpha\beta} a^{\lambda\sigma} e_{\alpha\lambda}^*(\mathbf{u}_1), e_{\beta\sigma}^*(\mathbf{u}_1) \geq \lambda^2 \sum_{\alpha,\lambda=1}^2 |e_{\alpha\lambda}^*(\mathbf{u}_1)|^2.$$

应用流形上的 Korn 不等式^[3,8], 可以断定

$$2\mu \frac{\delta^3}{3} (a^{\alpha\beta} a^{\lambda\sigma} e_{\alpha\lambda}^*(\mathbf{u}_1), e_{\beta\sigma}^*(\mathbf{v})) \geq C_0 \delta^3 \|\mathbf{u}_1\|_{1,D}.$$

$$(2) \quad \frac{\delta^3}{3} (\nabla^* p_0, \nabla^* p_0) \geq \lambda \frac{\delta^3}{3} \|p_0\|_{1,D} \geq C_1 \delta^3 \|p_1\|_{1,D}^2.$$

$$\begin{aligned} (3) \quad (q_{\alpha\beta} u_1^\alpha, u_1^\beta) &= \mu \frac{29}{32} \delta (a_{\alpha\beta} u_1^\alpha, u_1^\beta) + \mu [(2(K - H^2) a_{\alpha\beta} - H b_{\alpha\beta}) u_1^\alpha, u_1^\beta] \\ &\geq \mu \left(\frac{29}{32} \lambda \delta - C_\delta \delta^3 \right) \|\mathbf{u}_1\|_{0,D}^2 = \mu \delta \left(\frac{29}{32} \lambda - C_\delta \delta^2 \right) \|\mathbf{u}_1\|_{0,D}^2 \geq \frac{1}{2} \mu \lambda \delta \|\mathbf{u}_1\|_{0,D}^2. \end{aligned}$$

(4) 利用流形上的 Gauss 定理和分步积分,

$$\begin{aligned} &\left| (a_{\alpha\beta} d^\alpha(p_0), u_1^\beta) + \frac{\delta^3}{3} (M_1(u_1), p_0) \right| \leq I_1 + I_2, \\ I_1 &= \frac{\mu}{16} |\delta^2 (\operatorname{div}^* \mathbf{u}_1, p_0)| \leq \frac{\mu}{16} |(u_1^\beta, \nabla_\beta^* p_0)| \leq \frac{\mu}{16} \delta^2 \|\mathbf{u}_1\|_{0,D} \|p_0\|_{1,D} \\ &\leq \frac{\mu}{32} \lambda \delta \|\mathbf{u}_1\|_{0,D}^2 + \frac{\mu}{32\lambda} \delta^3 \|p_0\|_{1,D}^2, \\ I_2 &= \frac{\delta^3}{3} (-H \nabla_\beta^* p_0 + 3 \nabla_\beta^* H p_0, u_1^\beta) + \frac{\delta^3}{3} 2\mu (H \operatorname{div}^* \mathbf{u}_1 + u_1^\alpha \nabla_\alpha^* H + b_\beta^\alpha \nabla_\alpha^* u_1^\beta, p_0) \\ &= \frac{\delta^3}{3} (u_1^\beta, -H \nabla_\beta^* p_0 + 3 \nabla_\beta^* H p_0 - 2\mu \nabla_\beta^* (H p_0) + p_0 \nabla_\beta^* H - \nabla_\alpha^* (b_\beta^\alpha p_0)), \\ |I_2| &\leq \delta^3 C_\delta \|\mathbf{u}_1\|_{0,D} \|p_0\|_{1,D} \leq \frac{1}{4} C_1 \delta^3 \|p_0\|_{1,D}^2 + C_3 \delta^3 \|\mathbf{u}_1\|_{0,D}^2, \end{aligned}$$

$$\begin{aligned} \left| (a_{\alpha\beta} d^\alpha(p_0), u_1^\beta) + \frac{\delta^3}{3} (M_1(u_1), p_0) \right| &\leq \left(\frac{\mu}{32} \delta + C_3 \delta^3 \right) \|u_1\|_{0,D}^2 + \left(\frac{\mu}{32} + \frac{1}{4} C_1 \right) \delta^3 \|p_0\|_{1,D}^2 \\ &\leq \left(\frac{\mu}{16} \lambda \delta \right) \|u_1\|_{0,D}^2 + \left(\frac{\mu}{32\lambda} + \frac{1}{4} C_1 \right) \delta^3 \|p_0\|_{1,D}^2. \end{aligned}$$

综合上述, 只要 δ 充分小, 就有

$$\begin{aligned} A(U, U) &\geq C_0 \delta^3 \|u_1\|_{1,D}^2 + C_1 \delta^3 \|p_1\|_{1,D}^2 + \left(\frac{1}{2} \mu \lambda \delta \right) \|u_1\|_{0,D}^2, \\ A(U, U) + (a_{\alpha\beta} d^\alpha(p_0), u_1^\beta) + \frac{\delta^3}{3} (M_1(u_1), p_0) & \\ &\geq C_0 \delta^3 \|u_1\|_{1,D}^2 + \frac{1}{2} C_1 \delta^3 \|p_1\|_{1,D}^2 + \frac{7}{16} \mu \lambda \delta \|u_1\|_{0,D}^2. \end{aligned} \quad (4.47)$$

在 (4.45) 中, 令 $V = U$, 并利用 (4.47) 和

$$|b(u_1, u_1, u_1)| \leq M |u_1|_{1,D}^3,$$

有

$$\begin{aligned} 0 &\geq b(u_1, u_1, u_1) + C_0 \delta^3 \|u_1\|_{1,D}^2 + \frac{1}{2} C_1 \delta^3 \|p_1\|_{1,D}^2 + \frac{7}{16} \mu \lambda \delta \|u_1\|_{0,D}^2 - (F^{(1)}, u_1) + (f_p, p_0) \\ &\geq C_0 \delta^3 \|u_1\|_{1,D}^2 + \frac{7}{16} \mu \lambda \delta \|u_1\|_{0,D}^2 - M |u_1|_{1,D}^3 - \|F^{(1)}\|_{-1,D} |u_1|_{1,D} \\ &\quad + \frac{1}{2} C_1 \delta^3 \|p_1\|_{1,D}^2 - \|f_p\|_{-1,D} \|p_0\|_{1,D} \\ &= (C_0 \delta^3 \|u_1\|_{1,D} - M |u_1|_{1,D}^2 - \|F^{(1)}\|_{-1,D}) |u_1|_{1,D} + \left(\frac{1}{2} C_1 \delta^3 \|p_1\|_{1,D} - \|f_p\|_{-1,D} \right) \|p_0\|_{1,D}. \end{aligned}$$

所以, 如果令 $y = |u_1|_{1,D}$, 有

$$\begin{cases} (C_0 \delta^3 \|u_1\|_{1,D} - M |u_1|_{1,D}^2 - \|F^{(1)}\|_{-1,D}) \leq 0, \\ \frac{1}{2} C_1 \delta^3 \|p_1\|_{1,D} - \|f_p\|_{-1,D} \leq 0. \end{cases}$$

从而, 当 $F^{(1)}$ 满足

$$\|F^{(1)}\|_{-1,D} \leq \frac{C_0^2 \delta^6}{4M}$$

时, 就有

$$\left(y - \frac{C_0 \delta^3}{2M} \right)^2 \geq \left(\frac{C_0 \delta^3}{2M} \right)^2 - \frac{\|F^{(1)}\|_{-1,D}}{M} \Rightarrow y \leq \frac{C_0 \delta^3}{2M} - \sqrt{\left(\frac{C_0 \delta^3}{2M} \right)^2 - \frac{\|F^{(1)}\|_{-1,D}}{M}}.$$

最后, 我们得到

$$\begin{cases} |u_1|_{1,D} \leq \frac{C_0 \delta^3}{2M} - \sqrt{\left(\frac{C_0 \delta^3}{2M} \right)^2 - \frac{\|F^{(1)}\|_{-1,D}}{M}}, \\ \|p_1\|_{1,D} \leq \frac{2}{C_1 \delta^3} \|f_p\|_{-1,D}. \end{cases} \quad (4.48)$$

根据 (4.48), 应用文献 [6, 7] 中相同的方法, 可以证明 (4.46) 存在唯一的解. 定理得证. $\square$

应用文献 [6, 7] 中相同的方法, 对于充分小的 δ 以及右端满足一定条件, (4.46) 存在唯一的解.

5 曲面变形后的边界层方程和关于形状 Gateaux 导数方程

5.1 变形后的边界层方程

曲面变形, 即意味曲面 $\mathfrak{S} = \mathbf{r}(D)$ 上任一点, 忍受一个位移 $\boldsymbol{\eta}$, 变形后的曲面为 $\mathfrak{S}(\boldsymbol{\eta}) = \mathbf{r}(\boldsymbol{\eta})(D)$, 这里 $\mathbf{r}(\boldsymbol{\eta}) = \mathbf{r} + \boldsymbol{\eta}$, 即映射 $\forall \mathbf{r} \in \mathfrak{S} \Rightarrow \mathbf{r}(\boldsymbol{\eta}) \in \mathfrak{S}(\boldsymbol{\eta})$ 是光滑的, 使得曲面 $\mathfrak{S}(\boldsymbol{\eta})$ 上的基向量 $\mathbf{e}_\alpha(\boldsymbol{\eta}) = \partial_\alpha \mathbf{r}(\boldsymbol{\eta})$ 在 D 上的每一点都是线性无关. 并令

$$\mathbf{n}(\boldsymbol{\eta}) = \frac{\mathbf{e}_1(\boldsymbol{\eta}) \times \mathbf{e}_2(\boldsymbol{\eta})}{|\mathbf{e}_1(\boldsymbol{\eta}) \times \mathbf{e}_2(\boldsymbol{\eta})|},$$

那么, 边界层内任意向量可以表示为

$$\mathbf{u}(\boldsymbol{\eta}) = u^\alpha(\boldsymbol{\eta})\mathbf{e}_\alpha(\boldsymbol{\eta}) + u^3(\boldsymbol{\eta})\mathbf{n}(\boldsymbol{\eta}),$$

也就是, $u^\alpha(\boldsymbol{\eta})$ 和 $u^3(\boldsymbol{\eta})$ 是向量 $\mathbf{u}(\boldsymbol{\eta})$ 在 $\mathfrak{S}(\boldsymbol{\eta})$ -半测地坐标系下的逆变分量. 因为边界层方程 (4.43) 是张量形式, 具有不变性质. 因此, 我们不加证明地给出在变形后的边界曲面 $\mathfrak{S}(\boldsymbol{\eta})$ 上的边界层方程. 类似 (4.43), 有下面定理.

定理 14 在以建立在 $\mathfrak{S}(\boldsymbol{\eta})$ -半测地坐标系下, $\mathfrak{S}(\boldsymbol{\eta})$ 上流体速度法向梯度和压力 $(\mathbf{u}_1(\boldsymbol{\eta}), p_0(\boldsymbol{\eta}))$ 满足下列边值问题:

$$\left\{ \begin{array}{l} \text{求 } (\mathbf{u}_1(\boldsymbol{\eta}), p_0(\boldsymbol{\eta})) \in \mathbf{V}(D) \times L^2(D), \quad \forall \mathbf{v}_1 \in \mathbf{V}(D), \text{ 使得} \\ 2\mu(a^{\alpha\beta}(\boldsymbol{\eta})a^{\lambda\sigma}(\boldsymbol{\eta})\underline{e}_{\alpha\lambda}(\mathbf{u}_1(\boldsymbol{\eta})), \underline{e}_{\beta\sigma}(\mathbf{v}_1))\frac{\delta^3}{3} + \frac{\delta^3}{3}(a^{\alpha\beta}(\boldsymbol{\eta})\nabla_\alpha^* p_0(\boldsymbol{\eta}), \nabla_\beta^* q) \\ + \frac{5}{2}\frac{\delta^3}{3}(\underline{\text{div}}\mathbf{u}_1, \underline{\text{div}}\mathbf{v}_1) + (\underline{q}_{\alpha\beta}u_1^\alpha(\boldsymbol{\eta}), v_1^\beta) + \frac{\delta^3}{3}(\underline{M}_1(\mathbf{u}_1(\boldsymbol{\eta})), \mathbf{v}_1) \\ + (a_{\alpha\beta}(\boldsymbol{\eta})\underline{d}^\alpha(p_0)(\boldsymbol{\eta}), v_1^\beta) + \frac{\delta^3}{3}(p_2(\boldsymbol{\eta}), v_1^3) + \underline{b}(\mathbf{u}_1(\boldsymbol{\eta}), \mathbf{u}_1(\boldsymbol{\eta}), \mathbf{v}_1) \\ = (\mathbf{F}^{(1)}(\boldsymbol{\eta}), \mathbf{v}_1) + (f_p(\boldsymbol{\eta}), q), \end{array} \right. \quad (5.1)$$

对应的边值问题为

$$\left\{ \begin{array}{l} -\frac{\delta^3}{3}\mu(\underline{\Delta}u_1^\alpha(\boldsymbol{\eta}) + \frac{7}{2}a^{\alpha\lambda}(\boldsymbol{\eta})\nabla_\lambda^* \underline{\text{div}}\mathbf{u}_1(\boldsymbol{\eta}) + K(\boldsymbol{\eta})u_1^\alpha(\boldsymbol{\eta})) + \underline{q}_\beta^\alpha(\boldsymbol{\eta})u_1^\beta(\boldsymbol{\eta}) \\ + \underline{B}^\alpha(\mathbf{u}_1(\boldsymbol{\eta}), \mathbf{u}_1(\boldsymbol{\eta})) + \underline{d}^\alpha(p_0(\boldsymbol{\eta})) = a^{\alpha\beta}(\boldsymbol{\eta})F_\beta^{(1)}(\boldsymbol{\eta}), \\ -\frac{\delta^3}{3}\underline{\Delta}p_0(\boldsymbol{\eta}) + \frac{\delta^3}{3}\underline{M}_1(\mathbf{u}_1(\boldsymbol{\eta})) = -a^{\alpha\beta}(\boldsymbol{\eta})\nabla_\beta^* h_\alpha^2(\boldsymbol{\eta}) := f_p(\boldsymbol{\eta}), \\ u_1^3(\boldsymbol{\eta}) = 0, \\ \text{在 } \partial D \text{ 上满足周期性边界条件.} \end{array} \right. \quad (5.2)$$

而 $\mathbf{u}_2(\boldsymbol{\eta})$, $p_1(\boldsymbol{\eta})$ 和 $p_2(\boldsymbol{\eta})$ 由下列公式得到,

$$\left\{ \begin{array}{l} u_2^\alpha(\boldsymbol{\eta}) = K_\beta^\alpha(\boldsymbol{\eta})u_1^\beta(\boldsymbol{\eta}) - \frac{1}{2\mu}a^{\alpha\beta}(\boldsymbol{\eta})\nabla_\beta^* p_0(\boldsymbol{\eta}) + F_2^\alpha(\boldsymbol{\eta}), \quad u_2^3(\boldsymbol{\eta}) = -\frac{1}{2}\underline{\text{div}}\mathbf{u}_1(\boldsymbol{\eta}), \\ p_1(\boldsymbol{\eta}) = \left(3H(\boldsymbol{\eta}) - \frac{3}{2}\delta^{-1}\right)p_0(\boldsymbol{\eta}) - \frac{9}{4}\mu\underline{\text{div}}\mathbf{u}_1(\boldsymbol{\eta}) - \frac{3}{2}\delta^{-1}h_3^2(\boldsymbol{\eta}), \\ p_2(\boldsymbol{\eta})\frac{\delta^3}{3} = 2\mu\frac{\delta^3}{3}\underline{\beta}_0(\mathbf{u}_1(\boldsymbol{\eta})) - \mu\left(\frac{3}{4}\frac{\delta^2}{3} - 2H(\boldsymbol{\eta})\frac{\delta^3}{3}\right)\underline{\text{div}}\mathbf{u}_1(\boldsymbol{\eta}) \\ + \mu\frac{\delta^3}{3}\underline{\text{div}}(H(\boldsymbol{\eta})\mathbf{u}_1(\boldsymbol{\eta})) + d_0p_0(\boldsymbol{\eta}) + \frac{\delta^4}{4}b_{\alpha\beta}(\boldsymbol{\eta})u_1^\alpha(\boldsymbol{\eta})u_1^\beta(\boldsymbol{\eta}) + F_3^{(1)}(\boldsymbol{\eta}), \end{array} \right. \quad (5.3)$$

其中带下横线的几何符号或跟 (η) 的几何量, 表示在 $\mathfrak{S}(\eta)$ 半测地坐标系下计算的. 如 (2.10) 和 (2.12), 应用定理 2 并且去掉 η 高阶项, 有

$$\left\{ \begin{aligned} \underline{B}^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) &= (u_1^\beta(\eta) \underline{\nabla}_\beta u_1^\alpha(\eta) - u_1^\alpha(\eta) \underline{\operatorname{div}} \mathbf{u}_1(\eta)) \frac{\delta^4}{4} \\ &= B^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) + \widehat{B}^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)), \\ \widehat{B}^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta) &= u_1^\beta(\eta) [\nabla_\beta (\Phi_{\sigma\lambda}^\alpha(\eta) u_1^\sigma(\eta)) - \delta_\beta^\alpha \Phi_{\sigma\lambda}^\lambda(\eta) u_1^\sigma(\eta)] \frac{\delta^4}{4}, \\ \underline{d}^\alpha(p_0(\eta)) &= -\frac{\delta^3}{3} H(\eta) a^{\alpha\beta}(\eta) \nabla_\beta p_0(\eta) - 3a^{\alpha\gamma}(\eta) \nabla_\beta H(\eta) p_0(\eta) \frac{\delta^3}{3} \\ &= d^\alpha(p_0(\eta)) + \widehat{d}^\alpha(p_0(\eta)), \\ \widehat{d}^\alpha(p_0(\eta), \eta) &= -\frac{\delta^3}{3} a^{\alpha\beta}(\widehat{H}(\eta) + 2H\gamma_0(\eta)) \nabla_\beta p_0(\eta) \\ &\quad + \frac{3\delta^3}{3} a^{\alpha\beta}(2\gamma_0(\eta) \nabla_\beta H + \nabla_\beta \widehat{H}(\eta)) p_0(\eta), \\ q_\beta^\alpha(\eta) &= \mu \left(\frac{29}{32} \delta_\beta^\alpha \delta - 2((H^2(\eta) - K(\eta)) + H(\eta) b_{\alpha\beta}(\eta)) \frac{\delta^3}{3} \right) = q_\beta^\alpha + \widehat{q}_\beta^\alpha(\eta), \\ \widehat{q}_\beta^\alpha(\eta) &:= \mu \left(\frac{29}{16} \gamma_0(\eta) \delta - 2((2H\widehat{H}(\eta) - \widehat{K}(\eta)) \delta_\beta^\alpha + \widehat{b}_\beta^\alpha) \frac{\delta^3}{3} \right), \\ \underline{M}_1(\mathbf{u}_1(\eta), \eta) &= \frac{\mu}{2} \left[\left(2H(\eta) - \frac{3}{8\delta} \right) \underline{\operatorname{div}} \mathbf{u}_1(\eta) + u_1^\alpha(\eta) \nabla_\alpha H(\eta) + b_\beta^\alpha(\eta) \underline{\nabla}_\alpha u_1^\beta(\eta) \right] \\ &= M_1(\mathbf{u}_1(\eta)) + \widehat{M}_1(\mathbf{u}_1(\eta), \eta), \\ \widehat{M}_1(\mathbf{u}_1(\eta), \eta) &= \frac{\mu}{2} \left[2\widehat{H}(\eta) \operatorname{div} \mathbf{u}_1(\eta) + \left(2H - \frac{3}{8\delta} \right) \Phi_{\alpha\beta}^\alpha(\eta) u_1^\beta(\eta) \right. \\ &\quad \left. + u_1^\alpha(\eta) \nabla_\alpha \widehat{H}(\eta) + \widehat{b}_\beta^\alpha(\eta) \nabla_\alpha u_1^\beta(\eta) + b_\beta^\alpha \Phi_{\alpha\lambda}^\beta(\eta) u_1^\lambda(\eta) \right], \\ \underline{\nabla}_\alpha u^\lambda(\eta) &= \partial_\alpha u^\lambda(\eta) + \Gamma_{\alpha\sigma}^\lambda u^\sigma(\eta), \quad \underline{\operatorname{div}} \mathbf{u}(\eta) = \underline{\nabla}_\alpha u^\alpha(\eta), \\ \underline{\Delta} &= a^{\alpha\beta}(\eta) \underline{\nabla}_\alpha \underline{\nabla}_\beta, \quad F_2^\alpha = \frac{a^{\alpha\beta}(\eta)}{2\mu\delta^3} h_\beta^2(\eta), \\ (\underline{\nabla} p_0(\eta), \underline{\nabla} q) &= \int_D \left[a^{\alpha\beta} \nabla_\alpha p_0(\eta) \nabla_\beta q \left(1 + \frac{1}{2} \gamma_0(\eta) \right) \right] \sqrt{a} dx. \end{aligned} \right. \quad (5.4)$$

$$(5.5)$$

定理中的方程和公式以及几何量, 都是在 $\mathfrak{S}(\eta)$ - 半测地坐标系下的. 但是实际计算, 必须在 $\mathfrak{S}$ - 半测地坐标系下进行. 为此, 我们进行坐标变换. 在 $\mathfrak{S}$ - 半测地坐标系下, 首先计算 Christoffel 记号

$$\begin{aligned} \underline{\Gamma}_{\beta\sigma}^* &= (\mathbf{r}^\alpha + \boldsymbol{\eta}^\alpha)(\mathbf{r}_{\beta\sigma} + \boldsymbol{\eta}_{\beta\sigma}) = \mathbf{r}^\alpha \mathbf{r}_{\beta\sigma} + \mathbf{r}^\alpha \boldsymbol{\eta}_{\beta\sigma} + \boldsymbol{\eta}^\alpha \mathbf{r}_{\beta\sigma} + o(|\boldsymbol{\eta}|^2) \\ &= \Gamma_{\beta\sigma}^* + \mathbf{r}^\alpha \boldsymbol{\eta}_{\beta\sigma} + \boldsymbol{\eta}^\alpha \mathbf{r}_{\beta\sigma} + o(|\boldsymbol{\eta}|^2) \stackrel{(\text{Gauss 公式})}{=} \Gamma_{\beta\sigma}^* + \mathbf{r}^\alpha \boldsymbol{\eta}_{\beta\sigma} + \boldsymbol{\eta}^\alpha (\Gamma_{\beta\sigma}^\nu \mathbf{r}_\nu + b_{\beta\sigma} \mathbf{n}) \\ &= \Gamma_{\beta\sigma}^* + (a^{\alpha\nu} \boldsymbol{\eta}_{\beta\sigma} + \boldsymbol{\eta}^\alpha \Gamma_{\beta\sigma}^\nu) \mathbf{r}_\nu + b_{\beta\sigma} \boldsymbol{\eta}^\alpha \mathbf{n}. \end{aligned}$$

所以, 有

$$\left\{ \begin{aligned} \underline{\Gamma}_{\beta\sigma}^* &= \Gamma_{\beta\sigma}^* + \Phi_{\beta\sigma}^\alpha(\eta), \\ \Phi_{\beta\sigma}^\alpha(\eta) &= a^{\alpha\lambda}(\eta_{\beta\sigma, \lambda} + \Gamma_{\beta\sigma}^\nu \eta_{\lambda, \nu} + b_{\beta\sigma} \eta_{\lambda, 3}), \\ \boldsymbol{\eta}_\sigma &= \partial_\sigma \boldsymbol{\eta}, \quad \boldsymbol{\eta}^\alpha = a^{\alpha\beta} \boldsymbol{\eta}_\beta, \quad \eta_{\lambda\sigma} = \partial_{\lambda\sigma}^2 \boldsymbol{\eta}, \\ \eta_{\alpha, \nu} &= \partial_\alpha \boldsymbol{\eta} \cdot \mathbf{e}_\nu, \quad \eta_{\alpha\beta, \nu} = \partial_{\alpha\beta} \boldsymbol{\eta} \cdot \mathbf{e}_\nu, \quad \eta_{\alpha, 3} = \partial_\alpha \boldsymbol{\eta} \cdot \mathbf{n}, \end{aligned} \right. \quad (5.6)$$

在 $\mathfrak{S}(\eta)$ - 半测地坐标系下的协变导数和变形速度张量与在 $\mathfrak{S}$ - 半测地坐标系下的关系服从张量变换规律, 所以,

$$\begin{cases} \nabla_{\beta}^* u_1^{\alpha}(\eta) = \nabla_{\beta}^* u_1^{\alpha}(\eta) + \Phi_{\beta\sigma}^{\alpha}(\eta) u_1^{\sigma}(\eta), & \underline{\text{div}} \mathbf{u}(\eta) = \text{div} \mathbf{u}(\eta) + \Phi_{\alpha\sigma}^{\alpha}(\eta) u_1^{\sigma}(\eta), \\ \underline{e}_{\alpha\lambda}^*(\mathbf{u}_1(\eta)) = (1 + 2\gamma_0(\eta)) \underline{e}_{\alpha\lambda}^*(\mathbf{u}_1(\eta)) + \frac{1}{2}(a_{\alpha\lambda} \Phi_{\beta\sigma}^{\lambda}(\eta) + a_{\beta\lambda} \Phi_{\alpha\sigma}^{\lambda}(\eta)) u_1^{\sigma}. \end{cases} \quad (5.7)$$

下面引理给出 Laplace 算子的换算公式.

引理 15 在 $\mathfrak{S}(\eta)$ 和 $\mathfrak{S}$ - 两个半测地坐标系下, 迹算子 Laplace 和 Laplace-Betrami 算子有如下关系:

$$\begin{aligned} \underline{\Delta} u_1^{\alpha} &= \Delta u_1^{\alpha} + A^{\alpha}(\mathbf{u}_1, \eta), \\ A^{\alpha}(\mathbf{u}_1, \eta) &= 2\gamma_0(\eta) \Delta u_1^{\alpha} + a^{\beta\sigma} [\Phi_{\beta\nu}^{\alpha}(\eta) \nabla_{\sigma}^* u_1^{\nu} + \Phi_{\sigma\nu}^{\alpha}(\eta) \nabla_{\beta}^* u_1^{\nu} - \Phi_{\sigma\beta}^{\nu}(\eta) \nabla_{\nu}^* u_1^{\alpha}] \\ &\quad + a^{\beta\sigma} \nabla_{\sigma}^* \Phi_{\beta\nu}^{\alpha}(\eta) u_1^{\nu}, \\ \underline{\Delta} p_0(\eta) &= \Delta p_0(\eta) + \phi(p_0(\eta), \eta), \\ \phi(p_0(\eta), \eta) &= 2\gamma_0(\eta) \Delta p_0(\eta) - a^{\alpha\beta} \Phi_{\alpha\beta}^{\nu}(\eta) \nabla_{\nu}^* p_0(\eta). \end{aligned} \quad (5.8)$$

证明 只要证明第一式就够了. 第二式子证明类似. 实际上, 应用 (5.7) 和二阶协变导数定义, 有

$$\begin{aligned} \underline{\nabla}_{\alpha}^* \underline{\nabla}_{\beta}^* u_1^{\alpha} &= \partial_{\sigma} (\nabla_{\beta}^* u_1^{\alpha}) + (\Gamma_{\sigma\nu}^{\alpha} + \Phi_{\sigma\nu}^{\alpha}(\eta)) \nabla_{\beta}^* u_1^{\nu} - (\Gamma_{\beta\sigma}^{\nu} + \Phi_{\beta\sigma}^{\nu}(\eta)) \nabla_{\nu}^* u_1^{\alpha} \\ &= \nabla_{\sigma}^* (\nabla_{\beta}^* u_1^{\alpha}) + (\Phi_{\sigma\nu}^{\alpha}(\eta) \nabla_{\beta}^* u_1^{\nu} - \Phi_{\sigma\beta}^{\nu}(\eta) \nabla_{\nu}^* u_1^{\alpha}) \\ &= \nabla_{\sigma}^* \nabla_{\beta}^* u_1^{\alpha}(\eta) + \nabla_{\sigma}^* (\Phi_{\beta\nu}^{\alpha}(\eta) u_1^{\nu}) + \Phi_{\sigma\nu}^{\alpha}(\eta) (\nabla_{\beta}^* u_1^{\nu} + \Phi_{\beta\mu}^{\nu}(\eta) u_1^{\mu}) \\ &\quad - \Phi_{\sigma\beta}^{\nu}(\eta) (\nabla_{\nu}^* u_1^{\alpha} + \Phi_{\nu\mu}^{\alpha}(\eta) u_1^{\mu}). \end{aligned}$$

经过整理, 去掉高阶项, 就可得到 (5.8) 第一式. 证毕. $\square$

对于非线性算子有如下公式:

$$\begin{aligned} \underline{b}(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \mathbf{v}_1) &= \int_D \left[a_{\alpha\beta}(\eta) \underline{B}^{\alpha}(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) v^{\beta} \frac{\delta^4}{4} + \underline{B}^3(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) v^3 \frac{\delta^4}{4} \right] \left(1 + \frac{1}{2} \gamma_0(\eta) \right) \sqrt{a} dx, \\ \underline{B}^{\alpha}(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) &= [\nabla_{\lambda}^* u_1^{\alpha}(\eta) u_1^{\lambda}(\eta) - u_1^{\alpha}(\eta) \underline{\text{div}} \mathbf{u}_1(\eta)] \frac{\delta^4}{4} = B^{\alpha}(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) + \hat{B}^{\alpha}(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta), \\ \underline{B}^3(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) &= b_{\alpha\beta}(\eta) u_1^{\alpha}(\eta) u_1^{\beta}(\eta) \frac{\delta^4}{4} = B^3(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) + \hat{B}^3(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta), \end{aligned} \quad (5.9)$$

$$\begin{aligned} B^{\alpha}(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) &= u_1^{\lambda}(\eta) (\nabla_{\lambda}^* u_1^{\alpha}(\eta) - \delta_{\lambda}^{\alpha} \text{div} \mathbf{u}_1(\eta)) \frac{\delta^4}{4}, \\ \hat{B}^{\alpha}(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta) &= [\Phi_{\lambda\sigma}^{\alpha}(\eta) u_1^{\sigma}(\eta) u_1^{\lambda}(\eta) - u_1^{\alpha}(\eta) \Phi_{\lambda\sigma}^{\lambda}(\eta) u_1^{\sigma}(\eta)] \frac{\delta^4}{4}, \\ B^3(\mathbf{u}_1, \mathbf{u}_1) &= b_{\alpha\beta} u_1^{\alpha}(\eta) u_1^{\beta}(\eta), \quad \hat{B}^3(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta) = \rho_{\alpha\beta}(\eta) u_1^{\alpha}(\eta) u_1^{\beta}(\eta), \\ F_{\beta}^{(1)}(\eta) &= h_{\beta}^1(\eta) + \frac{1}{2} \nabla_{\beta}^* h_3^2(\eta), \quad F_3^{(1)}(\eta) = h_3^1(\eta) - \left(\frac{3}{4} \delta^{-1} - 2H \right) h_3^2(\eta). \end{aligned} \quad (5.10)$$

为了简单, 令曲面 $\mathfrak{S}$ 上的边界层的微分算子

$$\begin{cases} \mathcal{A}^\alpha(\mathbf{u}_1, p_0) = -\frac{\delta^3}{3}\mu\left(\overset{*}{\Delta}u_1^\alpha + \frac{7}{2}a^{\alpha\lambda}\overset{*}{\nabla}_\lambda\overset{*}{\operatorname{div}}\mathbf{u}_1 + Ku_1^\alpha\right) + q_\beta^\alpha u_1^\beta - d^\alpha(p_0) - a^{\alpha\beta}F_\beta^{(1)}, \\ \mathcal{L}(p_0, \mathbf{u}_1) = -\frac{\delta^3}{3}\overset{*}{\Delta}p_0 + \frac{\delta^3}{3}M_1(\mathbf{u}_1) - f_b, \\ B^\alpha(\mathbf{u}_1, \mathbf{u}_1) = [\overset{*}{\nabla}_\lambda u_1^\alpha u_1^\lambda - u_1^\alpha \overset{*}{\operatorname{div}}\mathbf{u}_1]\frac{\delta^4}{4}, \end{cases} \quad (5.11)$$

那么, 边界层方程 (4.39) 可以表示为

$$\begin{cases} \mathcal{A}^\alpha(\mathbf{u}_1, p_0) + B^\alpha(\mathbf{u}_1, \mathbf{u}_1) = 0, \\ \mathcal{L}(p_0, \mathbf{u}_1) = 0. \end{cases} \quad (5.12)$$

同样, 对应于 $\mathfrak{S}(\eta)$ 上的边界层算子

$$\begin{cases} \underline{\mathcal{A}}(\mathbf{u}_1(\eta), p_0(\eta)) = -\frac{\delta^3}{3}\mu(\overset{*}{\Delta}u_1^\alpha(\eta) + \frac{7}{2}a^{\alpha\lambda}(\eta)\overset{*}{\nabla}_\lambda\overset{*}{\operatorname{div}}\mathbf{u}_1(\eta) + K(\eta)u_1^\alpha(\eta)) \\ \quad + q_\beta^\alpha(\eta)u_1^\beta(\eta) - \underline{d}^\alpha(p_0(\eta)) - a^{\alpha\beta}(\eta)F_\beta^{(1)}(\eta), \\ \underline{\mathcal{L}}(p_0(\eta), \mathbf{u}_1(\eta)) = -\frac{\delta^3}{3}\overset{*}{\Delta}p_0(\eta) + \frac{\delta^3}{3}\underline{M}_1(\mathbf{u}_1(\eta)) - f_b(\eta). \end{cases} \quad (5.13)$$

将 (5.5), (5.7) 和 (5.8) 代入 (5.12) 得到, 在 $\mathfrak{S}$ -半测地坐标系下对应于 $\mathfrak{S}(\eta)$ 的边界层算子

$$\begin{cases} \underline{\mathcal{A}}^\alpha(\mathbf{u}_1(\eta), p_0(\eta)) = \mathcal{A}^\alpha(\mathbf{u}_1(\eta), p_0(\eta)) + \widehat{\mathcal{A}}^\alpha(u_1(\eta), p_0(\eta), \eta), \\ \widehat{\mathcal{A}}^\alpha(u_1(\eta), p_0(\eta), \eta) := -\frac{\delta^3}{3}\mu(A^\alpha(\mathbf{u}_1(\eta), \eta) + \frac{7}{2}a^{\alpha\lambda}[\gamma_0(\eta)\overset{*}{\nabla}_\lambda\overset{*}{\operatorname{div}}\mathbf{u}_1(\eta) \\ \quad + \overset{*}{\nabla}_\lambda(\Phi_{\beta\sigma}^\beta(\eta)u_1^\sigma(\eta))] + \widehat{K}(\eta)u_1^\alpha(\eta) + \widehat{q}_\beta^\alpha(\eta)u_1^\beta(\eta) - \widehat{d}^\alpha(p_0(\eta)), \\ \underline{\mathcal{L}}(p_0(\eta), \mathbf{u}_1(\eta)) = \mathcal{L}(p_0(\eta), \mathbf{u}_1(\eta)) + \frac{\delta^3}{3}(-\phi(p_0(\eta), \eta) + \widehat{M}_1(\mathbf{u}_1(\eta), \eta)). \end{cases} \quad (5.14)$$

边界层方程 (5.2) 可以表示为

$$\begin{cases} \mathcal{A}^\alpha(\mathbf{u}_1(\eta), p_0(\eta)) + B^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta)) + \widehat{\mathcal{A}}^\alpha(u_1(\eta), p_0(\eta), \eta) + \widehat{B}^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta) \\ \quad = F^{(1)\alpha}(\eta), \\ \mathcal{L}(p_0(\eta), \mathbf{u}_1(\eta)) + \frac{\delta^3}{3}(-\phi(p_0(\eta), \eta) + \widehat{M}_1(\mathbf{u}_1(\eta), \eta)) = f_p(\eta), \end{cases} \quad (5.15)$$

对应的变分形式是

$$\begin{cases} \text{求 } U(\eta) := (\mathbf{u}_1(\eta), p_0(\eta)) \in \mathbf{Y}(D) = \mathbf{V}(D) \times L_p^2(D), \text{ 使得} \\ A_0(U(\eta), V) + b(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \mathbf{v}) + \frac{\delta^3}{3}(M_1(\mathbf{u}_1(\eta)), q) + (a_{\alpha\beta}d^\alpha(p_0)(\eta), v^\beta) \\ \quad + (a_{\alpha\beta}\widehat{\mathcal{A}}^\alpha(u_1(\eta), p_0(\eta), \eta), v^\beta) + (\widehat{B}^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta), v^\beta) \\ \quad + \frac{\delta^3}{3}(-\phi(p_0(\eta), \eta) + \widehat{M}_1(\mathbf{u}_1(\eta), \eta), q) = (\mathbf{F}^{(1)}(\eta), \mathbf{v}) + (f_p(\eta), q), \quad \forall V = (\mathbf{v}, q) \in \mathbf{Y}(D), \end{cases}$$

其中

$$A_0(U, V) = 2\mu\frac{\delta^3}{3}(a^{\alpha\beta}a^{\lambda\sigma}\overset{*}{e}_{\alpha\lambda}(\mathbf{u}_1), \overset{*}{e}_{\beta\sigma}(\mathbf{v})) + \frac{\delta^3}{3}(\overset{*}{\nabla}p_0, \overset{*}{\nabla}q) + \frac{5}{2}\frac{\delta^3}{3}(\overset{*}{\operatorname{div}}\mathbf{u}_1, \overset{*}{\operatorname{div}}\mathbf{v}) + (q_{\alpha\beta}u_1^\alpha, v^\beta),$$

$$\begin{aligned} (a_{\alpha\beta}d^\alpha(p_0), v^\beta) &= \frac{\delta^3}{3}(-H \overset{*}{\nabla}_\beta p_0 + 3 \overset{*}{\nabla}_\beta H p_0, v^\beta), \\ \frac{\delta^3}{3}(M_1(\mathbf{u}_1), q) &= -\frac{\mu}{16}\delta^2(\operatorname{div} \mathbf{u}_1, q) + \frac{\delta^3}{32}\mu(H \operatorname{div} \mathbf{u}_1 + u_1^\alpha \overset{*}{\nabla}_\alpha H + b_\beta^\alpha \overset{*}{\nabla}_\alpha u_1^\beta, q). \end{aligned}$$

与此类似, 可以得到流形 $\mathfrak{S}(t\eta)$ 上的边界层方程

$$\begin{cases} \mathcal{A}^\alpha(\mathbf{u}_1(t), p_0(t)) + B^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t)) + \widehat{\mathcal{A}}^\alpha(u_1(t), p_0(t), t\eta) + \widehat{B}^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t), t\eta) = F^{(1)\alpha}(t), \\ \mathcal{L}(p_0(t), \mathbf{u}_1(t)) + \frac{\delta^3}{3}(-\phi(p_0(t), t\eta) + \widehat{M}_1(\mathbf{u}_1(t), t\eta)) = f_p(t), \end{cases} \quad (5.16)$$

在这里我们简单记 $\mathbf{u}(t) = \mathbf{u}(t\eta)$ 等. 它的变分形式为

$$\begin{cases} \text{求 } U(t) := (\mathbf{u}_1(t), p_0(t)) \in \mathbf{Y}(D) = \mathbf{V}(D) \times L_p^2(D), \text{ 使得} \\ A_0(U(t), V) + b(\mathbf{u}_1(t), \mathbf{u}_1(t), \mathbf{v}) + \frac{\delta^3}{3}(M_1(\mathbf{u}_1(t)), q) + (a_{\alpha\beta}d^\alpha(p_0(t)), v^\beta) \\ + (a_{\alpha\beta}\widehat{\mathcal{A}}^\alpha(u_1(t), p_0(t), t\eta), v^\beta) + (\widehat{B}^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t), t\eta), v^\beta) \\ + \frac{\delta^3}{3}(-\phi(p_0(t), t\eta) + \widehat{M}_1(\mathbf{u}_1(t), t\eta), q) = (\mathbf{F}^{(1)}(t\eta), \mathbf{v}) + (f_p(t\eta), q), \quad \forall V = (\mathbf{v}, q) \in \mathbf{Y}(D). \end{cases} \quad (5.17)$$

与证明定理 13 一样, 可以证明 (5.17) 同样存在唯一解. 实际上, 与 (4.43) 类似, 可得

$$\begin{aligned} A_0(U(t), U(t)) + (a_{\alpha\beta}d^\alpha(p_0(t)), u_1^\beta(t)) + \frac{\delta^3}{3}(M_1(u_1(t)), p_0(t)) \\ \geq C_0\delta^3|\mathbf{u}_1(t)|_{1,D}^2 + \frac{7}{16}\mu\lambda\delta\|\mathbf{u}_1(t)\|_{0,D}^2 + \frac{1}{2}C_1\|p_0(t)\|_{1,D}^2. \end{aligned} \quad (5.18)$$

由于

$$\begin{aligned} (a_{\alpha\beta}\widehat{\mathcal{A}}^\alpha(\mathbf{u}_1(t), p_0(t), t\eta), u_1^\beta(t)) \\ := -\frac{\delta^3}{3}\mu\left[(a_{\alpha\beta}A^\alpha(\mathbf{u}_1(t), t\eta), u_1^\beta(t)) - \frac{7}{2}(\operatorname{div} \mathbf{u}_1(t), \operatorname{div}(\gamma_0(t\eta)\mathbf{u}_1(t))) \right. \\ \left. - (\Phi_{\beta\sigma}^\beta(t\eta)u_1^\sigma(t), \operatorname{div} \mathbf{u}_1(t)) + (a_{\alpha\beta}\widehat{K}(t\eta)u_1^\alpha(t), u_1^\beta(t)) \right. \\ \left. + (\widehat{q}_{\alpha\beta}(t\eta)u_1^\beta(t), u_1^\beta(t)) - (a_{\alpha\beta}\widehat{d}^\alpha(p_0(t), t\eta), u_1^\beta(t))\right], \end{aligned} \quad (5.19)$$

其中函数 $\gamma_0(t\eta)$, $\Phi(t\eta)$, $\widehat{K}(t\eta)$, $\widehat{q}(t\eta)$ 和 $\widehat{d}^\alpha(p_0(t), t\eta)$ 均含有 t 的线性项, 以及

$$\begin{aligned} (A^\alpha(\mathbf{u}_1(t), t\eta), a_{\alpha\beta}u_1^\beta(t)) &= -2(\overset{*}{\nabla} \mathbf{u}_1(t), \overset{*}{\nabla}(\gamma_0(t\eta)\mathbf{u}_1(t))) + (\Phi_{\beta\nu}^\alpha(t\eta) \overset{*}{\nabla}_\sigma u_1^\nu(t) \\ &\quad + \Phi_{\sigma\nu}^\alpha(t\eta) \overset{*}{\nabla}_\beta u_1^\nu(t) - \Phi_{\sigma\beta}^\nu(t\eta) \overset{*}{\nabla}_\nu u_1^\alpha(t), a^{\beta\sigma}a_{\alpha\lambda}u_1^\lambda(t)) \\ &\quad + (a^{\beta\sigma} \overset{*}{\nabla}_\sigma \Phi_{\beta\nu}^\alpha(t\eta)u_1^\nu, a_{\alpha\lambda}u_1^\lambda(t)), \end{aligned} \quad (5.20)$$

$$|(A^\alpha(\mathbf{u}_1(t), t\eta), a_{\alpha\beta}u_1^\beta(t))| \leq |t|C|\mathbf{u}_1(t)|_{1,D}^2,$$

所以, 有

$$|(a_{\alpha\beta}\widehat{\mathcal{A}}^\alpha(\mathbf{u}_1(t), p_0(t), t\eta), u_1^\beta(t))| \leq \frac{|t|\delta^3}{3}\mu C|\mathbf{u}_1(t)|_{1,D}^2. \quad (5.21)$$

由 (5.9), 注意 $\Phi(t\eta)$ 是线性的, 那么, 存在一个依赖于 η 的常数, 使得

$$\begin{aligned} (\widehat{B}^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t), t\eta), a_{\alpha\beta}u_1^\beta) &= \frac{t\delta^4}{4}([\Phi_{\lambda\sigma}^\alpha(t\eta)u_1^\sigma(t\eta)u_1^\lambda(t\eta) - u_1^\alpha(t\eta)\Phi_{\lambda\sigma}^\lambda(t\eta)u_1^\sigma(t\eta)], a_{\alpha\beta}u_1^\beta), \\ |(\widehat{B}^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t), t\eta), a_{\alpha\beta}u_1^\beta)| &\leq \frac{t\delta^4}{4}C|\mathbf{u}_1(t)|_{1,D}^3. \end{aligned} \quad (5.22)$$

下面考虑 $(-\phi(p_0(t), t\eta) + \widehat{M}_1(\mathbf{u}_1(t), t\eta), p_0(t))$, 应用 (5.4) 和 (5.8), 我们可以断定

$$\frac{\delta^3}{3}|(-\phi(p_0(t), t\eta) + \widehat{M}_1(\mathbf{u}_1(t), t\eta), p_0(t))| \leq \frac{t\delta^3}{3}C\|p_0(t)\|_{1,D}^2 + \frac{t\delta^3}{3}C|u_1(t)|_{1,D}\|p_0\|_{1,D}.$$

由以上讨论可知, 可以成立类似 (5.18) 的不等式

$$\begin{aligned} A_0(U(t), U(t)) + (a_{\alpha\beta}d^\alpha(p_0(t)), u_1^\beta(t)) + \frac{\delta^3}{3}(M_1(u_1(t)), p_0(t)) \\ + (a_{\alpha\beta}\widehat{A}^\alpha(u_1(t), p_0(t), t\eta), u_1^\beta) + \frac{\delta^3}{3}(-\phi(p_0(t), t\eta) + \widehat{M}_1(\mathbf{u}_1(t), t\eta), p_0(t)) \\ \geq (C_0 - tC)\delta^3|\mathbf{u}_1(t)|_{1,D}^2 + \frac{7}{16}\mu\lambda\delta\|\mathbf{u}_1(t)\|_{0,D}^2 + \frac{1}{2}(C_1 - tC)\|p_0(t)\|_{1,D}^2. \end{aligned} \quad (5.23)$$

只要 t 充分小, $C_0 - tC = \widehat{C}_0 > 0$, $C_1 - tC = \widehat{C}_1 > 0$. 为简单起见, 常数 $\widehat{C}$, $\widehat{C}_1$ 和 $\widehat{M}$ 仍然记 C , C_1 和 M . 从而, (5.23) 与 (5.18) 相类似. 另外, (5.17) 中增加的双线性项成立 (5.22), 所以, 不改变类似与 (4.43) 解的估计. 因此, 应用相同的方法可以证明 (5.17) 解的存在. 于是得到下列定理.

定理 16 假设流形 $\mathfrak{S}(t\eta)$ 充分光滑, 使得 $a_{\alpha\beta}(t\eta), a^{\alpha\beta}(t\eta), H(t\eta), K(t\eta) \in C^1(\overline{D})$,

$$\sum_{\alpha,\beta} \{\|a_{\alpha\beta}(t\eta)\|_{1,D} + \|a^{\alpha\beta}(t\eta)\|_{1,D}\} \leq C_\delta, \quad \{\|H(t\eta)\|_{1,D}, \|K(t\eta)\|_{1,D}, \|a(t\eta)\|_{1,D}\} \leq C_\delta,$$

这里 C_δ 是一个绝对常数, 在不同的地方, 可能表示不同的意义. 那么, 只要 δ 充分小, 那么下面 (5.24) 第一式的强制性不等式成立, 并且如果方程右端满足

$$\|F^{(1)}\|_{-1,D} \leq \frac{C_0^2\delta^6}{4M},$$

那么, (5.17) 存在唯一解 $U = (\mathbf{u}_1(t), p_0(t))$, 它满足 (5.24) 的第二式:

$$\begin{cases} A_0(U(t), U(t)) + (a_{\alpha\beta}d^\alpha(p_0(t)), u_1^\beta(t)) + \frac{\delta^3}{3}(M_1(u_1(t)), p_0(t)) \\ + (a_{\alpha\beta}\widehat{A}^\alpha(u_1(t), p_0(t), t\eta), u_1^\beta) + \frac{\delta^3}{3}(-\phi(p_0(t), t\eta) + \widehat{M}_1(\mathbf{u}_1(t), t\eta), p_0(t)) \\ \geq C_0\delta^3|\mathbf{u}_1(t)|_{1,D}^2 + \frac{7}{16}\mu\lambda\delta\|\mathbf{u}_1(t)\|_{0,D}^2 + \frac{1}{2}C_1\|p_0(t)\|_{1,D}^2, \\ |\mathbf{u}_1(t)|_{1,D} \leq \frac{C_0\delta^3}{2M} - \sqrt{\left(\frac{C_0\delta^3}{2M}\right)^2 - \frac{\|F^{(1)}(t\eta)\|_{-1,D}}{M}}, \\ \|p_1\|_{1,D} \leq \frac{2}{C_1\delta^3}\|f_p(t\eta)\|_{-1,D}, \end{cases} \quad (5.24)$$

其中常数 C 只依赖于 F 和 f_p 的常数. □

5.2 不同坐标系下的速度梯度分量

下面考查有了 (5.17) 的解 $(\mathbf{u}_1(t), p_0(t))$ (注意它是在 $\mathfrak{S}$ -半测地坐标系下) 如何得到 (5.1) 的解 $(\mathbf{u}_1(t\eta), p_0(t\eta))$ (它是在 $\mathfrak{S}(t\eta)$ -半测地坐标系下).

因为边界层方程 (5.1), (5.2) 和 (5.12) 都是不变形式. (5.12) 和 (5.16) 都是在 $\mathfrak{S}$ -半测地坐标系下的形式, 所以, $\mathbf{u}_1$, $\mathbf{u}_1(\eta)$ 和 $\mathbf{u}_1(t)$ 都是在这个坐标系下的速度分量, 分别是在 $\mathfrak{S}$, $\mathfrak{S}(\eta)$ 和 $\mathfrak{S}(t\eta)$ 对应边界层内的速度梯度场.

要特别指出, 方程 (5.12) 和 (5.15) 是在不同的坐标系下. 虽然微分算子和解的分量形式都不一样. 如果令 $\mathfrak{S}$, $\mathfrak{S}(\eta)$ 和 $\mathfrak{S}(t\eta)$ 的半测地坐标系的基向量分别为 $(\mathbf{e}_\alpha, \mathbf{n})$, $(\mathbf{e}_\alpha(\eta), \mathbf{n})$ 和 $(\mathbf{e}_\alpha(t\eta), \mathbf{n})$. 向量是不变量, 但它的分量与坐标系有关,

$$\mathbf{u} = u^\alpha \mathbf{e}_\alpha + u^3 \mathbf{n} = u^\alpha(\eta) \mathbf{e}_\alpha(\eta) + u^3(\eta) \mathbf{n} = u^\alpha(t\eta) \mathbf{e}_\alpha(t\eta) + u^3 \mathbf{n}. \quad (5.25)$$

由 (5.1) 和 (5.2) 得到的解 $u_1^\beta(\eta)$, $u^3(\eta)$ 和 $p_0(\eta)$ 是在 $\mathfrak{S}(\eta)$ -半测地坐标系下的逆变分量和数量. 我们最后要得到的, 应该是在 $\mathfrak{S}$ -半测地坐标系下的逆变分量和数量. 为了完成这个变换, 令 $\mathfrak{S}$ 和 $\mathfrak{S}(\eta)$ 上的基向量记为 $(\mathbf{e}_\alpha, \mathbf{n})$ 和 $(\mathbf{e}_\alpha(\eta), \mathbf{n}(\eta))$. 根据张量分析, u_1^k 和 $u_1^k(\eta)$ 的变换规律是由基向量 $(\mathbf{e}_\alpha, \mathbf{n})$ 和 $(\mathbf{e}_\alpha(\eta), \mathbf{n}(\eta))$ 变换规律决定的. 设

$$\begin{cases} \boldsymbol{\eta} = \eta^\lambda \mathbf{e}_\lambda + \eta^3 \mathbf{n}, \\ \boldsymbol{\eta}_\alpha := \partial_\alpha \boldsymbol{\eta} = \eta_\alpha^\lambda \mathbf{e}_\lambda + \eta^\lambda \mathbf{e}_{\lambda\alpha} + \eta_\alpha^3 \mathbf{n} + \eta^3 \mathbf{n}_\alpha, \end{cases} \quad (5.26)$$

应用 Gauss 和 Weingarten 公式,

$$\mathbf{e}_{\alpha\lambda} = \Gamma_{\alpha\lambda}^{\sigma*} \mathbf{e}_\sigma + b_{\alpha\lambda} \mathbf{n}, \quad \mathbf{n}_\alpha = -b_\alpha^\beta \mathbf{e}_\beta.$$

于是,

$$\begin{cases} \boldsymbol{\eta}_\alpha = (\eta_\alpha^\lambda - \eta^3 b_\alpha^\lambda + \eta^\sigma \Gamma_{\alpha\lambda}^{\sigma*}) \mathbf{e}_\lambda + (\eta^\lambda b_{\alpha\lambda} + \eta_\alpha^3) \mathbf{n}, \\ \mathbf{e}_\alpha(\eta) = \mathbf{e}_\alpha + \boldsymbol{\eta}_\alpha = A_\alpha^\lambda(\eta) \mathbf{e}_\lambda + A_\alpha^3 \mathbf{n}, \\ \mathbf{n}(\eta) = \mathbf{n}, \end{cases} \quad (5.27)$$

其中基向量的变换矩阵是

$$\begin{cases} A_\alpha^\lambda(\eta) = \delta_\alpha^\lambda + \eta_\alpha^\lambda - \eta^3 b_\alpha^\lambda + \eta^\sigma \Gamma_{\alpha\sigma}^{\lambda*}, \\ A_\alpha^3(\eta) = \eta^\lambda b_{\alpha\lambda} + \eta_\alpha^3, \\ A_3^\alpha = 0, \\ A_3^3 = 1. \end{cases} \quad (5.28)$$

最后得到速度梯度在坐标变换下的变换规律

$$\begin{cases} \mathbf{u}_1(\eta) = u_1^\alpha \mathbf{e}_\alpha + u_1^3 \mathbf{n} = u_1^\alpha(\eta) \mathbf{e}_\alpha(\eta) + u_1^3(\eta) \mathbf{n} \\ \quad = A_\alpha^\lambda(\eta) u_1^\alpha(\eta) \mathbf{e}_\lambda + A_\alpha^3(\eta) u_1^\alpha(\eta) \mathbf{n} + u_1^3(\eta) \mathbf{n}, \\ u_1^\alpha = A_\beta^\alpha(\eta) u_1^\beta(\eta), \\ u_1^3 = A_\beta^3(\eta) u_1^\beta(\eta) + u_1^3(\eta). \end{cases} \quad (5.29)$$

从而, 如果我们从 (5.2) 求出 $(\mathbf{u}_1(\eta), p_0(\eta))$, 那么, (5.14) 给出了变形后边界层方程在 $\mathfrak{S}$ -半测地坐标系下的解 $(\mathbf{u}_1, p_0)$. 为了不与变形前的边界层方程的解相混, 变形后边界层方程在 $\mathfrak{S}$ -半测地坐标系下的解记为 $(\underline{\mathbf{u}}_1(\eta), \underline{p}_0(\eta))$, 即

$$\mathbf{u}_1(\eta) = \underline{u}_1^\alpha(\eta) \mathbf{e}_\alpha + \underline{u}_1^3 \mathbf{n} = u_1^\alpha(\eta) \mathbf{e}_\alpha(\eta) + u_1^3 \mathbf{n}(\eta).$$

从而,

$$\begin{cases} \underline{u}_1^\alpha(\eta) = A_\beta^\alpha(\eta)u_1^\beta(\eta), \\ \underline{u}_1^3(\eta) = A_\beta^3(\eta)u_1^\beta(\eta) + u_1^3(\eta), \\ \begin{pmatrix} \underline{u}_1^\alpha(\eta) \\ \underline{u}_1^3(\eta) \end{pmatrix} = [\mathcal{A}(\eta)] \begin{pmatrix} u_1^\beta(\eta) \\ u_1^3(\eta) \end{pmatrix}, \end{cases} \quad (5.30)$$

这里变换矩阵

$$[\mathcal{A}(\eta)] = \begin{pmatrix} A_\beta^\alpha & 0 \\ A_\beta^3 & 1 \end{pmatrix}. \quad (5.31)$$

引理 17 假设变形向量 η 光滑和充分小, 曲面 $\mathfrak{S}$ 的度量张量和曲率张量满足

$$a_{\alpha\beta} \in C^1(D), \quad b_{\alpha\beta} \in C^1(D).$$

那么, (5.30) 中的坐标变换矩阵 $[\mathcal{A}(\eta)]$ 是可逆的, 它的行列式

$$A = \det([\mathcal{A}(\eta)]) = 1 + \gamma_0(\eta), \quad \det([\mathcal{A}(\eta)]^{-1}) = A^{-1} = 1 - \gamma_0(\eta) + o(|\eta|^2), \quad (5.32)$$

在 $\mathfrak{S}(\eta)$ 上流体速度梯度向量 $\mathbf{u}_1(\eta)$ 在 $\mathfrak{S}$ -半测地坐标系和 $\mathfrak{S}(\eta)$ -半测地坐标系下的分量服从变换规律 (去掉 η 的高阶小量),

$$\begin{cases} \begin{pmatrix} \underline{u}_1^\alpha(\eta) \\ \underline{u}_1^3(\eta) \end{pmatrix} = [\mathcal{A}(\eta)] \begin{pmatrix} u_1^\beta(\eta) \\ u_1^3(\eta) \end{pmatrix}, \\ \begin{pmatrix} u_1^\alpha(\eta) \\ u_1^3(\eta) \end{pmatrix} = [\mathcal{A}(\eta)]^{-1} \begin{pmatrix} \underline{u}_1^\beta(\eta) \\ \underline{u}_1^3(\eta) \end{pmatrix}, \end{cases} \quad (5.33)$$

其中逆矩阵

$$[\mathcal{A}(\eta)]^{-1} := \{\bar{A}_j^i(\eta)\}, \quad (5.34)$$

$$\begin{cases} \bar{A}_1^1 = \frac{1}{A}(1 + \eta_2^2 - \eta^3 b_2^2 + \eta^\sigma \Gamma_{2\sigma}^*), & \bar{A}_2^1 = -\frac{1}{A}(\eta_1^2 - \eta^3 b_1^2 + \eta^\sigma \Gamma_{1\sigma}^*), \\ \bar{A}_3^1 = \frac{1}{A}[(\eta_1^2 - \eta^3 b_1^2 + \eta^\sigma \Gamma_{1\sigma}^*)(\eta^\lambda b_{2\lambda} + \eta_2^3)] - (1 + \eta_2^2 - \eta^3 b_2^2 + \eta^\sigma \Gamma_{2\sigma}^*)(\eta^\lambda b_{1\lambda} + \eta_1^3) \\ \quad \approx -\frac{1}{A}(\eta^\lambda b_{1\lambda} + \eta_1^3), \\ \bar{A}_1^2 = -\frac{1}{A}(\eta_2^1 - \eta^3 b_2^1 + \eta^\sigma \Gamma_{2\sigma}^*), & \bar{A}_2^2 = \frac{1}{A}(1 + \eta_1^1 - \eta^3 b_1^1 + \eta^\sigma \Gamma_{1\sigma}^*), \\ \bar{A}_3^2 = -\frac{1}{A}[(1 + \eta_1^1 - \eta^3 b_1^1 + \eta^\sigma \Gamma_{1\sigma}^*)(\eta^\lambda b_{2\lambda} + \eta_2^3) - (\eta_2^1 - \eta^3 b_2^1 + \eta^\sigma \Gamma_{2\sigma}^*)(\eta^\lambda b_{1\lambda} + \eta_1^3)] \\ \quad \approx -\frac{1}{A}(\eta^\lambda b_{2\lambda} + \eta_2^3), \\ \bar{A}_1^3 = \bar{A}_2^3 = 0, & \bar{A}_3^3 = 1. \end{cases} \quad (5.35)$$

证明 首先证明 (5.32). 从 (5.31) 推出,

$$\det([\mathcal{A}(\eta)]) = \det(\{A_\beta^\alpha(\eta)\}). \quad (5.36)$$

利用循环张量

$$\begin{aligned} \det(A_\beta^\alpha(\eta)) &= \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{\lambda\sigma} A_\alpha^\lambda A_\beta^\sigma \\ &= \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{\lambda\sigma} [\delta_\alpha^\lambda \delta_\beta^\sigma + \delta_\alpha^\lambda \eta_\beta^\sigma + \delta_\beta^\sigma \eta_\alpha^\lambda - \eta^3 (\delta_\alpha^\lambda b_\beta^\sigma + \delta_\beta^\sigma b_\alpha^\lambda) + \eta^\nu (\delta_\alpha^\lambda \Gamma_{\beta\nu}^* + \delta_\beta^\sigma \Gamma_{\alpha\nu}^*)] \end{aligned}$$

和

$$\begin{cases} \varepsilon^{\alpha\beta} \varepsilon_{\alpha\lambda} = \delta_\lambda^\beta, \\ b_1^1 + b_2^2 = a^{\alpha\beta} b_{\alpha\beta} = 2H, \\ \eta_1^1 + \eta_2^2 = \partial_\sigma \eta^\sigma, \\ (\Gamma_{1\nu}^1 + \Gamma_{2\nu}^2) \eta^\nu = \Gamma_{\lambda\nu}^* = \partial_\nu \ln \sqrt{a}, \\ \eta_1^1 + \eta_2^2 + (\Gamma_{1\nu}^1 + \Gamma_{2\nu}^2) \eta^\nu = \partial_\sigma \eta^\sigma + \Gamma_{\lambda\nu}^* = \nabla_\sigma^* \eta^\sigma = \operatorname{div}^* \boldsymbol{\eta}, \end{cases}$$

容易推出,

$$\begin{aligned} \det(A_\beta^\alpha(\eta)) &= \frac{1}{2} [2 + \eta_\sigma^\sigma + \eta_\lambda^\lambda - \eta^3 (b_\alpha^\alpha + b_\beta^\beta) + \eta^\nu (\Gamma_{\beta\nu}^* + \Gamma_{\alpha\nu}^*)] \\ &= 1 + \operatorname{div}^* \boldsymbol{\eta} - 2H\eta^3 \\ &= 1 + \gamma_0(\eta) \\ &= \det([\mathcal{A}(\eta)]), \end{aligned}$$

这就得到 (5.31). 另外,

$$\det([\mathcal{A}(\eta)]^{-1}) = (1 + \gamma_0(\eta))^{-1} = 1 - \gamma_0(\eta) + o(|\eta|^2), \quad [\mathcal{A}(\eta)]^{-1} = \{\bar{A}_j^i\}.$$

利用逆矩阵定义, 简单计算得到

$$\begin{cases} \bar{A}_1^1 = (-1)^{1+1} \frac{1}{A} (A_2^2 \cdot 1 - A_2^3 \cdot 0) = \frac{1}{A} A_2^2, \\ \bar{A}_2^1 = -\frac{1}{A} A_1^2, \\ \bar{A}_1^2 = -\frac{1}{A} A_2^1, \\ \bar{A}_2^2 = \frac{1}{A} A_1^1, \\ \bar{A}_3^1 = \frac{1}{A} (A_1^2 A_2^3 - A_2^2 A_1^3), \\ \bar{A}_3^2 = -\frac{1}{A} (A_1^1 A_2^3 - A_2^1 A_1^3), \end{cases} \quad (5.37)$$

将 (5.28) 代入 (5.36), 即可推出 (5.35) 和 (5.34). 证毕. $\square$

引理 17 告诉我们, 如果得到在 $\Im$ 半测地坐标系下的 $\Im(t\eta)$ 边界层内的速度梯度和压力 $(\underline{u}_1(t), \underline{p}(t))$, 那么, 公式 (5.33) 给出了计算 $\Im(t\eta)$ 半测地坐标系下 $\Im(t\eta)$ 边界层内的速度梯度和压力.

5.3 边界层方程的解关于边界形状的 Gâteaux 导数

下面考查边界层方程的解关于边界形状的 Gâteaux 导数. 边界层方程的解关于边界形状的 Gâteaux 导数, 应在同一的 $\mathfrak{S}$ -半测地坐标系下的定义,

$$\frac{\mathcal{D}\mathbf{u}_1}{\mathcal{D}\eta}\eta = \lim_{t \rightarrow 0} \frac{\mathbf{u}_1(t\eta) - \mathbf{u}_1}{t} := \widehat{\mathbf{u}}_1\eta, \quad \frac{\mathcal{D}p_0}{\mathcal{D}\eta}\eta = \lim_{t \rightarrow 0} \frac{p_0(t\eta) - p_0}{t} := \widehat{p}_0\eta,$$

其中 $\mathbf{u}_1(t\zeta)$ 和 $\mathbf{u}_1$ 分别是 (5.16) 和 (5.12) 在 $\mathfrak{S}$ 半测地坐标系下. 在同一个坐标系中速度在边界变形后 η 沿任意方向 η 的 $\mathbf{G}$ -导数.

让 (5.16) 减去 (5.12), 并且为了简单起见, 仍然用 $\mathbf{u}_1(t)$ 代替 $\mathbf{u}_1(t)$, 于是, 有

$$\begin{cases} \mathcal{A}^\alpha(\mathbf{u}_1(t) - \mathbf{u}_1, p_0(t) - p_0) + B^\alpha(\mathbf{u}_1(t) - \mathbf{u}_1, \mathbf{u}_1(t)) + B^\alpha(\mathbf{u}_1, \mathbf{u}_1(t) - \mathbf{u}_1) \\ \quad + \widehat{\mathcal{A}}^\alpha(\mathbf{u}_1(t), p_0(t), t\eta) + \widehat{B}^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t), t\eta) = 0, \\ \mathcal{L}(p_0(t) - p_0, \mathbf{u}_1(t) - \mathbf{u}_1) + \frac{\delta^3}{3}(-\phi(p_0(t), t\eta) + \widehat{M}_1(\mathbf{u}_1(t), t\eta)) = 0. \end{cases} \quad (5.38)$$

下面估计 (5.38) 的扰动项. 由 (5.4), (5.8), (5.9) 和 (5.14) 可知,

$$\begin{cases} \widehat{\mathcal{A}}^\alpha(\mathbf{u}_1(t), p_0(t), t\eta) := -\frac{\delta^3}{3}\mu(A^\alpha(\mathbf{u}_1(t), t\eta) + \frac{7}{2}a^{\alpha\lambda}[\gamma_0(t\eta) \overset{*}{\nabla}_\lambda \text{div} \mathbf{u}_1(t) + \overset{*}{\nabla}_\lambda(\Phi_{\beta\sigma}^\beta(t\eta)u_1^\sigma(t))]) \\ \quad + \widehat{K}(t\eta)u_1^\alpha(t) + \widehat{q}_\beta^\alpha(t\eta)u_1^\beta(t) - \widehat{d}^\alpha(p_0(t), t\eta), \\ \widehat{B}^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t), t\eta) = [\Phi_{\lambda\sigma}^\alpha(t\eta)u_1^\sigma(t)u_1^\lambda(t) - u_1^\alpha(t)\Phi_{\lambda\sigma}^\lambda(t\eta)u_1^\sigma(t)]\frac{\delta^4}{4}, \\ \widehat{M}_1(\mathbf{u}_1(t), t\eta) = \frac{\mu}{2}\left[2\widehat{H}(t\eta) \overset{*}{\text{div}} \mathbf{u}_1(t) + \left(2H - \frac{3}{8\delta}\right)\Phi_{\alpha\beta}^\alpha(t\eta)u_1^\beta(t) \right. \\ \quad \left. + u_1^\alpha(t) \overset{*}{\nabla}_\alpha \widehat{H}(t\eta) + \widehat{b}_\beta^\alpha(t\eta) \overset{*}{\nabla}_\alpha u_1^\beta(t) + b_\beta^\alpha\Phi_{\alpha\lambda}^\beta(t\eta)u_1^\lambda(t)\right], \\ \phi(p_0(t), t\eta) = 2\gamma_0(t\eta) \overset{*}{\Delta} p_0(t) - a^{\alpha\beta}\Phi_{\alpha\beta}^\nu(t\eta) \overset{*}{\nabla}_\nu p_0(t). \end{cases} \quad (5.39)$$

从 (5.8),

$$\begin{aligned} A^\alpha(\mathbf{u}_1(t), t\eta) &= 2\gamma_0(t\eta) \overset{*}{\Delta} u_1^\alpha(t) + a^{\beta\sigma}[\Phi_{\beta\nu}^\alpha(t\eta) \overset{*}{\nabla}_\sigma u_1^\nu(t) + \Phi_{\sigma\nu}^\alpha(t\eta) \overset{*}{\nabla}_\beta u_1^\nu(t) - \Phi_{\sigma\beta}^\nu(t\eta) \overset{*}{\nabla}_\nu u_1^\alpha(t)] \\ &\quad + a^{\beta\sigma} \overset{*}{\nabla}_\sigma \Phi_{\beta\nu}^\alpha(t\eta)u_1^\nu(t) \\ &= t\{2\gamma_0(\eta) \overset{*}{\Delta} u_1^\alpha(t) + a^{\beta\sigma}(\Phi_{\beta\nu}^\alpha(\eta) \overset{*}{\nabla}_\sigma u_1^\nu(t) + \Phi_{\sigma\nu}^\alpha(\eta) \overset{*}{\nabla}_\beta u_1^\nu(t) - \Phi_{\sigma\beta}^\nu(\eta) \overset{*}{\nabla}_\nu u_1^\alpha(t)) \\ &\quad + a^{\beta\sigma} \overset{*}{\nabla}_\sigma \Phi_{\beta\nu}^\alpha(\eta)u_1^\nu(t)\}, \\ \widehat{d}^\alpha(p_0(t), t\eta) &= -\frac{\delta^3}{3}a^{\alpha\beta}(\widehat{H}(t\eta) + 2H\gamma_0(t\eta)) \overset{*}{\nabla}_\beta p_0(t) + \frac{3\delta^3}{3}a^{\alpha\beta}(2\gamma_0(t\eta) \overset{*}{\nabla}_\beta H + \overset{*}{\nabla}_\beta \widehat{H}(t\eta))p_0(t) \\ &= t\left\{-\frac{\delta^3}{3}a^{\alpha\beta}(\widehat{H}(\eta) + 2H\gamma_0(\eta)) \overset{*}{\nabla}_\beta p_0(t) + \frac{3\delta^3}{3}a^{\alpha\beta}(2\gamma_0(\eta) \overset{*}{\nabla}_\beta H + \overset{*}{\nabla}_\beta \widehat{H}(\eta)p_0(t))\right\}, \\ \frac{7}{2}a^{\alpha\lambda}[\gamma_0(t\eta) \overset{*}{\nabla}_\lambda \text{div} \mathbf{u}_1(t) + \overset{*}{\nabla}_\lambda(\Phi_{\beta\sigma}^\beta(t\eta)u_1^\sigma(t))] + \widehat{K}(t\eta)u_1^\alpha(t) + \widehat{q}_\beta^\alpha(t\eta)u_1^\beta(t) \\ &= t\left\{\frac{7}{2}a^{\alpha\lambda}[\gamma_0(\eta) \overset{*}{\nabla}_\lambda \text{div} \mathbf{u}_1(t) + \overset{*}{\nabla}_\lambda(\Phi_{\beta\sigma}^\beta(\eta)u_1^\sigma(t)) + \widehat{K}(\eta)u_1^\alpha(t) + \widehat{q}_\beta^\alpha(\eta)u_1^\beta(t)]\right\}, \\ \phi(p_0(t), t\eta) &= 2\gamma_0(t\eta) \overset{*}{\Delta} p_0(t) - a^{\alpha\beta}\Phi_{\alpha\beta}^\nu(t\eta) \overset{*}{\nabla}_\nu p_0(t) \end{aligned}$$

$$= t\{2\gamma_0(\eta) \overset{*}{\Delta} p_0(t) + t[-a^{\alpha\beta}\Phi_{\alpha\beta}^\nu(\eta) \overset{*}{\nabla}_\nu p_0(t)]\},$$

在 (5.30) 中其他各项因为均含 t 的线性项, 因此, 有

$$\left\{ \begin{aligned} \widehat{\mathcal{A}}^\alpha(u_1(t), p_0(t), t\eta) &:= t\left\{ -\frac{\delta^3}{3}\mu\left(A^\alpha(\mathbf{u}_1(t), \eta) + \frac{7}{2}a^{\alpha\lambda}[\gamma_0(\eta) \overset{*}{\nabla}_\lambda \operatorname{div} \mathbf{u}_1(t) \right. \right. \\ &\quad \left. \left. + \overset{*}{\nabla}_\lambda (\Phi_{\beta\sigma}^\beta(\eta) u_1^\sigma(t))\right) + \widehat{K}(\eta) u_1^\alpha(t) \right\} + \widehat{q}_\beta^\alpha(\eta) u_1^\beta(t) - \widehat{d}^\alpha(p_0(t), \eta) \Big\}, \\ \widehat{B}^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t), t\eta) &= t[\Phi_{\lambda\sigma}^\alpha(\eta) u_1^\sigma(t) u_1^\lambda(t) - u_1^\alpha(t) \Phi_{\lambda\sigma}^\lambda(\eta) u_1^\sigma(t)] \frac{\delta^4}{4}, \\ \widehat{M}_1(\mathbf{u}_1(t), t\eta) &= t\left\{ \frac{\mu}{2} \left[2\widehat{H}(\eta) \operatorname{div} \mathbf{u}_1(t) + \left(2H - \frac{3}{8\delta} \right) \Phi_{\alpha\beta}^\alpha(\eta) u_1^\beta(t) \right. \right. \\ &\quad \left. \left. + u_1^\alpha(t) \overset{*}{\nabla}_\alpha \widehat{H}(\eta) + \widehat{b}_\beta^\alpha(\eta) \overset{*}{\nabla}_\alpha u_1^\beta(t) + b_\beta^\alpha \Phi_{\alpha\lambda}^\beta(\eta) u_1^\lambda(t) \right] \right\}, \\ \phi(p_0(t), t\eta) &= t\{2\gamma_0(\eta) \overset{*}{\Delta} p_0(t) - a^{\alpha\beta}\Phi_{\alpha\beta}^\nu(\eta) \overset{*}{\nabla}_\nu p_0(t)\}, \end{aligned} \right. \quad (5.40)$$

在 (5.38) 两边同除以 t , 那么,

$$\left\{ \begin{aligned} \mathcal{A}^\alpha\left(\frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t}, \frac{p_0(t) - p_0}{t}\right) + B^\alpha\left(\frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t}, \mathbf{u}_1(t)\right) \\ + B^\alpha\left(\mathbf{u}_1, \frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t}\right) + \widehat{\mathcal{A}}^\alpha(\mathbf{u}_1(t), p_0(t), \eta) + \widehat{B}^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t), \eta) &= 0, \\ \mathcal{L}\left(\frac{p_0(t) - p_0}{t}, \frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t}\right) - \phi(p_0(t), \eta) + \widehat{M}_1(\mathbf{u}_1(t), \eta) &= 0, \end{aligned} \right. \quad (5.41)$$

其中

$$\left\{ \begin{aligned} \mathcal{A}^\alpha\left(\frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t}, \frac{p_0(t) - p_0}{t}\right) \\ = -\frac{\delta^3}{3}\mu\left(\overset{*}{\Delta} \frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t} + \frac{7}{2}a^{\alpha\lambda} \overset{*}{\nabla}_\lambda \operatorname{div} \frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t} + K \frac{u_1^\alpha(t) - u_1^\alpha}{t}\right) \\ + q_\beta^\alpha \frac{u_1^\beta(t) - u_1^\beta}{t} - d^\alpha\left(\frac{p_0(t) - p_0}{t}\right) - a^{\alpha\beta} F_\beta^{(1)}, \\ \mathcal{L}\left(\frac{p_0(t) - p_0}{t}, \frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t}\right) = -\frac{\delta^3}{3} \overset{*}{\Delta} \frac{p_0(t) - p_0}{t} + \frac{\delta^3}{3} M_1\left(\frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t}\right) - f_b, \\ B^\alpha\left(\frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t}, \mathbf{u}_1\right) + B^\alpha\left(\mathbf{u}_1, \frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t}\right) \\ = \left[\overset{*}{\nabla}_\lambda u_1^\alpha \frac{u_1^\lambda(t) - u_1^\lambda}{t} - \frac{u_1^\alpha(t) - u_1^\alpha}{t} \overset{*}{\operatorname{div}} \mathbf{u}_1 \right] \frac{\delta^4}{4} \\ + \left[u_1^\lambda \overset{*}{\nabla}_\lambda \frac{u_1^\alpha(t) - u_1^\alpha}{t} - u_1^\alpha \overset{*}{\operatorname{div}} \frac{\mathbf{u}_1(t) - \mathbf{u}_1}{t} \right] \frac{\delta^4}{4}, \end{aligned} \right. \quad (5.42)$$

对应于 (5.41) 的变分形式是

$$\widehat{F}^{(1)} = \frac{F^{(1)}(t\eta) - F^{(1)}}{t}, \quad \widetilde{f}_p = \frac{f_p(t\eta) - f_p}{t}.$$

为了研究 (5.41) 当 $t \rightarrow 0$ 时的极限过度, 我们必须讨论 (5.12) 和 (5.16) 解的估计和解关于 t 的 Lipschitz 连续性问题. 记

$$w(t) = \mathbf{u}_1(t) - \mathbf{u}_1, \quad s(t) = p_0(t) - p_0,$$

显然, (5.38) 的变分形式为

$$\begin{cases} \frac{2\mu\delta^3}{3}(a^{\alpha\beta}a^{\beta\sigma}e_{\alpha\lambda}^*(\mathbf{w}(t)), e_{\beta\sigma}^*(\mathbf{v})) + \frac{5}{2}(\operatorname{div}^*\mathbf{w}(t), \operatorname{div}^*\mathbf{v}) - (a_{\alpha\beta}d^\alpha(s(t), v^\beta) \\ + b(\mathbf{w}(t), \mathbf{u}_1(t), v) + b(\mathbf{u}_1, \mathbf{w}(t), \mathbf{v}) + (a_{\alpha\beta}\hat{A}^\alpha(\mathbf{u}_1(t), p_0(t), t\eta), v^\beta) \\ + (a_{\alpha\beta}\hat{B}^\alpha(\mathbf{u}_1(t), \mathbf{u}_1(t), t\eta)v^\beta) = ([F]_\beta^{(1)}, v^\beta), \quad \forall \mathbf{v} \in V(D), \\ \frac{\delta^3}{3}(\nabla^*s(t), \nabla^*q) + \frac{\delta^3}{3}(M_1(\mathbf{w}(t), q) - (\phi_0(p_0(t), t\eta) \\ + (\hat{M}_1(\mathbf{u}_1(t), t\eta), q) = ([f]_p, q), \quad \forall q \in L^2(D), \end{cases} \quad (5.43)$$

其中

$$[F]^{(1)} = F^{(1)}(t\eta) - F^{(1)}, \quad [f]_p = f_p(t\eta) - f_p.$$

对 (5.43) 应用定理 16, 只要 t 和 δ 充分小, 并且右端满足

$$\|[F]^{(1)}\|_{-1,D} \leq \frac{C_0^2\delta^6}{4M},$$

那么, (5.43) 存在唯一解 $(\mathbf{w}(t), s(t))$ 且有估计

$$\|\mathbf{w}(t)\|_{1,D}^2 + \|s(t)\|_{0,D}^2 \leq C(\|[F]^{(1)}\|_{1,D} + \|[f]_p\|_{0,D}) \Rightarrow 0 \quad \text{a.s. } t \Rightarrow 0. \quad (5.44)$$

将引理 17 代入 (5.28), 经过整理可直接得到如下定理.

定理 18 边界层方程的速度梯度和压力的 Gâteaux 导数满足下列边值问题:

$$\begin{cases} \frac{2\mu\delta^3}{3}(a^{\alpha\beta}a^{\beta\sigma}e_{\alpha\lambda}^*(\hat{\mathbf{u}}_1), e_{\beta\sigma}^*(\mathbf{v})) + \frac{5}{2}(\operatorname{div}^*\hat{\mathbf{u}}_1, \operatorname{div}^*\mathbf{v}) - a_{\alpha\beta}d^\alpha(\hat{p}_0, v^\beta) \\ + b(\hat{\mathbf{u}}_1, \mathbf{u}_1, v) + b(\mathbf{u}_1, \hat{\mathbf{u}}_1, \mathbf{v}) = (\mathbf{R}(\mathbf{u}_1, p_0), \mathbf{v}), \quad \forall \mathbf{v} \in V(D), \\ \frac{\delta^3}{3}(\nabla^*\hat{p}_0, \nabla^*q) + \frac{\delta^3}{3}(M_1(\hat{\mathbf{u}}_1), q) = (R_p(\mathbf{u}_1, p_0), q), \quad \forall q \in L^2(D), \\ (\hat{\mathbf{u}}_1, p_0(\eta, \zeta)) \text{ 满足周期性边界条件,} \end{cases} \quad (5.45)$$

其中

$$\begin{cases} (\mathbf{R}(\mathbf{u}_1, p_0), \mathbf{v}) := -(a_{\alpha\beta}\hat{A}^\alpha(\mathbf{u}_1, p_0, \eta), v^\beta) - (a_{\alpha\beta}\hat{B}^\alpha(\mathbf{u}_1, \mathbf{u}_1, \eta), v^\beta) \\ + ((F^{(1)})'_\beta, v^\beta), \quad \forall \mathbf{v} \in V(D), \\ (R_p(\mathbf{u}_1, p_0), q) := \frac{\delta^3}{3}[(\phi_0(p_0), \eta) - (\hat{M}_1(\mathbf{u}_1, \eta), q)] + (f'_p, q), \quad \forall q \in L^2(D). \end{cases} \quad (5.46)$$

5.4 在 $\mathfrak{S}(\eta)$ 上边界层方程的解关于形状的 Gâteaux 导数

下面最优控制问题的共轭梯度算法中, 需要考查边界层方程在 $\mathfrak{S}(\eta)$ 上的解沿着方向 ζ 的 Gâteaux 导数

$$\begin{aligned} \frac{\mathcal{D}\mathbf{u}_1(\eta)}{\mathcal{D}\eta}\zeta &= \lim_{t \rightarrow 0} \frac{\mathbf{u}_1(\eta + t\eta) - \mathbf{u}_1(\eta)}{t} := \hat{\mathbf{u}}_1(\eta)\zeta, \\ \frac{\mathcal{D}p_0(\eta)}{\mathcal{D}\eta}\zeta &= \lim_{t \rightarrow 0} \frac{p_0(\eta + t\eta) - p_0(\eta)}{t} := \hat{p}_0(\eta)\zeta, \end{aligned}$$

其中 $\underline{\mathbf{u}}_1(\eta)$ 和 $\underline{\mathbf{u}}_1(\eta + t\zeta)$ 分别是边界层方程在 $\Im(\eta)$ 和 $\Im(\eta + t\zeta)$ 上在 $\Im$ -半测地坐标系下的解. 这些方程分别是 (5.15),

$$\left\{ \begin{array}{l} \text{求 } U(\eta) := (\mathbf{u}_1(\eta), p_0(\eta)) \in \mathbf{Y}(D) = \mathbf{V}(D) \times L_p^2(D), \text{ 使得} \\ A_0(U(\eta), V) + b(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \mathbf{v}) + \frac{\delta^3}{3}(M_1(\mathbf{u}_1(\eta)), q) + (a_{\alpha\beta}d^\alpha(p_0)(\eta), v^\beta) \\ \quad + (a_{\alpha\beta}\hat{\mathcal{A}}^\alpha(u_1(\eta), p_0(\eta), \eta), v^\beta) + (\hat{B}^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta), v^\beta) + \frac{\delta^3}{3}(-\phi(p_0(\eta), \eta) \\ \quad + \widehat{M}_1(\mathbf{u}_1(\eta), \eta), q) = (\mathbf{F}^{(1)}(\eta), \mathbf{v}) + (f_p(\eta), q), \quad \forall V = (\mathbf{v}, q) \in \mathbf{Y}(D), \end{array} \right. \quad (5.47)$$

及

$$\left\{ \begin{array}{l} \mathcal{A}^\alpha(\mathbf{u}_1(\omega), p_0(\omega)) + B^\alpha(\mathbf{u}_1(\omega), \mathbf{u}_1(\omega)) + \hat{\mathcal{A}}^\alpha(u_1(\omega), p_0(\omega), \omega) \\ \quad + \hat{B}^\alpha(\mathbf{u}_1(\omega), \mathbf{u}_1(\omega), \omega) = F^{(1)}(\omega), \\ \mathcal{L}(p_0(\omega), \mathbf{u}_1(\omega)) + \frac{\delta^3}{3}(-\phi(p_0(\omega), \omega) + \widehat{M}_1(\mathbf{u}_1(\omega), \omega)) = f_p(\omega), \end{array} \right. \quad (5.48)$$

和对应的变分形式是

$$\left\{ \begin{array}{l} \text{求 } U(\omega) := (\mathbf{u}_1(\omega), p_0(\omega)) \in \mathbf{Y}(D) = \mathbf{V}(D) \times L_p^2(D), \text{ 使得} \\ A_0(U(\omega), V) + b(\mathbf{u}_1(\omega), \mathbf{u}_1(\omega), \mathbf{v}) + \frac{\delta^3}{3}(M_1(\mathbf{u}_1(\omega)), q) + (a_{\alpha\beta}d^\alpha(p_0)(\omega), v^\beta) \\ \quad + (a_{\alpha\beta}\hat{\mathcal{A}}^\alpha(u_1(\omega), p_0(\omega), \omega), v^\beta) + (a_{\alpha\beta}\hat{B}^\alpha(\mathbf{u}_1(\omega), \mathbf{u}_1(\omega), \omega), v^\beta) + \frac{\delta^3}{3}(-\phi(p_0(\omega), \omega) \\ \quad + \widehat{M}_1(\mathbf{u}_1(\omega), \omega), q) = (\mathbf{F}^{(1)}(\omega), \mathbf{v}) + (f_p(\omega), q), \quad \forall V = (\mathbf{v}, q) \in \mathbf{Y}(D), \end{array} \right. \quad (5.49)$$

这里令 $\omega = \eta + t\zeta$ 并且为了简单, 仍然记 $\mathbf{u}_1(\omega)$ 代替 $\underline{\mathbf{u}}_1(\omega)$, 其中

$$\begin{aligned} \hat{\mathcal{A}}^\alpha(u_1(\omega), p_0(\omega), \omega) &:= -\frac{\delta^3}{3}\mu \left(A^\alpha(\mathbf{u}_1(\omega), \omega) + \frac{7}{2}a^{\alpha\lambda}[\gamma_0(\omega) \overset{*}{\nabla}_\lambda \text{div} \mathbf{u}_1(\omega) + \overset{*}{\nabla}_\lambda (\Phi_{\beta\sigma}^\beta(\omega) u_1^\sigma(\omega))] \right. \\ &\quad \left. + \hat{K}(\omega) u_1^\alpha(\omega) \right) + \hat{q}_\beta^\alpha(\omega) u_1^\beta(\omega) - \hat{d}^\alpha(p_0(\omega), \omega), \\ \hat{B}^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\omega), \omega) &= [\Phi_{\lambda\sigma}^\alpha(\omega) u_1^\sigma(\omega) u_1^\lambda(\omega) - u_1^\alpha(\omega) \Phi_{\lambda\sigma}^\lambda(\omega) u_1^\sigma(\omega)] \frac{\delta^4}{4}, \\ \phi(p_0(\omega), \omega) &= 2\gamma_0(\omega) \overset{*}{\Delta} p_0(\omega) - a^{\alpha\beta} \Phi_{\alpha\beta}^\nu(\omega) \overset{*}{\nabla}_\nu p_0(\omega), \\ \hat{d}^\alpha(p_0(\omega), \omega) &= -\frac{\delta^3}{3} a^{\alpha\beta} (\hat{H}(\omega) + 2H\gamma_0(\omega)) \overset{*}{\nabla}_\beta p_0(\omega) + \frac{3\delta^3}{3} a^{\alpha\beta} (2\gamma_0(\omega) \overset{*}{\nabla}_\beta H + \overset{*}{\nabla}_\beta \hat{H}(\omega)) p_0(\omega), \\ \hat{q}_\beta^\alpha(\omega) &:= \mu \left(\frac{29}{16} \gamma_0(\omega) \delta - 2(2H\hat{H}(\omega) - \hat{K}(\omega)) \delta_\beta^\alpha + \hat{b}_\beta^\alpha \right) \frac{\delta^3}{3}, \\ \widehat{M}_1(\mathbf{u}_1(\omega), \omega) &= \frac{\mu}{2} \left[2\hat{H}(\omega) \overset{*}{\text{div}} \mathbf{u}_1(\omega) + \left(2H - \frac{3}{8\delta} \right) \Phi_{\alpha\beta}^\alpha(\omega) u_1^\beta(\omega) \right. \\ &\quad \left. + u_1^\alpha(\omega) \overset{*}{\nabla}_\alpha \hat{H}(\omega) + \hat{b}_\beta^\alpha(\omega) \overset{*}{\nabla}_\alpha u_1^\beta(\omega) + b_\beta^\alpha \Phi_{\alpha\lambda}^\beta(\eta) u_1^\lambda(\omega) \right], \end{aligned} \quad (5.50)$$

以及

$$A_0(U, V) = \frac{2\mu\delta^3}{3} (a^{\alpha\beta} a^{\lambda\sigma} \overset{*}{e}_{\alpha\lambda}(\mathbf{u}_1), \overset{*}{e}_{\beta\sigma}(\mathbf{v})) + \frac{\delta^3}{3} (\overset{*}{\nabla} p_0, \overset{*}{\nabla} q) + \frac{5\delta^3}{2 \cdot 3} (\text{div} \mathbf{u}_1, \text{div} \mathbf{v}) + (q_{\alpha\beta} u_1^\alpha, v^\beta),$$

$$(a_{\alpha\beta}d^\alpha(p_0), v^\beta) = \frac{\delta^3}{3}(-H \overset{*}{\nabla}_\beta p_0 + 3 \overset{*}{\nabla}_\beta H p_0, v^\beta),$$

$$\frac{\delta^3}{3}(M_1(\mathbf{u}_1), q) = \frac{\mu}{8} \left(16H \frac{\delta^3}{3} - \frac{\delta^2}{2} \right) (\operatorname{div} \mathbf{u}_1, q) + \frac{\delta^3}{3} 2\mu(u_1^\alpha \overset{*}{\nabla}_\alpha H + b_\beta^\alpha \overset{*}{\nabla}_\alpha u_1^\beta, q),$$

即算子是在半测地坐标系下的形式.

显然, 对 (5.47) 和 (5.49), 定理 16 仍然成立. (5.46)–(5.48) 除以 t 并令 $t \Rightarrow 0$, 得到类似于 (5.45) 方程. 实际上, (5.47)–(5.49) 并令 $\mathbf{w}(t\zeta) = \mathbf{u}_1(\omega) - \mathbf{u}_1(\eta)$, $s(t\zeta) = p_0(\omega) - p_0(\eta)$, 可以得到类似于 (5.43) 的方程

$$\left\{ \begin{aligned} & \frac{2\mu\delta^3}{3} (a^{\alpha\beta} a^{\beta\sigma} \overset{*}{e}_{\alpha\lambda}(\mathbf{w}(t\zeta)), \overset{*}{e}_{\beta\sigma}(\mathbf{v})) + \frac{5}{2} (\operatorname{div} \mathbf{w}(t\zeta), \operatorname{div} \mathbf{v}) \\ & - (a_{\alpha\beta} d^\alpha(s(t\zeta), v^\beta) + b(\mathbf{w}(t\zeta), \mathbf{u}_1(\omega), v) + b(\mathbf{u}_1(\eta), \mathbf{w}(t\zeta), v) \\ & + (a_{\alpha\beta}(\hat{\mathcal{A}}^\alpha(\mathbf{u}_1(\omega), p_0(\omega), \omega) - \hat{\mathcal{A}}^\alpha(\mathbf{u}_1(\eta), p_0(\eta), \eta)), v^\beta) \\ & + (a_{\alpha\beta}(\hat{B}^\alpha(\mathbf{u}_1(\omega), \mathbf{u}_1(\omega), \omega) - \hat{B}^\alpha(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta)), v^\beta) \\ & = (\mathbf{F}^{(1)}(\omega) - \mathbf{F}^{(1)}(\eta), \mathbf{v}), \quad \forall V = (\mathbf{v}, q) \in \mathbf{Y}(D), \\ & \frac{\delta^3}{3} (\overset{*}{\nabla} s(t\zeta), \overset{*}{\nabla} q) + \frac{\delta^3}{3} (M_1(\mathbf{w}(t\zeta), q)) + \frac{\delta^3}{3} ((-\phi(p_0(\omega), \omega) \\ & + \phi(p_0(\eta), \eta)) + \widehat{M}_1(\mathbf{u}_1(\omega), \omega) - \widehat{M}_1(\mathbf{u}_1(\eta), \eta), q) = (f_p(\omega) - f_p(\eta), q). \end{aligned} \right. \quad (5.51)$$

注意 (5.8), 有

$$\begin{aligned} & (a_{\alpha\beta}(\hat{\mathcal{A}}^\alpha(\mathbf{u}_1(\omega), p_0(\omega), \omega) - \hat{\mathcal{A}}^\alpha(\mathbf{u}_1(\eta), p_0(\eta), \eta)), v^\beta) \\ & := -\frac{\delta^3}{3} \mu \left\{ (a_{\alpha\beta}(A^\alpha(\mathbf{u}_1(\omega), \omega) - A^\alpha(\mathbf{u}_1(\eta), \eta)), v^\beta) \right. \\ & \quad + \frac{7}{2} ([\gamma_0(\omega) \overset{*}{\nabla}_\beta \operatorname{div} \mathbf{u}_1(\omega) - \gamma_0(\eta) \overset{*}{\nabla}_\beta \operatorname{div} \mathbf{u}_1(\eta)], v^\beta) \\ & \quad + (\overset{*}{\nabla}_\lambda (\Phi_{\beta\sigma}^\beta(\omega) u_1^\sigma(\omega) - \Phi_{\beta\sigma}^\beta(\eta) u_1^\sigma(\eta)), v^\lambda) + (\hat{K}(\omega) u_1^\alpha(\omega) - \hat{K}(\eta) u_1^\alpha(\eta), a_{\alpha\beta} v^\beta) \Big\} \\ & \quad + (\hat{q}_{\alpha\beta}(\omega) u_1^\beta(\omega) - \hat{q}_{\alpha\beta}(\eta) u_1^\beta(\eta), v^\beta) + (-\hat{d}^\alpha(p_0(\omega), \omega) + \hat{d}^\alpha(p_0(\eta), \eta), a_{\alpha\beta} v^\beta). \end{aligned} \quad (5.52)$$

由于

$$\gamma_0(\omega) - \gamma - 0(\eta) = \gamma_0(\omega - \eta) = t\gamma_0(\zeta), \quad \Phi_{\sigma\nu}^\alpha(\omega - \eta) = t\Phi_{\sigma\nu}^\alpha(\zeta),$$

并令

$$\mathbf{w}(t\zeta) = \mathbf{u}_1(\omega) - \mathbf{u}_1(\eta), \quad s(t\zeta) = p_0(\omega) - p_0(\eta),$$

那么,

$$\begin{aligned} & -\frac{\delta^3}{3} \mu (A^\alpha(\mathbf{u}_1(\omega), \omega) - A^\alpha(\mathbf{u}_1(\eta), \eta), a_{\alpha\beta} v^\beta) \\ & = -\frac{\delta^3}{3} \mu (2\gamma_0(\omega) \overset{*}{\Delta}(w^\alpha(t\zeta)) + 2t\gamma_0(\zeta) \overset{*}{\Delta}(u_1^\alpha(\eta)), a_{\alpha\beta} v^\beta) \\ & \quad - \frac{\delta^3}{3} \mu (a^{\beta\sigma} [t\Phi_{\beta\nu}^\alpha(\zeta) \overset{*}{\nabla}_\sigma u_1^\nu(\omega) + \Phi_{\beta\nu}^\alpha(\eta) \overset{*}{\nabla}_\sigma (w^\nu(t\zeta)) + t\Phi_{\sigma\nu}^\alpha(\zeta) \overset{*}{\nabla}_\beta u_1^\nu(\omega) \\ & \quad + \Phi_{\sigma\nu}^\alpha(\eta) \overset{*}{\nabla}_\beta (w^\nu(t\zeta)) - t\Phi_{\sigma\beta}^\nu(\zeta) \overset{*}{\nabla}_\nu u_1^\alpha(\omega) - \Phi_{\sigma\beta}^\nu(\eta) \overset{*}{\nabla}_\nu (w^\alpha(t\zeta))]) \end{aligned}$$

$$\begin{aligned}
& + a^{\beta\sigma} [t \nabla_{\sigma}^* \Phi_{\beta\nu}^{\alpha}(\zeta) u_1^{\nu}(\omega) + \nabla_{\sigma}^* \Phi_{\beta\nu}^{\alpha}(\eta) (w^{\nu}(t\zeta))] a_{\alpha\beta} v^{\beta}) \\
& = \frac{\delta^3}{3} 2\mu [(\nabla^* \mathbf{w}(t\zeta), \nabla^* (\gamma_0(\omega) \mathbf{v})) + t(\nabla^* \mathbf{u}_1(\eta), \nabla^* (\gamma_0(\zeta) \mathbf{v}))] \\
& \quad - \frac{\delta^3}{3} \mu [t[\Phi_{\beta\nu}^{\alpha}(\zeta) \nabla_{\sigma}^* u_1^{\nu}(\omega) + \Phi_{\sigma\nu}^{\alpha}(\zeta) \nabla_{\beta}^* u_1^{\nu}(\omega) - \Phi_{\sigma\beta}^{\nu}(\zeta) \nabla_{\nu}^* u_1^{\alpha}(\omega)] \\
& \quad + \Phi_{\beta\nu}^{\alpha}(\eta) \nabla_{\sigma}^* w^{\nu}(t\zeta) + \Phi_{\sigma\nu}^{\alpha}(\eta) \nabla_{\beta}^* w^{\nu}(t\zeta) - \Phi_{\sigma\beta}^{\nu}(\eta) \nabla_{\nu}^* w^{\alpha}(t\zeta), a_{\alpha\lambda} a^{\beta\sigma} v^{\lambda}). \quad (5.53)
\end{aligned}$$

类似地, 有

$$\begin{aligned}
\gamma_0(\omega) \nabla_{\beta}^* \operatorname{div} \mathbf{u}_1(\omega) - \gamma_0(\eta) \nabla_{\beta}^* \operatorname{div} \mathbf{u}_1(\eta) & = t\gamma_0(\zeta) \nabla_{\beta}^* \operatorname{div} \mathbf{u}_1(\omega) + \gamma_0(\eta) \nabla_{\beta}^* \operatorname{div} \mathbf{w}(t\zeta), \\
& \quad - \frac{\delta^3}{3} \frac{7}{2} \mu (\gamma_0(\omega) \nabla_{\beta}^* \operatorname{div} \mathbf{u}_1(\omega) - \gamma_0(\eta) \nabla_{\beta}^* \operatorname{div} \mathbf{u}_1(\eta), v^{\beta}) \\
& = \frac{\delta^3}{3} \frac{7}{2} \mu \{t(\operatorname{div} \mathbf{u}_1(\omega), \operatorname{div} (\gamma(\zeta) \mathbf{v})) + (\operatorname{div} \mathbf{w}(t\zeta), \operatorname{div} (\gamma(\eta) \mathbf{v})), \\
& \quad \nabla_{\lambda}^* (\Phi_{\beta\sigma}^{\alpha}(\omega) u_1^{\sigma}(\omega) - \Phi_{\beta\sigma}^{\alpha}(\eta) u_1^{\sigma}(\eta)) = \nabla_{\lambda}^* (t\Phi_{\beta\sigma}^{\alpha}(\zeta) u_1^{\sigma}(\omega) + \Phi_{\beta\sigma}^{\alpha}(\eta) w^{\sigma}(t\zeta)), \\
& \quad \widehat{K}(\omega) u_1^{\alpha}(\omega) - \widehat{K}(\eta) u_1^{\alpha}(\eta) = t\widehat{K}(\zeta) u_1^{\alpha}(\omega) + \widehat{K}(\eta) w^{\alpha}(t\zeta), \\
& \quad \widehat{q}_{\beta}^{\alpha}(\omega) u_1^{\beta}(\omega) - \widehat{q}_{\beta}^{\alpha}(\eta) u_1^{\beta}(\eta) = t\widehat{q}_{\beta}^{\alpha}(\zeta) u_1^{\beta}(\omega) + \widehat{q}_{\beta}^{\alpha}(\eta) w^{\beta}(t\zeta), \\
& \quad \widehat{d}^{\alpha}(p_0(\omega), \omega) - \widehat{d}^{\alpha}(p_0(\eta), \eta) = t\widehat{d}^{\alpha}(p_0(\omega), \zeta) + \widehat{d}^{\alpha}(s(t\zeta), \eta).
\end{aligned}$$

将以上各式代入 (5.52),

$$\begin{aligned}
& (a_{\alpha\beta} (\widehat{\mathcal{A}}^{\alpha}(\mathbf{u}_1(\omega), p_0(\omega), \omega) - \widehat{\mathcal{A}}^{\alpha}(\mathbf{u}_1(\eta), p_0(\eta), \eta)), v^{\beta}) \\
& = t \left\{ \frac{\delta^3}{32} \mu \left[\left(\nabla^* \frac{\mathbf{w}(t\zeta)}{t}, \nabla^* (\gamma_0(\eta) \mathbf{v}) \right) + \frac{7}{4} \left(\operatorname{div} \frac{\mathbf{w}(t\zeta)}{t}, \operatorname{div} (\gamma_0(\eta) \mathbf{v}) \right) \right. \right. \\
& \quad \left. \left. + (\nabla^* \mathbf{u}_1(\omega), \nabla^* (\gamma_0(\zeta) \mathbf{v})) + \frac{7}{4} (\operatorname{div} \mathbf{u}_1(\omega), \operatorname{div} (\gamma_0(\zeta) \mathbf{v})) \right] \right. \\
& \quad - \frac{\delta^3}{3} \mu \left([\Phi_{\beta\nu}^{\alpha}(\zeta) \nabla_{\sigma}^* u_1^{\nu}(\omega) + \Phi_{\sigma\nu}^{\alpha}(\zeta) \nabla_{\beta}^* u_1^{\nu}(\omega) - \Phi_{\sigma\beta}^{\nu}(\zeta) \nabla_{\nu}^* u_1^{\alpha}(\omega)] \right. \\
& \quad \left. + \Phi_{\beta\nu}^{\alpha}(\eta) \nabla_{\sigma}^* \frac{w^{\nu}(t\zeta)}{t} + \Phi_{\sigma\nu}^{\alpha}(\eta) \nabla_{\beta}^* \frac{w^{\nu}(t\zeta)}{t} - \Phi_{\sigma\beta}^{\nu}(\eta) \nabla_{\nu}^* \frac{w^{\alpha}(t\zeta)}{t}, a_{\alpha\lambda} a^{\beta\sigma} v^{\lambda} \right) \\
& \quad - \frac{\delta^3}{3} \mu \left(a_{\alpha\beta} \widehat{K}(\zeta) u_1^{\alpha}(\omega) + a_{\alpha\beta} \widehat{K}(\eta) \frac{w^{\alpha}(t\zeta)}{t}, v^{\beta} \right) \\
& \quad \left. + \left((\widehat{q}_{\alpha\beta}(\zeta) u_1^{\alpha}(\omega) + a_{\alpha\beta} \widehat{d}^{\alpha}(p_0(\omega), \zeta)) + \widehat{q}_{\alpha\beta}(\eta) \frac{w^{\alpha}(t\zeta)}{t} + a_{\alpha\beta} \widehat{d}^{\alpha} \left(\frac{s(t\zeta)}{t}, \eta \right), v^{\beta} \right) \right\}. \quad (5.54)
\end{aligned}$$

另外, 非线性项

$$\begin{aligned}
& B^{\alpha}(\mathbf{u}_1(\omega), \mathbf{u}_1(\omega), \omega) - B^{\alpha}(\mathbf{u}_1(\eta), \mathbf{u}_1(\eta), \eta) \\
& = \{[\Phi_{\lambda\sigma}^{\alpha}(\omega) u_1^{\sigma}(\omega) u_1^{\lambda}(\omega) - u_1^{\alpha}(\omega) \Phi_{\lambda\sigma}^{\alpha}(\omega) u_1^{\sigma}(\omega)] - [\Phi_{\lambda\sigma}^{\alpha}(\eta) u_1^{\sigma}(\eta) u_1^{\lambda}(\eta) - u_1^{\alpha}(\eta) \Phi_{\lambda\sigma}^{\alpha}(\eta) u_1^{\sigma}(\eta)]\} \frac{\delta^4}{4} \\
& = t \left\{ [\Phi_{\lambda\sigma}^{\alpha}(\zeta) u_1^{\lambda}(\omega) - \Phi_{\lambda\sigma}^{\alpha}(\eta) u_1^{\lambda}(\omega)] u_1^{\sigma}(\omega) - \Phi_{\beta\sigma}^{\alpha}(\eta) u_1^{\sigma}(\omega) \frac{w^{\alpha}(t\zeta)}{t} \right. \\
& \quad \left. + (2\Phi_{\lambda\sigma}^{\alpha}(\eta) u_1^{\lambda}(\omega) - \Phi_{\beta\sigma}^{\alpha}(\eta) u_1^{\alpha}(\eta)) \frac{w^{\sigma}(t\zeta)}{t} \right\} \frac{\delta^4}{4}. \quad (5.55)
\end{aligned}$$

另外, 我们还需考虑

$$\begin{aligned}
& \phi(p_0(\omega), \omega) - \phi(p_0(\eta), \eta) \\
&= 2\gamma_0(\omega) \overset{*}{\Delta} p_0(\omega) - a^{\alpha\beta} \Phi_{\alpha\beta}^\nu(\omega) \overset{*}{\nabla} p_0(\omega) - [2\gamma_0(\eta) \overset{*}{\Delta} p_0(\eta) - a^{\alpha\beta} \Phi_{\alpha\beta}^\nu(\eta) \overset{*}{\nabla} p_0(\eta)] \\
&= 2t\gamma_0(\zeta) \overset{*}{\Delta} p_0(\omega) + 2\gamma_0(\eta) \overset{*}{\Delta} (p_0(\omega) - p_0(\eta)) - a^{\alpha\beta} [t\Phi_{\alpha\beta}^\nu(\zeta) \overset{*}{\nabla} p_0(\omega) \\
&\quad + \Phi_{\alpha\beta}^\nu \overset{*}{\nabla} (p_0(\omega) - p_0(\eta))], \\
&(\phi(p_0(\omega), \omega) - \phi(p_0(\eta), \eta) - 2(\gamma_0(\omega) \overset{*}{\Delta} p_0(\omega), q) \\
&= t\{2(\gamma_0(\zeta) \overset{*}{\nabla} p_0(\omega), \overset{*}{\nabla} q) - 2(\overset{*}{\nabla} \gamma_0(\zeta) \overset{*}{\nabla} p_0(\omega), q) - (a^{\alpha\beta} \Phi_{\alpha\beta}^\nu(\zeta) \overset{*}{\nabla} p_0(\omega), q) \\
&\quad + 2\left(\gamma_0(\eta) \overset{*}{\nabla} \frac{s(t\zeta)}{t}, \overset{*}{\nabla} q\right) - 2\left(\overset{*}{\nabla} \gamma_0(\eta) \overset{*}{\nabla} \frac{s(t\zeta)}{t}, q\right) - \left(a^{\alpha\beta} \Phi_{\alpha\beta}^\nu(\eta) \overset{*}{\nabla} \frac{s(t\zeta)}{t}, q\right), \\
&\widehat{M}_1(\mathbf{u}_1(\omega), \omega) - \widehat{M}_1(\mathbf{u}_1(\eta), \eta) \\
&= \frac{\mu}{2} \left[2\widehat{H}(\omega) \operatorname{div} \mathbf{u}_1(\omega) + \left(2H - \frac{3}{8\delta}\right) \Phi_{\alpha\beta}^\alpha(\omega) u_1^\beta(\omega) \right. \\
&\quad \left. + u_1^\alpha(\omega) \overset{*}{\nabla}_\alpha \widehat{H}(\omega) + \widehat{b}_\beta^\alpha(\omega) \overset{*}{\nabla}_\alpha u_1^\beta(\omega) + b_\beta^\alpha \Phi_{\alpha\lambda}^\beta(\omega) u_1^\lambda(\omega) \right] - \frac{\mu}{2} \left[2\widehat{H}(\eta) \operatorname{div} \mathbf{u}_1(\eta) \right. \\
&\quad \left. + \left(2H - \frac{3}{8\delta}\right) \Phi_{\alpha\beta}^\alpha(\eta) u_1^\beta(\eta) + u_1^\alpha(\eta) \overset{*}{\nabla}_\alpha \widehat{H}(\eta) + \widehat{b}_\beta^\alpha(\eta) \overset{*}{\nabla}_\alpha u_1^\beta(\eta) + b_\beta^\alpha \Phi_{\alpha\lambda}^\beta(\eta) u_1^\lambda(\eta) \right] \\
&= \frac{t\mu}{2} \left\{ \left[2\widehat{H}(\zeta) \operatorname{div} \mathbf{u}_1(\omega) + \left(2H - \frac{3}{8\delta}\right) \Phi_{\alpha\beta}^\alpha(\zeta) u_1^\beta(\omega) + \overset{*}{\nabla}_\beta \widehat{H}(\zeta) u_1^\beta(\eta) \right. \right. \\
&\quad \left. \left. + \widehat{b}_\beta^\alpha(\zeta) \overset{*}{\nabla}_\alpha u_1^\beta(\omega) + \Phi_{\alpha\lambda}^\beta(\zeta) u_1^\lambda(\omega) \right] + \left(2\widehat{H}(\eta) - \frac{3}{8\delta}\right) \operatorname{div} \frac{\mathbf{w}(t\zeta)}{t} \right. \\
&\quad \left. + \widehat{b}_\beta^\alpha(\eta) \overset{*}{\nabla}_\alpha \frac{\mathbf{w}^\beta(t\zeta)}{t} + \left[\left(2H - \frac{3}{8\delta}\right) \Phi_{\beta\alpha}^\beta(\eta) + \overset{*}{\nabla}_\alpha \widehat{H}(\omega) + b_\beta^\lambda \Phi_{\alpha\lambda}^\beta(\eta) \right] \frac{\mathbf{w}^\alpha(t\zeta)}{t} \right\}.
\end{aligned}$$

将 (5.55) 和 (5.54) 代入 (5.51), 得

$$\begin{aligned}
& 2\mu \frac{\delta^3}{3} \left[(a^{\alpha\beta} a^{\beta\sigma} \overset{*}{e}_{\alpha\lambda}(\mathbf{w}(t\zeta)), \overset{*}{e}_{\beta\sigma}(\mathbf{v})) + \frac{5}{2} (\operatorname{div} \mathbf{w}(t\zeta), \operatorname{div} \mathbf{v}) \right] \\
& - a_{\alpha\beta} d^\alpha(s(t\zeta), v^\beta) + b(\mathbf{w}(t\zeta), \mathbf{u}_1(\omega), v) + b(\mathbf{u}_1(\eta), \mathbf{w}(t\zeta), v) \\
& + t \left\{ \frac{\delta^3}{32} \mu \left[\left(\overset{*}{\nabla} \frac{\mathbf{w}(t\zeta)}{t}, \overset{*}{\nabla} (\gamma_0(\eta) \mathbf{v}) \right) + \frac{7}{4} \left(\operatorname{div} \frac{\mathbf{w}(t\zeta)}{t}, \operatorname{div} (\gamma_0(\eta) \mathbf{v}) \right) \right. \right. \\
& \quad \left. \left. + \left(\Phi_{\beta\nu}^\alpha(\eta) \overset{*}{\nabla}_\sigma \frac{\mathbf{w}^\nu(t\zeta)}{t} + \Phi_{\sigma\nu}^\alpha(\eta) \overset{*}{\nabla}_\beta \frac{\mathbf{w}^\nu(t\zeta)}{t} - \Phi_{\sigma\beta}^\nu(\eta) \overset{*}{\nabla}_\nu \frac{\mathbf{w}^\alpha(t\zeta)}{t}, a_{\alpha\lambda} a^{\beta\sigma} v^\lambda \right) \right. \right. \\
& \quad \left. \left. + \left(a_{\alpha\beta} \widehat{K}(\eta) \frac{\mathbf{w}^\alpha(t\zeta)}{t}, v^\beta \right) \right] + \left(\widehat{q}_{\alpha\beta}(\eta) \frac{\mathbf{w}^\alpha(t\zeta)}{t} + a_{\alpha\beta} \widehat{d}^\alpha \left(\frac{s(t\zeta)}{t}, \eta \right), v^\beta \right) \right. \\
& \quad \left. + \left((2\Phi_{\lambda\sigma}^\alpha(\eta) u_1^\lambda(\omega) - \Phi_{\nu\sigma}^\nu(\eta) u_1^\alpha(\eta)) \frac{\mathbf{w}^\sigma(t\zeta)}{t} - \Phi_{\nu\sigma}^\nu(\eta) u_1^\sigma(\omega) \frac{\mathbf{w}^\alpha(t\zeta)}{t}, a_{\alpha\beta} v^\beta \right) \frac{\delta^4}{4} \right\} \\
& = t \left\{ \left(\frac{\mathbf{F}^{(1)}(\omega) - \mathbf{F}^{(1)}(\eta)}{t}, \mathbf{v} \right) - \frac{\delta^3}{3} \mu \left[(\overset{*}{\nabla} \mathbf{u}_1(\omega), \overset{*}{\nabla} (\gamma_0(\zeta) \mathbf{v})) + \frac{7}{4} (\operatorname{div} \mathbf{u}_1(\omega), \operatorname{div} (\gamma_0(\zeta) \mathbf{v})) \right. \right. \\
& \quad \left. \left. - ([\Phi_{\beta\nu}^\alpha(\zeta) \overset{*}{\nabla}_\sigma u_1^\nu(\omega) + \Phi_{\sigma\nu}^\alpha(\zeta) \overset{*}{\nabla}_\beta u_1^\nu(\omega) - \Phi_{\sigma\beta}^\nu(\zeta) \overset{*}{\nabla}_\nu u_1^\alpha(\omega)], a^{\beta\sigma} a_{\alpha\lambda} v^\lambda) \right. \right. \\
& \quad \left. \left. - (a_{\alpha\beta} \widehat{K}(\zeta) u_1^\alpha(\omega), v^\beta) \right] - (\widehat{q}_{\alpha\beta}(\zeta) u_1^\alpha(\omega) + a_{\alpha\beta} \widehat{d}^\alpha(p_0(\omega), \zeta), v^\beta) \right\}
\end{aligned}$$

$$\begin{aligned}
& -t([\Phi_{\lambda\sigma}^\alpha(\zeta)u_1^\lambda(\omega) - \Phi_{\lambda\sigma}^\lambda(\eta)u_1^\alpha(\omega)]u_1^\sigma(\omega), a_{\alpha\beta}v^\beta)\frac{\delta^4}{4}, \\
& \frac{\delta^3}{3}t\left[\left(\nabla^* \frac{s(t\zeta)}{t}, \nabla^* q\right) + 2\left(\gamma_0(\eta) \nabla^* \frac{s(t\zeta)}{t}, \nabla^* q\right) - 2\left(\nabla^* \gamma_0(\eta) \nabla^* \frac{s(t\zeta)}{t}, q\right)\right. \\
& \quad \left. - \left(a^{\alpha\beta}\Phi_{\alpha\beta}^\nu(\eta) \nabla_\nu^* \frac{s(t\zeta)}{t}, q\right)\right] + t\frac{\delta^3}{3}\left\{M_1\left(\frac{\mathbf{w}(t\zeta)}{t}, q\right) - \left[\left(2\widehat{H}(\eta) - \frac{3}{8\delta}\right) \operatorname{div}^* \frac{\mathbf{w}(t\zeta)}{t}\right.\right. \\
& \quad \left. + \widehat{b}_\beta^\alpha(\eta) \nabla_\alpha^* \frac{\mathbf{w}^\beta(t\zeta)}{t} + \left[\left(2H - \frac{3}{8\delta}\right)\Phi_{\beta\alpha}^\beta(\eta) + \nabla_\alpha^* \widehat{H}(\omega) + b_\beta^\lambda \Phi_{\alpha\lambda}^\beta(\eta)\right] \frac{\mathbf{w}^\alpha(t\zeta)}{t}, q\right)\Big\} \\
& = -\frac{t\delta^3}{3}[2(\gamma_0(\zeta) \nabla^* p_0(\omega), \nabla^* q) - 2(\nabla^* \gamma_0(\zeta) \nabla^* p_0(\omega), q) - (a^{\alpha\beta}\Phi_{\alpha\beta}^\nu(\zeta) \nabla_\nu^* p_0(\omega), q)] \\
& \quad + \frac{t\delta^3}{3}\frac{\mu}{2}\left[\left(2\widehat{H}(\zeta) \operatorname{div}^* \mathbf{u}_1(\omega) + \left(2H - \frac{3}{8\delta}\right)\Phi_{\alpha\beta}^\alpha(\zeta)u_1^\beta(\omega) + \nabla_\beta^* \widehat{H}(\zeta)\right)u_1^\beta(\eta)\right. \\
& \quad \left. + \widehat{b}_\beta^\alpha(\zeta) \nabla_\alpha^* u_1^\beta(\omega) + \Phi_{\alpha\lambda}^\beta(\zeta)u_1^\lambda(\omega)\right] + t\left(\frac{f_p(\omega) - f_p(\eta)}{t}, q\right), \quad \forall V = (\mathbf{v}, q) \in \mathbf{Y}(D). \quad (5.56b)
\end{aligned}$$

用类似方法, 对 (5.51), 并且进行 $t \Rightarrow 0$ 极限过度, 可以证明定理 18 仍然成立, 于是有下面的定理.

定理 19 在定理 16 的假设下, 边界层方程 (5.47) 的速度梯度和压力 $(\mathbf{u}_1(\eta), p_0(\eta))$ 的 Gâteaux 导数 $(\widehat{\mathbf{u}}_1(\eta), \widehat{p}_0(\eta))$ 满足下列变分问题:

$$\left\{ \begin{aligned}
& 2\mu\frac{\delta^3}{3}\left[(a^{\alpha\beta}a^{\beta\sigma}e_{\alpha\lambda}^*(\widehat{\mathbf{u}}_1(\eta)), e_{\beta\sigma}^*(\mathbf{v})) + \frac{5}{2}(\operatorname{div}^* \widehat{\mathbf{u}}_1(\eta), \operatorname{div}^* \mathbf{v})\right] \\
& \quad - (a_{\alpha\beta}d^\alpha(\widehat{p}_0(\eta), v^\beta) + b(\widehat{\mathbf{u}}_1(\eta), \mathbf{u}_1, v) + b(\mathbf{u}_1, \widehat{\mathbf{u}}_1(\eta), v) \\
& \quad + \frac{\delta^3}{32}\mu\left[(\nabla^* \widehat{\mathbf{u}}_1(\eta), \nabla^* (\gamma_0(\eta)\mathbf{v})) + \frac{7}{4}(\operatorname{div}^* \widehat{\mathbf{u}}_1(\eta), \operatorname{div}^* (\gamma_0(\eta)\mathbf{v}))\right. \\
& \quad \left. + (\Phi_{\beta\nu}^\alpha(\eta) \nabla_\sigma^* \widehat{u}_1^\nu(\eta) + \Phi_{\sigma\nu}^\alpha(\eta) \nabla_\beta^* \widehat{u}_1^\nu(\eta) - \Phi_{\sigma\beta}^\nu(\eta) \nabla_\nu^* \widehat{u}_1^\alpha(\eta), a_{\alpha\lambda}a^{\beta\sigma}v^\lambda) \right. \\
& \quad \left. + (a_{\alpha\beta}\widehat{K}(\eta)\widehat{u}_1^\alpha(\eta), v^\beta)\right] + (\widehat{q}_{\alpha\beta}(\eta)\widehat{u}_1^\alpha(\eta) + a_{\alpha\beta}\widehat{d}^\alpha(\widehat{p}_0(\eta), \eta), v^\beta) \\
& \quad \left. + ((2\Phi_{\lambda\sigma}^\alpha(\eta)u_1^\lambda(\omega) - \Phi_{\nu\sigma}^\nu(\eta)u_1^\alpha(\eta))\widehat{u}^\sigma(\eta) - \Phi_{\nu\sigma}^\nu(\eta)u_1^\sigma(\omega)\widehat{u}^\alpha(\eta), a_{\alpha\beta}v^\beta)\frac{\delta^4}{4}\right\} \\
& = (\mathbf{Q}(\mathbf{u}_1(\eta), p_0(\eta)), \mathbf{v}), \quad \forall \mathbf{v} \in V(D), \\
& \frac{\delta^3}{3}\left[(\nabla^* \widehat{p}_0(\eta), \nabla^* q) + 2(\gamma_0(\eta) \nabla^* \widehat{p}_0(\eta), \nabla^* q) - 2(\nabla^* \gamma_0(\eta) \nabla^* \widehat{p}_0(\eta), q) \right. \\
& \quad \left. - (a^{\alpha\beta}\Phi_{\alpha\beta}^\nu(\eta) \nabla_\nu^* \widehat{p}_0(\eta), q) + \frac{\delta^3}{3}\left\{(M_1(\widehat{\mathbf{u}}(\eta)), q) - \left[\left(2\widehat{H}(\eta) - \frac{3}{8\delta}\right) \operatorname{div}^* \widehat{\mathbf{u}}(\eta) \right.\right.\right. \\
& \quad \left. \left. + \widehat{b}_\beta^\alpha(\eta) \nabla_\alpha^* \widehat{\mathbf{u}}(\eta) + \left[\left(2H - \frac{3}{8\delta}\right)\Phi_{\beta\alpha}^\beta(\eta) + \nabla_\alpha^* \widehat{H}(\omega) + b_\beta^\lambda \Phi_{\alpha\lambda}^\beta(\eta)\right] \widehat{\mathbf{u}}(\eta), q\right]\right\} \\
& = (\mathbf{Q}_p(\mathbf{u}_1(\eta), p_0(\eta)), q), \quad \forall q \in L^2(D), \\
& (\widehat{\mathbf{u}}_1(\eta), p_0(\eta)) \text{ 满足周期性边界条件,}
\end{aligned} \right. \quad (5.57)$$

其中

$$\begin{aligned}
& (\mathbf{Q}(\mathbf{u}_1(\eta), p_0(\eta)), \mathbf{v}) \\
& := ((F^{(1)})'_\beta(\eta), v^\beta) - \frac{\delta^3}{3}\mu\left[(\nabla^* \mathbf{u}_1(\eta), \nabla^* (\gamma_0(\zeta)\mathbf{v})) + \frac{7}{4}(\operatorname{div}^* \mathbf{u}_1(\eta), \operatorname{div}^* (\gamma_0(\zeta)\mathbf{v}))\right] \\
& \quad - \frac{\delta^3}{3}\mu[(\Phi_{\beta\nu}^\alpha(\zeta) \nabla_\sigma^* u_1^\nu(\eta) + \Phi_{\sigma\nu}^\alpha(\zeta) \nabla_\beta^* u_1^\nu(\eta) - \Phi_{\sigma\beta}^\nu(\zeta) \nabla_\nu^* u_1^\alpha(\eta)], a^{\beta\sigma}a_{\alpha\lambda}v^\lambda)
\end{aligned}$$

$$\begin{aligned}
& -\frac{\delta^3}{3}\mu(a_{\alpha\beta}\widehat{K}(\zeta)u_1^\alpha(\eta),v^\beta)-(\widehat{q}_{\alpha\beta}(\zeta)u_1^\alpha(\eta)+a_{\alpha\beta}\widehat{d}^\alpha(p_0(\eta),\zeta),v^\beta) \\
& -((\Phi_{\lambda\sigma}^\alpha(\zeta)u_1^\lambda(\eta)-\Phi_{\lambda\sigma}^\lambda(\eta)u_1^\alpha(\eta))u_1^\sigma(\eta),a_{\alpha\beta}v^\beta)\frac{\delta^4}{4}, \quad \forall v \in V(D), \\
& (Q_p(u_1,p_0),q) \\
& := -\frac{\delta^3}{3}[2(\gamma_0(\zeta) \overset{*}{\nabla} p_0(\omega), \overset{*}{\nabla} q) - 2(\overset{*}{\nabla} \gamma_0(\zeta) \overset{*}{\nabla} p_0(\omega), q) - (a^{\alpha\beta}\Phi_{\alpha\beta}^\nu(\zeta) \overset{*}{\nabla}_\nu p_0(\omega), q)] \\
& + \frac{\mu}{2}\frac{\delta^3}{3}\left(\left[2\widehat{H}(\zeta) \overset{*}{\text{div}} u_1(\omega) + \left(2H - \frac{3}{8\delta}\right)\Phi_{\alpha\beta}^\alpha(\zeta)u_1^\beta(\omega) + u_1^\alpha(\eta) \overset{*}{\nabla}_\alpha \widehat{H}(\zeta) \right. \right. \\
& \left. \left. + \widehat{b}_\beta^\alpha(\zeta) \overset{*}{\nabla}_\alpha u_1^\beta(\omega) + \Phi_{\alpha\lambda}^\beta(\zeta)u_1^\lambda(\omega)\right], q\right) + (f'_p(\eta), q), \quad \forall q \in L^2(D).
\end{aligned} \tag{5.58}$$

6 形状优化

流动控制和形状优化, 如飞行器阻力或升力的优化, 高速火车头、汽车和叶轮叶片等形状优化问题等, 是现代数学家、科学家和工程师所极度关注的问题. “一个沿着旋转体轴向匀速流动的流体 (如图 2 所示), 一个怎样的回转曲面, 使得受到最小的阻力?” 其中作一定假设后, 这个问题等价于求一个函数 $y(x)$, $0 \leq x \leq x_2$, 使得它是极值问题

$$J(y) = \inf_{z \in C^1([x_1, x_2])} \int_{x_1}^{x_2} \frac{z(x)(z'(x))^3}{1 + (z'(x))^2} dx$$

的解. 这个最优解可以表示为参数方程

$$x = \frac{c}{p}(1 + p^2)^2, \quad y = a + c\left(-\ln p + p^2 + \frac{3}{4}p^4\right).$$

这个问题是 Newton^[9] 于 1687 提出的. 我们考虑另一个在工业上重要例子是减少机翼阻力的形状优化问题. 流体在机翼 $\mathfrak{S}$ 上的流动产生反向应力, 它是一个向量, 在飞行方向的分量就是阻力, 剩余部分就是升力. 只要减少很少阻力, 在商用飞机上可以节约很大的费用. 以后假设无限远来流的速度 v_∞ . 记流动中固壁边界的法向全应力为

$$\Sigma_n = \left\{ \int_{\mathfrak{S}} \left[-pg^{ij} + 2\mu e^{ij}(\mathbf{u}) - \frac{1}{3} \frac{\mu g^{ij}}{\mathbf{u}} \right] n_i dv, j = 1, 2, 3 \right\},$$

其中 g^{ij} 为在附体坐标系 (y^1, y^2, y^3) 下的度量张量逆变分量, $\mathbf{n}$ 是机翼 $\mathfrak{S}$ 表面外法向单位法向量. 包含压力积分部分称为波阻力升力: $\mathbf{I}_w$, 其余部分称为黏性阻力升力: $\mathbf{I}_\nu$,

$$\Sigma_n = \mathbf{I}_w + \mathbf{I}_\nu.$$

极小化泛函 (估值函数)

$$\begin{cases} J(\mathfrak{S}) = \frac{C_d}{C_d^0} + 0.05 \frac{C_l - C_l^0}{C_l^0} + 0.1 \frac{\text{Vol} - \text{Vol}^0}{\text{Vol}^0}, \\ C_d := \Sigma_n \cdot \frac{\mathbf{v}_\infty}{v_\infty}, \quad C_l = \Sigma_n - C_d, \end{cases} \tag{6.1}$$

其中上标 “0” 表示参考值. Vol 是飞行器表面所包含的体积, 系数 0.05 和 0.1 是权数, 可以改变. 为简单, 设阻力泛函

$$J(\mathfrak{S}) = \int_D \left[-pg^{ij} + 2\mu e^{ij}(\mathbf{u}) - \frac{1}{3} \frac{\mu g^{ij}}{\mathbf{u}} \right] n_i v_{\infty j} \sqrt{a} dx,$$

图 2 旋转椭圆绕流流线图

而叶轮叶片做功率泛函

$$I(\mathfrak{S}) = \int_D \left[-pg^{ij} + 2\mu e^{ij}(\mathbf{u}) - \frac{1}{3} \frac{\mu g^{ij}}{\mathbf{u}} \right] n_i(\mathbf{e}_\theta)_j r \omega \sqrt{a} dx,$$

其中 ω 是叶轮旋转角速度, $\mathbf{e}_\theta$ 是旋转方向单位基向量. 假设 W_0 是叶轮的额定功率, 那么,

$$J(\mathfrak{S}) = W_0 - I(\mathfrak{S}) = W_0 - \int_D \left[-pg^{ij} + 2\mu e^{ij}(\mathbf{u}) - \frac{1}{3} \mu g^{ij} \nabla \cdot \mathbf{u} \right] n_i(\mathbf{e}_\theta)_j r \omega \sqrt{a} dx$$

就是极小化泛函. 形状最优控制问题是在一个恰当的允许设计空间中, 求一个闭曲面

$$J(\mathfrak{S}) = \inf_{S \in \mathcal{F}} J(S), \quad (6.2)$$

极小化问题 (6.2) 可以称为广义极小值曲面问题, 因为这时求的并不是物体表面的面积最小, 而是物体所受流体的阻力最小.

这里有几个约束, 几何约束: 闭曲面所包围的体积 V 必须大于一个确定的值 $V > V_0$; 空气动力学约束: 升力 l_i 必须大于一个确定的数值 $|l_i| > l_0$. 问题的难点在于包含求解高 Reynolds 数下的无界区域内的可压缩的 Navier-Stokes 方程和目标泛函关于边界形状的梯度计算平台的建立.

定义 20 一个定义在 $V(D)$ 中的边界 $\mathfrak{S}$ 的泛函 $J(\mathfrak{S})$ 对任意方向 $\eta \in V(D)$ 极限

$$\delta J(\mathfrak{S})\eta = \lim_{t \rightarrow 0} \frac{1}{t} (J(\mathfrak{S}(t\eta)) - J(\mathfrak{S})) \quad (6.3)$$

称为 $J(\mathfrak{S})$ 沿方向 η 的第一变分. 如果对所有方向第一变分存在, 映照 $\eta \rightarrow \delta J(\mathfrak{S})\eta : V \Rightarrow \mathbb{R}$ 是连续的. 那么, 称 J 是形状可微的. 如果存在一个广义函数 $\text{Grad}J(\mathfrak{S}) \in V'(D)$ (对偶空间) 使得

$$\delta J(\mathfrak{S})\eta = \langle \text{Grad}J(\mathfrak{S}), \eta \rangle, \quad (6.4)$$

则称 $\text{Grad}J(\mathfrak{S})$ 为 $\mathfrak{S}$ 的形状梯度. 由 Riesz 表示定理, 存在一个 $g \in V(D)$ 使得

$$((g, \eta)) = \langle \text{Grad}J(\mathfrak{S}), \eta \rangle, \quad (6.5)$$

g 即为通常的梯度, 这里 $(\mathbf{u}, \mathbf{v}) = \int_D a_{\alpha\beta} u^\alpha v^\beta \sqrt{a} dx$.

下面考查在分别以 $\mathfrak{S}$ 和 $\mathfrak{S}(\eta)$ 为基础的半测地坐标系下的法向面应力.

引理 21 上述 $\mathfrak{S}$ -半测地坐标系下, $\mathfrak{S}$ 和 $\mathfrak{S}(\eta)$ 的法向面应力

$$\sigma^{ij}(p, \mathbf{u})n_i = (-pg^{ij} + 2\mu e^{ij}(\mathbf{u}))n_i = -pg^{ij}n_i + \mu(g^{ik}\nabla_k u^j + g^{jk}\nabla_k u^i)n_i$$

分别是

$$\begin{cases} \sigma^{ij}(\mathbf{u}, p)n_i|_{\mathfrak{S}} = \{\sigma^{3\alpha}(\mathbf{u}, p)|_{\mathfrak{S}} = \mu u_1^\alpha, \sigma^{33}(\mathbf{u}, p)|_{\mathfrak{S}} = -p_0\}, \\ \sigma^{ij}(\mathbf{u}, p)n_i|_{\mathfrak{S}(\eta)} = \{\sigma^{3\alpha}(\mathbf{u}, p)|_{\mathfrak{S}(\eta)} = \mu \underline{u}_1^\alpha(\eta), \sigma^{33}(\mathbf{u}, p)|_{\mathfrak{S}(\eta)} = -\underline{p}_0(\eta)\}, \end{cases} \quad (6.6)$$

其中 $(\mathbf{u}_1, p_0)$ 和 $(\underline{\mathbf{u}}_1(\eta), \underline{p}_0(\eta))$ 分别是对应于 $\mathfrak{S}$ 和 $\mathfrak{S}(\eta)$ 的边界层方程 (4.42) 和 (5.17) 的解.

证明 在以 $\mathfrak{S}$ 为基础的半测地坐标系下, $\mathbf{n} = (0, 0, 1)$. 另外, 由 (2.1) 和 (2.7), 在 $\mathfrak{S}$ 上, 即 $\xi = 0$. 首先证明在边界面上

$$e^{33}(\mathbf{u}) = \frac{\partial u^3}{\partial \xi} = u_1^3 = 0, \quad e^{3\alpha}(\mathbf{u}) = \frac{1}{2}u_1^\alpha. \quad (6.7)$$

实际上, 由 (2.8) 以及在 $\xi = 0$, $\mathbf{u} = 0$, $u_1^3 = 0$,

$$\begin{aligned} e^{33}(\mathbf{u}) &= g^{33}g^{33}e_{33}(\mathbf{u}) = u_1^3 = 0, \\ e^{3\alpha}(\mathbf{u}) &= g^{33}g^{\alpha\beta}e_{3\beta}(\mathbf{u}) \stackrel{(2.8)}{=} g^{\alpha\beta}\frac{1}{2}\left(g_{\beta\lambda}\frac{\partial u^\lambda}{\partial \xi} + \nabla_\beta^* u^3\right) \\ &= u_1^\alpha + a^{\alpha\beta}\nabla_\beta^* u^3 = u_1^\alpha, \end{aligned}$$

这就推出 (6.7). 所以, 有

$$\begin{aligned} \sigma^{ij}(\mathbf{u}, p)n_i &= \sigma^{3j}(\mathbf{u}, p) = \{\sigma^{3\alpha}(\mathbf{u}, p), \sigma^{33}(\mathbf{u}, p)\}, \\ \sigma^{33}(\mathbf{u}, p) &= -g^{33}p + 2\mu e^{33}(\mathbf{u}) = -p + 2\mu u_1^3 = -p, \\ \sigma^{\alpha 3}(\mathbf{u}, p) &= -pg^{\alpha 3} + 2\mu e^{\alpha 3}(\mathbf{u}) = \mu u_1^\alpha. \end{aligned}$$

下面考查 $\mathfrak{S}(\eta)$ 上的面应力. 应用定理 1 和 2, 边界条件以及边界层方程仍然有

$$g^{ij}(\eta)g_{jk}(\eta) = \delta_k^i, \quad \mathbf{n}(\eta) = (0, 0, 1), \quad |\mathbf{u}(\eta)| = 1, \quad u_1^3(\eta) = \frac{\partial u^3(\eta)}{\partial \xi} = 0,$$

那么, 同样成立

$$e^{33}(\mathbf{u}(\eta)) = \frac{\partial u^3(\eta)}{\partial \xi} = u_1^3(\eta) = 0, \quad e^{3\alpha}(\mathbf{u}(\eta)) = \frac{1}{2}u_1^\alpha(\eta). \quad (6.8)$$

为此, 应用下式,

$$e^{3\alpha}(\mathbf{u}) = g^{33}(\eta)g^{\alpha\beta}(\eta)e_{3\beta}(\mathbf{u}(\eta)) = g^{\alpha\beta}(\eta)\frac{1}{2}(g_{\beta\lambda}(\eta)\nabla_3 u^\lambda(\eta) + g_{33}(\eta)\nabla_\beta u^3(\eta))$$

$$= \frac{1}{2} g^{\alpha\beta} (g_{\beta\lambda}(\eta) \nabla_3 u^\lambda(\eta) + \nabla_\beta u^3(\eta)),$$

由于 $g_{33}(\eta) = 1$, 应用 (2.7),

$$\begin{aligned} e^{3\alpha}(\mathbf{u}) &= \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\lambda}(\eta) \left(\frac{\partial u^\lambda(\eta)}{\partial \xi} + \theta^{-1}(\eta) I_\sigma^\lambda(\eta) u^\sigma(\eta) \right) + \frac{\partial u^3(\eta)}{\partial x^\beta} + J_{\beta\sigma}(\eta) u^\sigma(\eta) \right) \\ &= \frac{1}{2} g^{\alpha\beta} \left[g_{\beta\lambda} \frac{\partial u^\lambda(\eta)}{\partial \xi} + \nabla_\beta^* u^3 + \theta^{-1}(\eta) (g_{\beta\lambda}(\eta) I_\sigma^\lambda(\eta) + \theta(\eta) J_{\beta\sigma}(\eta)) u^\sigma(\eta) \right]. \end{aligned} \quad (6.9)$$

下面证明公式

$$\begin{cases} g_{\alpha\lambda}(\eta) I_\sigma^\lambda(\eta) + \theta(\eta) J_{\alpha\sigma}(\eta) = 0, \\ I_\lambda^\alpha(\eta) = -b_\lambda^\alpha(\eta) + K(\eta) \xi \delta_\lambda^\alpha, \\ J_{\beta\lambda}(\eta) = b_{\beta\lambda}(\eta) - \xi c_{\beta\lambda}(\eta), \\ \theta(\eta) = 1 - 2H(\eta) \xi + K(\eta) \xi^2. \end{cases} \quad (6.10)$$

实际上,

$$\begin{aligned} g_{\alpha\lambda}(\eta) I_\sigma^\lambda(\eta) &= (a_{\alpha\lambda}(\eta) - 2\xi b_{\alpha\lambda}(\eta) + \xi^2 c_{\alpha\lambda}(\eta)) (-b_\sigma^\lambda(\eta) + K(\eta) \xi \delta_\sigma^\lambda) \\ &= -b_{\alpha\sigma}(\eta) + K(\eta) \xi a_{\alpha\sigma}(\eta) + 2\xi b_{\alpha\lambda}(\eta) b_\sigma^\lambda(\eta) - 2\xi^2 K(\eta) b_{\alpha\sigma}(\eta) \\ &\quad - \xi^2 c_{\alpha\lambda}(\eta) b_\sigma^\lambda(\eta) + \xi^3 K(\eta) c_{\alpha\sigma}(\eta). \end{aligned}$$

应用

$$\begin{aligned} b_{\alpha\lambda}(\eta) b_\sigma^\lambda(\eta) &= c_{\alpha\sigma}(\eta), \quad K(\eta) a_{\alpha\lambda}(\eta) - 2H(\eta) b_{\alpha\lambda}(\eta) + c_{\alpha\lambda}(\eta) = 0, \\ c_{\alpha\lambda}(\eta) b_\sigma^\lambda(\eta) &= (2H(\eta) b_{\alpha\lambda}(\eta) - K(\eta) a_{\alpha\lambda}(\eta)) b_\sigma^\lambda(\eta) = 2H(\eta) c_{\alpha\sigma}(\eta) - K(\eta) b_{\alpha\sigma}(\eta), \end{aligned}$$

代入前式得

$$\begin{aligned} g_{\alpha\lambda}(\eta) I_\sigma^\lambda(\eta) &= -b_{\alpha\sigma}(\eta) + K(\eta) \xi a_{\alpha\sigma} + 2\xi c_{\alpha\sigma}(\eta) - 2\xi^2 K(\eta) b_{\alpha\sigma}(\eta) \\ &\quad - \xi^2 [2H(\eta) c_{\alpha\sigma}(\eta) - K(\eta) b_{\alpha\sigma}(\eta)] + \xi^3 K(\eta) c_{\alpha\sigma}(\eta) \\ &= -b_{\alpha\sigma}(\eta) (1 - 2H(\eta) \xi + K(\eta) \xi^2) + \xi c_{\alpha\sigma} (1 - 2H(\eta) \xi + K(\eta) \xi^2) \\ &= -J_{\alpha\sigma}(\eta) \theta(\eta). \end{aligned}$$

这就推出 (6.10). 代入 (6.9) 得

$$e^{3\alpha}(\mathbf{u}) = \frac{1}{2} g^{\alpha\beta} \left[g_{\beta\lambda} \frac{\partial u^\lambda(\eta)}{\partial \xi} + \nabla_\beta^* u^3 \right] = \frac{1}{2} \delta_\lambda^\alpha u_1^\lambda(\eta) + 0 = \frac{1}{2} u_1^\alpha(\eta).$$

所以, 注意到 $p_0(\eta) = \underline{p}_0(\eta)$,

$$\begin{aligned} \sigma^{ij}(\mathbf{u}, p) n_i |_{\mathfrak{S}(\eta)} &= \sigma^{3j}(\mathbf{u}, p) |_{\mathfrak{S}(\eta)} = \{\sigma^{3\alpha}(\mathbf{u}, p), \sigma^{33}(\mathbf{u}, p)\} |_{\mathfrak{S}(\eta)}, \\ \sigma^{33}(\mathbf{u}, p) |_{\mathfrak{S}(\eta)} &= -g^{33}(\eta) p_0(\eta) + 2\mu \underline{e}^{33}(\mathbf{u}) \\ &= -p_0(\eta) + \mu (g^{33}(\eta) \underline{\nabla}_3 u^3(\eta) + g^{33}(\eta) \underline{\nabla}_3 u^3(\eta)) \end{aligned}$$

$$= -p_0(\eta) + 2\mu \frac{\partial u^3}{\partial \xi}(\eta) = -p_0(\eta) + 2\mu u_1^3(\eta) = -p_0(\eta),$$

应用 (6.10) 和 $\mathbf{u}(\eta)|_{\mathfrak{S}(\eta)} = 0$ 得

$$\begin{aligned} \sigma^{\alpha 3}(\mathbf{u}, p)|_{\mathfrak{S}(\eta)} &= -p_0(\eta)g^{\alpha 3}(\eta) + 2\mu \underline{e}^{\alpha 3}(\mathbf{u}(\eta)) \\ &= \mu[g^{3k}(\eta)\nabla_k u^\alpha(\eta) + g^{\alpha k}(\eta)\nabla_k u^3(\eta)] = \mu(\nabla_3 u^\alpha(\eta) + g^{\alpha\beta}(\eta)\nabla_\beta u^3(\eta)) \\ &= \mu\left[\frac{\partial u^\alpha(\eta)}{\partial \xi} + \theta^{-1}(\eta)I_\beta^\alpha(\eta)u^\beta(\eta) + g^{\alpha\beta}(\eta)(\partial_\beta u^3(\eta) + J_{\beta\lambda}(\eta)u^\lambda(\eta))\right] \\ &= \mu[\underline{u}_1^\alpha(\eta) + g^{\alpha\beta}(\eta)\partial_\beta u^3(\eta) + \theta^{-1}(\eta)g^{\alpha\beta}(\eta)(g_{\beta\lambda}(\eta)I_\sigma^\lambda(\eta) + \theta(\eta)J_{\beta\sigma}(\eta))\underline{u}^\sigma(\eta)] \\ &= \mu \underline{u}_1^\alpha(\eta), \end{aligned}$$

这就推出 (6.6) 的第二式. 证毕. $\square$

下面考查阻力泛函的形状梯度 $\text{Grad}J(\mathfrak{S})$. 对不可压情形, 有如下引理.

引理 22 阻力泛函沿 ζ 方向的梯度

$$\begin{aligned} \delta J(\mathfrak{S})\zeta &= \langle \text{Grad}J(\mathfrak{S}), \zeta \rangle \\ &= \int_D [-(\widehat{p}\zeta + \gamma_0(\zeta)p_0)V_\infty^3 + \mu(a_{\alpha\beta}\widehat{u}_1^\alpha\zeta + (a_{\alpha\beta}\gamma_0(\zeta) + 2\gamma_{\alpha\beta}(\zeta))u_1^\alpha)V_\infty^\beta]\sqrt{a}dx, \end{aligned} \quad (6.11)$$

其中 $(p_0, \mathbf{u}_1)$ 和 $(\widehat{p}_0, \widehat{\mathbf{u}}_1)$ 分别是边界层方程 (4.42) 和它关于形状的导数的方程 (5.45) 的解.

证明 从 (6.3) 和 (6.6) 可得

$$\begin{aligned} \delta J(\mathfrak{S})\zeta &= \langle \text{Grad}J(\mathfrak{S}), \zeta \rangle \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{\mathfrak{S}(t\zeta)} [\sigma^{ij}(\mathbf{u}(t\zeta), p(t\zeta))n_i(t\zeta)g_{jk}(t\zeta)V_\infty^k] dS(t\zeta) \right. \\ &\quad \left. - \int_{\mathfrak{S}} [\sigma^{ij}(\mathbf{u}(t\zeta), p(t\zeta))n_i(t\zeta)g_{jk}(t\zeta)V_\infty^k] dS \right) \\ &= \lim_{t \rightarrow 0} \left(\int_D [-p_0(t\zeta)V_\infty^3 + \mu \underline{u}_1^\alpha(t\zeta)a_{\alpha\beta}(t\zeta)V_\infty^\beta]\sqrt{a(t\zeta)}dx - \int_D [-p_0V_\infty^3 + \mu a_{\alpha\beta}u_1^\alpha V_\infty^\beta]\sqrt{a}dx \right), \end{aligned}$$

因为

$$\begin{aligned} dS(t\zeta) &= \sqrt{a(t\zeta)}dx = \left(1 + \frac{t}{2}\gamma_0(\zeta)\right)\sqrt{a}dx, \quad a_{\alpha\beta}(t\zeta) = a_{\alpha\beta} + 2t\gamma_{\alpha\beta}(\zeta), \\ dS &= \sqrt{a}dx, \quad \gamma_0(t\eta) = t\gamma_0(\eta) = t(\text{div}^* \eta - 2H\eta^3), \\ \left(1 + \frac{t}{2}\gamma_0(\zeta)\right)(a_{\alpha\beta} + 2t\gamma_{\alpha\beta}(\zeta)) &= a_{\alpha\beta} + t\left(\frac{1}{2}a_{\alpha\beta}\gamma_0(\zeta) + 2\gamma_{\alpha\beta}(\zeta)\right) + t^2\gamma_0(\zeta)\gamma_{\alpha\beta}(\zeta), \end{aligned}$$

代入上式后, 由定理 16, 可以进行极限过度, 得到

$$\begin{aligned} \delta J(\mathfrak{S})\zeta &= \langle \text{Grad}J(\mathfrak{S}), \zeta \rangle \\ &= \lim_{t \rightarrow 0} \left(\int_D [-(p_0(t\zeta) - p_0) + t\gamma_0(\zeta)p_0(t\zeta)]V_\infty^3 \right. \\ &\quad \left. + (\mu a_{\alpha\beta}(u_1^\alpha(t\zeta) - u_1^\alpha) + t\mu(a_{\alpha\beta}\gamma_0(\zeta) + 2\gamma_{\alpha\beta}(\zeta))u_1^\alpha(t\zeta))V_\infty^\beta]\sqrt{a}dx + o(t^2) \right) \end{aligned}$$

$$= \int_D [-\widehat{p}(\zeta) + \gamma_0(\zeta)p_0)V_\infty^3 + \mu(a_{\alpha\beta}\widehat{u}_1^\alpha(\zeta) + (a_{\alpha\beta}\gamma_0(\zeta) + 2\gamma_{\alpha\beta}(\zeta))u_1^\alpha)V_\infty^\beta]\sqrt{a}dx.$$

这就推出 (6.12). 证毕. $\square$

下面给出任何的在变形后 $\mathfrak{S}(\eta)$ 沿任何方向 ζ 的梯度.

引理 23 $\forall \zeta, \eta \in V(D)$, 在 $\mathfrak{S}$ -半测地坐标系下, 阻力泛函在 η 处沿方向 ζ 的梯度计算公式如下:

$$\begin{aligned} \langle \text{Grad}J(\mathfrak{S}(\eta)), \zeta \rangle &= (J(\mathfrak{S}(\eta + t\zeta)))'|_{t=0} \\ &= \int_D \left[-\left(\left(1 + \frac{1}{2}\gamma_0(\eta) \right) \widehat{p}_0(\eta)\zeta + \frac{1}{2}\gamma_0(\zeta)p_0(\eta) \right) V_\infty^3 \right. \\ &\quad + \mu \left(1 + \frac{1}{2}\gamma_0(\eta) \right) (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta)) \widehat{u}_1^\alpha(\eta)\zeta \\ &\quad \left. + \mu \left[2 \left(1 + \frac{1}{2}\gamma_0(\eta) \right) \gamma_{\alpha\beta}(\zeta) + \frac{1}{2}\gamma_0(\zeta)(a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta)) \right] u_1^\alpha(\eta) V_\infty^\beta \right] \sqrt{a} dx, \end{aligned} \quad (6.12)$$

这里 $(u_1(\eta), p_0(\eta))$ 是边界层方程 (5.15) 的解, $(\widehat{u}_1^\alpha(\eta)\zeta, \widehat{p}_0(\eta)\zeta)$ 是边界层方程的解关于形状的 Gâteaux 导数方程 (5.57) 的解.

证明 与引理 22 的证明一样,

$$\begin{aligned} \langle \text{Grad}J(\mathfrak{S}(\eta)), \zeta \rangle &= (J(\mathfrak{S}(\eta + t\zeta)))'|_{t=0} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left[\int_D (\mu u_1^\alpha(\eta + t\zeta) a_{\alpha\beta}(\eta + t\zeta) V_\infty^\beta - p_0(\eta + t\zeta) V_\infty^3) \sqrt{a(\eta + t\zeta)} dx \right. \\ &\quad \left. - \int_D (\mu u_1^\alpha(\eta) a_{\alpha\beta}(\eta) V_\infty^\beta - p_0(\eta) V_\infty^3) \sqrt{a(\eta)} dx \right]. \end{aligned} \quad (6.13)$$

应用定理 1 和 2,

$$\begin{aligned} \sqrt{a(\eta + t\zeta)} &= \left(1 + \frac{1}{2}\gamma_0(\eta + t\zeta) \right) \sqrt{a} = \left(1 + \frac{1}{2}\gamma_0(\eta) + \frac{t}{2}\gamma_0(\zeta) \right) \sqrt{a}, \\ \sqrt{a(\eta)} &= \left(1 + \frac{1}{2}\gamma_0(\eta) \right) \sqrt{a}, \\ a_{\alpha\beta}(\eta + t\zeta) &= a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta + t\zeta) = a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta) + 2t\gamma_{\alpha\beta}(\zeta), \\ a_{\alpha\beta}(\eta) &= a_{\alpha\beta} = 3\gamma_{\alpha\beta}(\eta). \end{aligned}$$

经过简单计算,

$$\begin{aligned} &p_0(\eta + t\zeta) \left(1 + \frac{1}{2}\gamma_0(\eta) + \frac{t}{2}\gamma_0(\zeta) \right) - p_0(\eta) \left(1 + \frac{1}{2}\gamma_0(\eta) \right) \\ &= \left(1 + \frac{1}{2}\gamma_0(\eta) \right) (p_0(\eta + t\zeta) - p_0(\eta)) + \frac{1}{2}t\gamma_0(\zeta)p_0(\eta + t\zeta), \\ &\left(1 + \frac{1}{2}\gamma_0(\eta + t\zeta) \right) (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta) + 2t\gamma_{\alpha\beta}(\zeta)) \mu u_1^\alpha(\eta + t\zeta) - \left(1 + \frac{1}{2}\gamma_0(\eta) \right) (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta)) \mu u_1^\alpha(\eta) \\ &= \mu \left(1 + \frac{1}{2}\gamma_0(\eta) \right) (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta)) (u_1^\alpha(\eta + t\zeta) - u_1^\alpha(\eta)) \\ &\quad + t\mu \left[2 \left(1 + \frac{1}{2}\gamma_0(\eta) \right) \gamma_{\alpha\beta}(\zeta) + \frac{1}{2}\gamma_0(\zeta)(a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta)) \right] u_1^\alpha(\eta + t\zeta), \end{aligned}$$

代入 (6.14), 应用定理 16 进行极限过度, 得

$$\langle \text{Grad} J(\mathfrak{S}(\eta)), \zeta \rangle \quad (6.14)$$

$$\begin{aligned} &= (J(\mathfrak{S}(\eta + t\zeta)))' |_{t=0} \\ &= \int_D \left[- \left(\left(1 + \frac{1}{2} \gamma_0(\eta) \right) \widehat{p}_0(\eta, \zeta) + \frac{1}{2} \gamma_0(\zeta) p_0(\eta) \right) V_\infty^3 + \mu \left(1 + \frac{1}{2} \gamma_0(\eta) \right) (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta)) \widehat{u}_1^\alpha(\eta, \zeta) \right. \\ &\quad \left. + \mu \left[2 \left(1 + \frac{1}{2} \gamma_0(\eta) \right) \gamma_{\alpha\beta}(\zeta) + \frac{1}{2} \gamma_0(\zeta) (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta)) \right] u_1^\alpha(\eta) V_\infty^\beta \right] \sqrt{a} dx. \end{aligned} \quad (6.15)$$

这就推出 (6.13). 证毕. $\square$

7 阻力泛函的共轭梯度算法

考查极小化问题 (6.2),

$$J(\mathfrak{S}) = \inf_{S \in \mathcal{F}} J(S) = \inf_{\eta \in V} J(\mathfrak{S}(\eta)), \quad (7.1)$$

其中阻力泛函 $J(\mathfrak{S}) = \int_{\mathfrak{S}} \sigma^{ij}(\mathbf{u}, p) n_i g_{jk} V_\infty^j dS$, 这里 V_∞ 是无穷远来流的速度. 从引理 21,

$$\begin{aligned} \varphi(t) := J(\mathfrak{S}(\eta + t\zeta)) &= \int_D \left[\mu \left(1 + \frac{1}{2} \gamma_0(\eta + t\zeta) \right) (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta + t\zeta)) u_1^\alpha(\eta + t\zeta) V_\infty^\beta \right. \\ &\quad \left. - \left(1 + \frac{1}{2} \gamma_0(\eta + t\zeta) \right) p_0(\eta + t\zeta) V_\infty^3 \right] \sqrt{a} dx, \end{aligned} \quad (7.2)$$

因为

$$\begin{cases} \mu \left(1 + \frac{1}{2} \gamma_0(\eta + t\zeta) \right) (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta + t\zeta)) = \sum_{k=0}^2 \rho_{\alpha\beta}^k(\eta, \zeta) t^k, \\ \rho_{\alpha\beta}^0(\eta, \zeta) = \mu (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta)) \left(1 + \frac{1}{2} \gamma_0(\eta) \right), \\ \rho_{\alpha\beta}^1(\eta, \zeta) = \mu \left[2 \left(1 + \frac{1}{2} \gamma_0(\eta) \right) \gamma_{\alpha\beta}(\zeta) + \frac{1}{2} \gamma_0(\zeta) (a_{\alpha\beta} + 2\gamma_{\alpha\beta}(\eta)) \right], \\ \rho_{\alpha\beta}^2(\eta, \zeta) = \mu \gamma_0(\zeta) \gamma_{\alpha\beta}(\zeta), \quad q^0 = 1 + \frac{1}{2} \gamma_0(\eta), \quad q^1 = \frac{1}{2} \gamma_0(\zeta), \end{cases} \quad (7.3)$$

以及

$$u_1^\alpha(\eta + t\zeta) = \widehat{u}_1^\alpha(\eta, \zeta)t + u_1^\alpha(\eta), \quad p_0(\eta + t\zeta) = t\widehat{p}_0(\eta, \zeta) + p_0(\eta),$$

从而, 有

$$\varphi(t) = \sum_{k=0}^3 \varphi_k(\eta, \zeta) t^k, \quad (7.4)$$

其中

$$\begin{cases} \varphi_1(\eta, \zeta) = \int_D [(\rho_{\alpha\beta}^0(\eta, \zeta)\widehat{u}_1^\alpha(\eta, \zeta) + \rho_{\alpha\beta}^1(\eta, \zeta)u_1^\alpha(\eta))V_\infty^\beta \\ \quad - (q^1(\eta, \zeta)p_0(\eta) + q^0(\eta)\widehat{p}_0(\eta, \zeta))V_\infty^3]\sqrt{a}dx, \\ \varphi_2(\eta, \zeta) = \int_D [(\rho_{\alpha\beta}^1(\eta, \zeta)\widehat{u}_1^\alpha(\eta, \zeta) + \rho_{\alpha\beta}^2(\eta, \zeta)u_1^\alpha(\eta))V_\infty^\beta \\ \quad - (q^1(\eta, \zeta)\widehat{p}_0(\eta, \zeta) + q^0(\eta)p_0(\eta))V_\infty^3]\sqrt{a}dx, \\ \varphi_3(\eta, \zeta) = \int_D [(\rho_{\alpha\beta}^2(\eta, \zeta)\widehat{u}_1^\alpha(\eta, \zeta)V_\infty^\beta]\sqrt{a}dx, \end{cases} \quad (7.5)$$

曲面变形是由任一曲面 $\mathfrak{S}$ 的每一点得到一个任意位移 η 后, 而形成一个新的曲面 $\mathfrak{S}(\eta)$. 优化曲面以便减小阻力, 从 $\mathfrak{S}$ 到 $\mathfrak{S}(\eta)$, 其中 η 是变形向量. 最优解就在于搜索 η .

下面给出共轭梯度算法来得到最优曲面. 设 $\eta^0 \in V$ 给定, 求梯度

$$(g^0, v) = \langle \text{Grad}J(\eta^0), v \rangle, \quad \forall v \in V. \quad (7.6)$$

令 $\zeta^0 = g^0$. 设 (η^n, g^n, ζ^n) 已知, 用下列程序计算 $(\eta^{n+1}, g^{n+1}, \zeta^{n+1})$,

(1) 最速下降方向:

$$\begin{cases} \rho_n = \arg \min_{\rho} \{J(\eta^n - \rho\zeta^n)\}, \\ \eta^{n+1} = \eta^n - \rho_n\zeta^n; \end{cases} \quad (7.7)$$

(2) 新的下降方向:

$$\text{求 } g^{n+1}, \text{ 使得 } (g^{n+1}, v) = \langle \text{Grad}J(\eta^{n+1}), v \rangle, \quad \forall v \in V. \quad (7.8)$$

如果 $\|g^{n+1}\|/\|g^0\| \leq \varepsilon$, 取 $\eta^* = \eta^{n+1}$, 否则计算

$$\gamma_n = \|g^{n+1}\|^2 / \|g^n\|^2 \quad (\text{Fletcher-Reeves 原则})$$

或

$$\gamma_n = \langle g^{n+1} - g^n, g^{n+1} \rangle / \|g^n\|^2 \quad (\text{Polak-Ribiere 原则}), \quad \zeta^{n+1} = g^n + \gamma_n \zeta^n.$$

返回到 (1), 存在很多类型的共轭梯度方法, 它们的不同点在于共轭方法步长的选择.

毫无疑问, 在共轭梯度算法中, 最关键和最繁重的工作是梯度算子计算和一维极小化问题求解. 在梯度算子计算可利用 (6.12), 这其中, 我们不但要求 $u_1^\alpha(\eta)$ (求解 (5.16)), 而且还要求 Gâteaux 导算子 $\widehat{u}_1^\alpha(\eta, \zeta)$ (求解 (5.37)). 求解一维极小化问题 (7.7), 有下面的引理.

引理 24 极小值问题 (7.7) 的解是下列三次多项式方程的根 $\frac{\partial \varphi(\rho)}{\partial \rho} = \varphi_1 + 2\varphi_2\rho + \varphi_3\rho^2 + \varphi_4\rho^3 = 0$. 它至少有一个根. 其中 $\varphi(\rho)$ 是由 (7.2) 所定义的. $\varphi_k(\eta, \zeta)$ 是由 (7.5) 所定义.

证明 综合 (7.2)–(7.5) 就可以得到引理的结论. $\square$

8 计算实例

假设关于曲面 S 上的升阻比函数为 $f(S)$, 我们的目标是优化 S 使得 $f(S)$ 到达最大. 令 $\mathfrak{S}'(\eta) = \mathfrak{S}_0 + \eta$, $\eta = (\delta \sin(s_1) \sin(s_2)) \cdot \mathbf{n}$, 其中 S_0 为初始曲面, s_1 和 s_2 为曲面的正交局部坐标 (为了保证 S' 和 S_0

具有相同的边界, 需要对 s_1 和 s_2 乘以适当的因子), 这样升阻比函数 $f(S)$ 关于曲面的优化问题就转换为关于参数 δ 的优化问题.

如果认为优化参数过少不够精确, 可以多取几项, 如

$$\mathfrak{S}'(\eta) = \mathfrak{S}_0 + \eta, \quad \eta = (\delta_1 \sin(s_1) \sin(s_2) + \delta_2 \sin(2s_1) \sin(2s_2) + \cdots) \cdot \mathbf{n}.$$

我们令来流攻角 $\alpha = 0$, 远场压力 $p = 12219.74$ pa, 来流速度 $U = 260.39$ m/s, 来流温度 $T = 300$ K. 接下来给出优化结果. 其中图 3 为优化后曲面比较; 图 4 为边界层方程计算下, 优化后升阻力系数; 图 5 为有限体积方法下两种不同网格得到的升阻力系数和攻角关系; 图 6 和 7 为边界层方程计算下, 优化后压力分布; 图 8 为边界层方程与有限体积方法的机身压力分布对比; 图 9 为边界层方程计算下, 机身表面优化的流线图; 图 10 为边界层方程与有限体积方法的机身表面流线分布对比.

图 3 (a) 机身曲面 (红色部分), (b) 1 次优化后曲面, (c) 2 次优化后曲面, (d) 3 次优化后曲面

图 4 边界层方程结果, (a) 升力系数 C_l , (b) 阻力系数 C_d , (c) 升阻比 C_l/C_d

图 5 有限体积结果, (a) 和 (b) 是两种网格下的: (a) 升力系数 C_l , (b) 阻力系数 C_d

图 6 (a) 原始机身压力分布, (b) 1 次优化压力分布, (c) 2 次优化压力分布, (d) 3 次优化压力分布

图 7 (a) 原始机身压力分布, (b) 1 次优化压力分布, (c) 2 次优化压力分布, (d) 3 次优化压力分布

图 8 (a) 边界层方程计算机身压力分布, (b) 有限体积计算机身压力分布

图 9 (a) 原始机身表面流线, (b) 1 次优化表面流线, (c) 2 次优化表面流线, (d) 3 次优化表面流线

图 10 (a) 边界层方程计算机身表面流线, (b) 有限体积计算机身表面流线

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A new boundary layer equation and its applications to the shape optimal control

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Abstract In this paper, the second order partial differential equations of the velocity gradient and pressure for the flow on a solid boundary (two-dimensional manifold) are proposed. Thus, the normal stress on the boundary, the drag and lift of the moving body in the flow are obtained. The key innovation of our method is that the velocity gradient on the boundary which must be computed in the shape optimization is obtained not from numerical difference of velocity at the boundary layer, but from solving new boundary layer equations which are proposed by us. Moreover, we prove the posedness of boundary layer equations and give the equations satisfied by the Gâteaux derivative of the solutions to boundary layer equations with respond to the shape. At last, we give an application, which applies our method to the shape optimal control of the airplane, and the calculated form for the first variation of the drag functional to the shape. Numerical example shows the accuracy of drag obtained by using our method higher than by standard software.

Keywords boundary layer equations, drag, shape optimal control, Navier-Stokes equations

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