Vol. 45 No. 2 SCIENCE IN CHINA (Series A) February 2002

Holomorphic map with unipotent Jacobian matrices in C^3

CHEN Zhihua (陈志华) & CHEN Wenge (陈文革)

Department of Mathematics,Tongji University, Shanghai 200092, China
Correspondence should be addressed to Chen Zhihua (email: zzzhhc@online.sh.cn)

Received February 20, 2001

Abstract We prove that the holomorphic unipotent Jacobian conjecture is valid when $n = 3$.

Keywords: Jacobian conjecture, holomorphic unipotent map, analytic variety.

The Jacobian conjecture seems to have been first formulated by O.H. Keller in 1939. Aside from the trivial case $n = 1$, it remains an open problem for all $n \geqslant 2$.

In 1982, Bass et al.[1] proved the reduction theorem; i.e. if for all n and all $F \in MA_n(k)$ of the form $F = X + N$ with N cubic homogeneous and $J(N)$ nilpotent F is invertible, for all $F \in MA_n(k)$ with $J(F)$ invertible, F is invertible. And they have also proved that for all n and all $F \in MA_n(k)$ of the form $F = X + N$ with N homogeneous and $(J(N))^2 = 0$, F is invertible.

In 1999, Chen[2] formulated the holomorphic unipotent Jacobian conjecture; i.e. if $H : C^n \longrightarrow C^n$ is a holomorphic map with $(JH(z))^2 = 0$, then $Z + H$ is invertible, and he proved that it is valid in the case $n = 2$. In this paper, we give a proof of the Unipotent Jacobian Conjecture for holomorphic map in the case $n = 3$.

1 Preliminaries[3—6]

We start this section with Theorem A[2] that can be proved directing by the Taylor expansion.

Theorem A. Let $H : C^n \longrightarrow C^n$ be a holomorphic map. Then the following statements are equivalent:

(i) $(JH(z))^2 = 0$ for all $z \in C^n$;

(ii) $H(z + JH(z)z') = H(z)$ for all $z, z' \in C^n$;

(iii) $JH(z + JH(z)z')JH(z) = 0$ for all $z, z' \in C^n$.

Proposition 1. Let A be an $n \times n$ matrix and $A^2 = 0$. Then rank$A \leqslant [n/2]$.

Proof. We use the reduction to absurdity. Suppose rank$A = m > [n/2]$. Then there are invertible matrices P, Q such that

$$A = P \cdot \begin{pmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{pmatrix} \cdot Q, \tag{1}$$

where $A^2 = 0$. Hence

$$0 = A^2$$

$$= P \cdot \begin{pmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{pmatrix} \cdot Q \cdot P \cdot \begin{pmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{pmatrix} \cdot Q.$$

Set

$$R = \begin{pmatrix} r_{11} & \dots & r_{1n} \\ \dots & \dots & \dots \\ r_{n1} & \dots & r_{nn} \end{pmatrix} \equiv Q \cdot P \cdot \begin{pmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{pmatrix}.$$

It is obvious that $\text{rank} R = m$ and

$$\begin{pmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{pmatrix} \cdot R = 0. \tag{2}$$

By eq. (2), we have $r_{ij} = 0, 1 \leqslant i \leqslant m$. This contradicts $m > [n/2]$, so that $\text{rank} A \leqslant [n/2]$.

Proposition 2. Let $H : C^n \to C^n$ be a holomorphic map with $(JH(z))^2 = 0$; $G = A \cdot z^{\mathrm{T}} + \gamma^{\mathrm{T}}$, where $\gamma = (\gamma_1, ..., \gamma_n)$ is a constant vector and A is an invertible matrix. Let $N := G \circ H \circ G^{-1}$. Then

(i) $(JN(z))^2 = 0$,

(ii) $Z + H$ is invertible if and only if $Z + N$ is invertible.

Proof. (i) $(JN(z))^2 = (A \cdot JH(G^{-1}(z)) \cdot A^{-1})^2$
$= A \cdot (JH(G^{-1}(z))^2 \cdot A^{-1} = 0.$

(ii) Suppose $Z+H$ is invertible. Let F^{-1} be the invertible map of $Z+H$. Then $F^{-1} \circ (Z+H) = Z$. Therefore

$$Z = F^{-1} \circ G^{-1} \circ G \circ (Z + H) \circ G^{-1} \circ G = F^{-1} \circ G^{-1} \circ (Z + N) \circ G.$$

Then

$$G \circ F^{-1} \circ G^{-1} \circ (Z + N) = Z.$$

Proposition 3. Let $H : C^n \to C^n$ be a holomorphic map, $S := \{z \in C^3 | JH(z) = 0\}$. Then H is a constant on every connected component of S.

Proof. Without losing generality, we assume S is connected. $S = \cup_{\alpha \in I} S_\alpha$, where every S_α is the irreducible component of S, then $S_\alpha = R(S_\alpha) \cup S(S_\alpha)$, where $R(S_\alpha)$ is the regular points set of $S_\alpha, S(S_\alpha)$ is the singular points set of S_α. Because $R(S_\alpha)$ is a complex manifold and $JH(z) = 0$, H is a constant on $R(S_\alpha)$. By the continuity, H is a constant on S_α. Since S is connected, H is a constant on S.

2 Main result

Suppose $JH(z) \equiv 0$. Then $Z + H$ is a translation. Evidently, $Z + H$ is invertible. So we only need to prove the case for $JH(z) \not\equiv 0$.

Let $H : C^3 \to C^3$ be a holomorphic map with $(JH(z))^2 = 0$. If $JH(z_0) \neq 0$, $z_0 \in C^3$, without losing generality, we assume

$$(D_1H_1(z_0), D_1H_2(z_0), D_1H_3(z_0)) \neq 0.$$

By Proposition 1, we have $\mathrm{rank}JH(z_0)=1$, so there are $\lambda_1,\lambda_2\in C$ such that

$$(D_2H_1(z_0),D_2H_2(z_0),D_2H_3(z_0))=\lambda_1\cdot(D_1H_1(z_0),D_1H_2(z_0),D_1H_3(z_0)),$$

$$(D_3H_1(z_0),D_3H_2(z_0),D_3H_3(z_0))=\lambda_2\cdot(D_1H_1(z_0),D_1H_2(z_0),D_1H_3(z_0)).$$

Therefore

$$z_0^{\mathrm{T}}+JH(z_0)(z')^{\mathrm{T}}=z_0^{\mathrm{T}}+(z_1'+\lambda_1z_2'+\lambda_2z_3')(D_1H_1(z_0),D_1H_2(z_0),D_1H_3(z_0))^{\mathrm{T}},$$

where $z'\in C^3$ must be a complex line through the point z_0.

Lemma 1. Let $H:C^n\to C^n$ be a holomorphic map with $(JH(z))^2=0$. Suppose holomorphic unipotent Jacobian conjecture is valid in C^{n-1}, and there exists a vector $(a_1,a_2,...,a_n)\neq 0$ such that $(a_1,a_2,...,a_n)\cdot JH(z)=0\ \forall z\in C^n$. Then $Z+H$ is invertible.

Proof. Let $G(z)=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & \iddots & \vdots\\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n}\\ a_1 & a_2 & \cdots & a_n\end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_n\end{pmatrix}$,

where

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & \iddots & \vdots\\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n}\\ a_1 & a_2 & \cdots & a_n\end{pmatrix}$$

is an invertible matrix.

Let $M:=(M_1,M_2,\cdots M_n)=G\circ H\circ G^{-1}$. Then

$$JM(z)=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & \iddots & \vdots\\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n}\\ a_1 & a_2 & \cdots & a_n\end{pmatrix}\cdot JH(G^{-1}(z))\cdot\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & \iddots & \vdots\\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n}\\ a_1 & a_2 & \cdots & a_n\end{pmatrix}^{-1}.$$

Obviously, we have $(JM(z))^2=0$.

On the other hand, $(a_1,a_2,...,a_n)\cdot JH(z)=0\ \forall z\in C^n$. So M_n must be a constant. We define $T=(T_1,\cdots,T_n):=G'\circ M\circ(G')^{-1}$ where G' is a translation map on C^n

$$G'(z_1,z_2,\cdots,z_n)=(z_1,z_2,\cdots,z_n-M_n),$$

then

$$(JT(z))^2=(JM((G')^{-1}(z)))^2=0,$$

and

$$T_n=0.$$

By Proposition 2, $Z+H$ is invertible if and only if $Z+T$ is invertible. So it is enough to prove $Z+T$ is invertible. Set $z_n = c$ and $z' = (z_1, z_2, \cdots, z_{n-1})$. Let $T' = (T_1(z', c), T_2(z', c), \cdots, T_{n-1}(z', c))$. It is obvious that T' is a holomorphic map and

$$(JT(z',c))^2 = \begin{pmatrix} D_1T_1(z',c) & D_2T_1(z',c) & \cdots & D_nT_1(z',c) \\ \vdots & \vdots & \iddots & \vdots \\ D_1T_{n-1}(z',c) & D_2T_{n-1}(z',c) & \cdots & D_nT_{n-1}(z',c) \\ 0 & 0 & \cdots & 0 \end{pmatrix}^2$$

$$= \begin{pmatrix} \begin{pmatrix} D_1T_1(z',c) & \cdots & D_{n-1}T_1(z',c) \\ \vdots & \ddots & \vdots \\ D_1T_{n-1}(z',c) & \cdots & D_{n-1}T_{n-1}(z',c) \end{pmatrix}^2 & \begin{matrix} * \\ \vdots \end{matrix} \\ 0 \quad \cdots \quad 0 & 0 \end{pmatrix} = \begin{pmatrix} (JT'(z',c))^2 & * \\ 0 \ \cdots \ 0 & 0 \end{pmatrix}.$$

From $(JT'(z',c))^2 = 0$ and that the holomorphic unipotent Jacobian conjecture holds in C^{n-1}, there exists a holomorphic map $N'(z',c) = (N'_1(z',c), \cdots, N'_{n-1}(z',c))$ such that

$$N'(z',c) \circ (Z' + T') = Z',$$

i.e.

$$N'_i(z_1 + T_1(z',c), z_2 + T_2(z',c), \cdots, Z_{n-1} + T_{n-1}(z',c), c) = z_i; \quad i = 1, 2, \cdots, n-1.$$

Let

$$N(z', z_n) = (N'_1(z', z_n), \cdots, N'_{n-1}(z', z_n), z_n) : C^n \to C^n.$$

We have

$$N'_i(z_1 + T_1(z', z_n), z_2 + T_2(z', z_n), \cdots, Z_{n-1} + T_{n-1}(z', z_n), z_n) = z_i; \quad i = 1, 2, \cdots, n-1,$$

i.e.

$$N \circ (Z + T) = Z. \tag{3}$$

Next, we will prove that $N(z)$ is a holomorphic map. Let $z_0 \in C^n$. Since, $\det(I + JT(z_0)) \neq 0$, there exists an open neighbourhood $U(z_0)$ of z_0 such that $(Z+T)|_{U(z_0)}$ is biholomorphic. By eq. (3), $N(z)$ is holomorphic at z_0, so that $N(z)$ is a holomorphic map.

According to the identity theorem of several complex variables, if there exists a nonempty open set $U \subset C^n$ such that $(a_1, a_2, \cdots, a_n)JH(z) = 0$; $\forall z \in U$, then

$$(a_1, a_2, \cdots, a_n)JH(z) = 0; \quad \forall z \in C^n.$$

Hence, if we can prove that a holomorphic map $H : C^3 \to C^3$ with $(JH(z))^2 = 0$ there exists $a = (a_1, a_2, a_3) \neq 0$ and a nonempty open set $U \subset C^3$ such that $(a_1, a_2, a_3)JH(z) = 0; \ \forall z \in U$, then $Z + H$ is invertible.

Now we use L_z to denote the complex line $z^{\mathrm{T}} + JH(z)(\omega)^{\mathrm{T}}$, $\omega \in C^3$, and $V(z)$ the direction vector of the complex line L_z, in fact $V(z)$ is the column vector of $JH(z)$.

Definition 1. $z \in C^3 \backslash S$. The complex line L_z has the property (P), if $V(y) = \lambda(y)V(z)$, $\lambda(y) \in C, \forall y \in L_z$.

Lemma 2. Let $H : C^3 \to C^3$ be a holomorphic map with $(JH(z))^2 = 0$. Sequence $\{z_n\}_{n\in N} \subset C^3\backslash S$ and $z \in C^3\backslash S$, $\lim_{n\to\infty} z_n = z$. If all L_{z_n} have the property (P), then L_z has the property (P).

Proof. Since L_{z_n} has the property (P), there exist $\alpha_n, \beta_n \in C^3$, $|\alpha_n| = |\beta_n| = 1, \alpha_n \cdot \bar{\beta}_n = 0$ such that $\alpha_n \cdot V(y) = \beta_n \cdot V(y) = 0, \quad \forall y \in L_{z_n}$, where the point operator is Euclidean inner product. We can choose the subsequence $\{\alpha_{n_i}\}$ and $\{\beta_{n_i}\}$ of $\{\alpha_n\}_{n\in N}$ and $\{\beta_n\}_{n\in N}$ respectively, such that $\lim_{n_i\to\infty}\alpha_{n_i} = \alpha$, $\lim_{n_i\to\infty}\beta_{n_i} = \beta$ then

$$\alpha \cdot \bar{\beta} = 0, \quad |\alpha| = |\beta| = 1,$$

$\forall y \in L_z$, since $\lim_{n_i\to\infty} z_{n_i} = z$, there exists $y_{n_i} \in L_{z_{n_i}}$, such that $\lim_{n_i\to\infty} y_{n_i} = y$ and

$$\alpha \cdot V(y) = \lim_{n_i\to\infty}\alpha_{n_i} \cdot V(y_{n_i}) = 0, \ \ \beta \cdot V(y) = \lim_{n_i\to\infty}\beta_{n_i} \cdot V(y_{n_i}) = 0.$$

Therefore, L_z has the (P) property.

Theorem 1. Let $H : C^3 \to C^3$ be a holomorphic map with $(JH(z))^2 = 0$. If there exists $z_0 \in C^3\backslash S$, such that L_{z_0} does not have property (P), then $Z + H$ is invertible.

Proof. Let $\alpha, \beta \in C^3$, $|\alpha| = |\beta| = 1$, $\alpha \cdot \bar{\beta} = 0$, such that $\alpha \cdot V(z_0) = \beta \cdot V(z_0) = 0$. Since $\alpha \cdot V(y)$ and $\beta \cdot V(y)$, $y \in L_{z_0}$ is a holomorphic function on L_{z_0}.

$$T = \{y \in L_{z_0} \mid \alpha \cdot V(y) = \beta \cdot V(y) = 0\}$$

is a discrete set of L_{z_0}. Hence there exists $y_0 \in L_{z_0}\backslash T$ such that complex line L_{y_0} does not have the property (P). Otherwise, by Lemma 2, L_{z_0} possesses the property (P), which contradicts the assumption that L_{z_0} does not have property (P).

Now we choose a disk $D(y, \varepsilon)$ on L_{z_0}, such that $JH(y) \neq 0$; $\forall y \in D(y_0, \varepsilon)$. So we can get a complex line set $M_\varepsilon = \{L_y \mid y \in D(y_0, \varepsilon)\}$. By Theorem A, $H|_{M_\varepsilon}$ is a constant map. Let M be the complex plane spanned by L_{y_0} and L_{z_0}.

1) If there exists $\varepsilon > 0$ such that $M_\varepsilon \subset M$, obviously, M_ε includes a nonempty open set of M, by the identity theorem, H restricted on M is a constant map.

Noting $(a_1, a_2, a_3) \neq 0$ is the normal vector of complex plane M, $(a_1, a_2, a_3)JH(z_0) = (a_1, a_2, a_3)JH(y_0) = 0$.

Next, we will prove that there are

$$(a_1, a_2, a_3)JH(z) = 0, \quad \forall z \in C^3. \tag{4}$$

If there exists $\omega \in C^3\backslash S$ such that $(a_1, a_2, a_3)JH(\omega) \neq 0$, then complex line L_ω must be intersected with complex plane M. By the continuity, we can get an open neighbourhood U of ω such that $JH(z) \neq 0; \quad \forall z \in U$ and L_z intersects with M for all $z \in U$. Theorem A shows that H restricted on U is a constant map. Hence H must be a constant map, so it contradicts the assumption $JH(z) \not\equiv 0$. By Lemma 1 and (4), $Z + H$ is invertible.

2) If $M_\varepsilon \not\subset M$ for all $\varepsilon > 0$, then L_{z_0} only intersects with M_ε at point y_0, so that L_{z_0} is transverse to M_ε. Thus we can get a little open neighborhood $U(z_0) \subset C^3 \backslash S$ of z_0, and construct a complex line set

$$\xi = \{L_z \mid z \in U(z_0)\},$$

such that every complex line in ξ intersects with M_ε. Thus H restricted on $U(z_0)$ is a constant map. So that $JH(z) \equiv 0$ on $U(z_0)$, by the identity theorem of several complex variables, $JH(z) \equiv 0$ on C^3, it contradicts the assumption $JH(z) \not\equiv 0$. Therefore, there exists $\varepsilon > 0$ such that $M_\varepsilon \subset M$.

According to the proof of Theorem 1, let $H : C^3 \to C^3$ be a holomorphic map with $(JH(z))^2 = 0$. If for some $z_0 \in C^3 \backslash S$, L_{z_0} does not have property (P), $Z + H$ is invertible. So we only need to consider the situation of all complex lines L_z with the property (P).

Theorem 2. Let $H : C^3 \to C^3$ be a holomorphic map with $(JH(z))^2 = 0$. If L_z has property (P) for all $z \in C^3 \backslash S$ then for any $z \in C^3 \backslash S$, $L_z \cap S = \emptyset$.

Proof. Suppose there is a point $z \in C^3 \backslash S$ and some $z_0 \in L_z$ such that $JH(z_0) = 0$. Since $JH(z) \neq 0$, without losing generality, we can assume $(D_1H_1(z), D_1H_2(z), D_1H_3(z)) \neq 0$. Then we have

$$JH(z) = \begin{pmatrix} D_1H_1(z) \\ D_1H_2(z) \\ D_1H_3(z) \end{pmatrix} \begin{pmatrix} 1 & \lambda_2(z) & \lambda_3(z) \end{pmatrix}; \quad \lambda_2(z), \lambda_3(z) \in C,$$

and

$$0 = (JH(z))^2 = (D_1H_1(z) + \lambda_2(z) \cdot D_1H_2(z) + \lambda_3(z) \cdot D_1H_3(z)) \cdot JH(z).$$

Hence

$$D_1H_1(z) + \lambda_2(z) \cdot D_1H_2(z) + \lambda_3(z) \cdot D_1H_3(z) = 0. \tag{5}$$

From $(D_1H_1(z), D_1H_2(z), D_1H_3(z)) \neq 0$ and (5), we have

$$(D_1H_2(z), D_1H_3(z)) \neq 0,$$

so there must exist $a, b \in C$ such that

$$aD_1H_2(z) + bD_1H_3(z) \neq 0.$$

By the continuity, we can get an open neighborhood $U(z)$ of z such that

$$aD_1H_2(y) + bD_1H_3(y) \neq 0; \quad \forall y \in U(z).$$

Consider the holomorphic function

$$g(y) = aD_1H_2(y) + bD_1H_3(y); \quad y \in C^3.$$

By the Weierstrass preparation theorem and $g(z_0) = 0$, $S_1 = \{y \in C^3 \mid g(y) = 0\}$ is an analytic variety of complex 2 dimension. Since $aD_1H_2(z) + bD_1H_3(z) \neq 0$, y_0 is the isolated zero point of the holomorphic function $g(y)$ restricted on L_z. Now we get an open neighborhood $U(y_0)$ of y_0 in C^3 such that $S \cap U(y_0)$ is connected and $U(y_0) \cap S_1 \cap L_z = \{y_0\}$. By the continuity, we can choose a little open neighborhood $U'(z) \subset U(z)$ such that complex line $L_y \cap S_1 \cap U(y_0)$ is nonempty for all $y \in U'(z)$. If x_0 is the intersect point of L_y and $S_1 \cap U(y_0)$, since L_y has the property (P), we have

$$(D_1H_1(x_0), D_1H_2(x_0), D_1H_3(x_0)) = \lambda(D_1H_1(y), D_1H_2(y), D_1H_3(y)); \quad \lambda \in C.$$

Since $x_0 \in S_1$

$$0 = aD_1H_2(x_0) + bD_1H_3(x_0) = \lambda(aD_1H_2(y) + bD_1H_3(y)),$$

but $aD_1H_2(y) + bD_1H_3(y) \neq 0$, so that $\lambda = 0$, i.e.

$$D_1H_1(x_0) = D_1H_2(x_0) = D_1H_3(x_0) = 0.$$

Hence $x_0 \in S \cap U(y_0)$. By Proposition 3 and Theorem A, H is a constant map on $\{L_y \mid y \in U'(z)\}$. Since $U'(z)$ is an open set of C^3, by the identity, H is a constant map. This contradicts the assumption $JH(z) \not\equiv 0$ on C^3, so $L_z \cap S = \emptyset$.

Theorem 3. Let $H : C^3 \to C^3$ be a holomorphic map with $(JH(z))^2 = 0$. Then $Z + H$ is invertible.

Proof. By Theorems 1 and 2, we can assume that L_z has the property (P) and $L_z \cap S = 0$ for all $z \in C^3 \backslash S$.

For any point $z^{(1)} = (z_1^{(1)}, z_2^{(1)}, z_3^{(1)}) \in C^3 \backslash S$, we can assume $D_1H_3(z^{(1)}) \neq 0$. Considering the following holomorphic function on complex plane $z_3 = z_3^{(1)}$,

$$\begin{aligned} f(z_1, z_2) = \quad & (z_1 - z_1^{(1)})(D_1H_2(z_1, z_2, z_3^{(1)})D_1H_3(z^{(1)}) - D_1H_2(z^{(1)})D_1H_3(z_1, z_2, z_3^{(1)})) \\ & -(z_2 - z_2^{(1)})(D_1H_1(z_1, z_2, z_3^{(1)})D_1H_3(z^{(1)}) - D_1H_1(z^{(1)})D_1H_3(z_1, z_2, z_3^{(1)})), \end{aligned}$$

since $f(z_1^{(1)}, z_2^{(1)}) = 0$, $S_{z^{(1)}} = \{(z_1, z_2, z_3^{(1)}) \in C^3 \mid f(z_1, z_2) = 0\}$ is a complex analytic variety of complex 1 dimension on complex plane $z_3 = z_3^{(1)}$. If there is some point $y = (y_1, y_2, z_3^{(1)}) \in S_{z^{(1)}}$ such that

$$D_1H_1(y_1, y_2, z_3^{(1)})D_1H_3(z^{(1)}) - D_1H_1(z^{(1)})D_1H_3(y_1, y_2, z_3^{(1)}) \neq 0, \tag{6}$$

or

$$D_1H_2(y_1, y_2, z_3^{(1)})D_1H_3(z^{(1)}) - D_1H_2(z^{(1)})D_1H_3(y_1, y_2, z_3^{(1)}) \neq 0, \tag{7}$$

then complex line $L_{z^{(1)}}$ and L_y must meet at the point

$$(y_1 + \varphi(y)D_1H_1(y), y_2 + \varphi(y)D_1H_2(y), z_3^{(1)} + \varphi(y)D_1H_3(y))$$

or

$$(y_1 + \psi(y)D_1H_1(y), y_2 + \psi(y)D_1H_2(y), z_3^{(1)} + \psi(y)D_1H_3(y)),$$

where

$$\varphi(y) = \frac{(z_1^{(1)} - y_1) \cdot D_1H_3(z^{(1)})}{D_1H_1(y)D_1H_3(z^{(1)}) - D_1H_1(z^{(1)})D_1H_3(y)},$$

$$\psi(y)=\frac{(z_2^{(1)}-y_2)\cdot D_1H_3(z^{(1)})}{D_1H_2(y)D_1H_3(z^{(1)})-D_1H_2(z^{(1)})D_1H_3(y)}.$$

Therefore, at least one of L_y and $L_z^{(1)}$ does not have the property (P). By Theorem 1, $Z+H$ is invertible.

So we only need to discuss the following case where

$$D_1H_1(y_1,y_2,z_3^{(1)})D_1H_3(z^{(1)})-D_1H_1(z^{(1)})D_1H_3(y_1,y_2,z_3^{(1)})=0, \tag{8}$$

and

$$D_1H_2(y_1,y_2,z_3^{(1)})D_1H_3(z^{(1)})-D_1H_2(z^{(1)})D_1H_3(y_1,y_2,z_3^{(1)})=0 \tag{9}$$

are both valid for any $y=(y_1,y_2,z_3^{(1)})\in S_z^{(1)}$, which implies the complex line L_y is parallel to $L_z^{(1)}$ for any $y=(y_1,y_2,z_3^{(1)})\in S_z^{(1)}\cap C^3\backslash S$. We set

$$N_z^{(1)}=\{L_{(y_1,y_2,z_3^{(1)})}\mid (y_1,y_2,z_3^{(1)})\in S_z^{(1)}\},$$

where $N_z^{(1)}$ is a complex ruled surface. Now we choose a ball $B(z^{(1)},\delta)\subset C^3\backslash S$, such that $D_1H_3(z)\neq 0$ for any $z\in B(z^{(1)},\delta)$. Then, for every point $z\in B(z^{(1)},\delta)$, we can construct a similar complex analytic variety S_z and complex ruled surfaces N_z.

If there exist $z^{(2)},z^{(3)}\in B(z^{(1)},\delta)$ such that $V(z^{(1)})$, $V(z^{(2)})$ and $V(z^{(3)})$ are linearly independent, among the complex ruled surfaces $N_z^{(1)}$, $N_z^{(2)}$ and $N_z^{(3)}$, at least two of them meet at some point, so that, we can find a point $z\in C^3\backslash S$ such that L_z does not have the property (P). Then by Theorem 1, $Z+H$ is invertible; otherwise, we can find a vector $(a_1,a_2,a_3)\neq 0$ such that

$$(a_1,a_2,a_3)JH(z)=0, \qquad \forall z\in B(z^{(1)},\delta).$$

By Lemma 1, $Z+H$ is invertible.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant No. 19631010).

References

1. Bass, H., Connell, E. H., Wright, D., The Jacobian conjecture: Reduction of degree and formal expansion of the inverse, Bull. AMS, 1982, 7: 287—330.
2. Chen, Y. Q., A note on holomorphic maps with unipotent Jacobian matrices, Proc. AMS, 1999, 127: 2041—2044.
3. Cheng, C. C., Sakkalis, T., Wang, S. S., A case of the Jacobian conjecture, J. Pure Appl. Algebra, 1994, 90: 15—18.
4. Moh, T. T., On the Jacobian conjecture and the configurations of roots, J. Reine Angew. Math, 1983, 340: 140—212.
5. Wang, S., A Jacobian criterion for separability, J. Algebra, 1980, 65: 453—494.
6. Wright, D., The amalgamated free product structure of and the weak Jacobian theorem for two variables, J. Pure Appl. Algebra, 1978, 12: 235—257.