A RELATION BETWEEN CHARACTERISTIC VALUES AND SINGULAR VALUES OF LINEAR INTEGRAL EQUATIONS*

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I. INTRODUCTION

Let $K(x, y)$ be a L^2 kernel so that

$$\int_a^b \int_a^b |K(x, y)|^2 \, dx \, dy < +\infty ,$$

there are two kinds of eigen values connected with $K(x, y)$, the characteristic values and the singular values. The characteristic values of $K(x, y)$ are the complex numbers μ such that

$$\varphi(x) = \mu \int_a^b K(x, y) \, \varphi(y) \, dy ,$$

or

$$\varphi(x) = \mu K \varphi[x] ,$$

for at least one real non-null L^2 function $\varphi(x)$. The singular values of $K(x, y)$ are the non-negative numbers λ such that λ^2 is one of the characteristic values of

$$K K^* [x, y] = \int_a^b K(x, \xi) \overline{K(y, \xi)} \, d\xi .$$

Arranging the μ's as well as the λ's in the usual way so that $|\mu_1| \leqslant |\mu_2| \leqslant \cdots$ and $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$, we have the inequality

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$$\lambda_1 \leqslant |\mu_1| \tag{1}$$

and its immediate consequence

$$\lambda_1 \lambda_2 \cdots \lambda_n \leqslant |\mu_1 \mu_2 \cdots \mu_n|. \quad (n = 1, 2, 3, \cdots) \tag{2}$$

(1) and (2) were proved by the author[1] in 1949. Later, by means of these inequalities H. Weyl[2] proved a theorem of the author[3]. In this note, the author presents the original proof as follows.

As a further deduction, the author also proves the following theorem which gives a new property of determinants.

Theorem. *For any L^2 function $K(x, y)$ there exists a decomposition of the determinant* $\det|K(s_i, t_j)|$ *of the n-th order into at least n factors in the sense that*

$$\det|K(s_i, t_j)| = \mathfrak{K}_1 \mathfrak{K}_2 \cdots \mathfrak{K}_n [S, T], \tag{3}$$

where $\mathfrak{K}_1 \mathfrak{K}_2 [S, T]$ denotes

$$\int_a^b \cdots \int_a^b \mathfrak{K}_1(s_1, s_2, \cdots, s_n; \xi_1, \xi_2, \cdots, \xi_n) \mathfrak{K}_2(\xi_1, \xi_2, \cdots, \xi_n; t_1, t_2, \cdots, t_n)\, d\xi_1\, d\xi_2 \cdots d\xi_n,$$

$\mathfrak{K}_1 \mathfrak{K}_2 \mathfrak{K}_3 [S, T]$ denotes $(\mathfrak{K}_1 \mathfrak{K}_2) \mathfrak{K}_3 [S, T]$ etc. and each $\mathfrak{K}_i (S, T)$ is a L^2 function so that

$$\int_a^b \cdots \int_a^b |\mathfrak{K}_i(s_1, \cdots, s_n; t_1, \cdots, t_n)|^2\, ds_1 \cdots ds_n\, dt_1 \cdots dt_n < +\infty.$$

If further

$$\frac{\partial^\rho K(x, y)}{\partial x^\rho} \quad (\rho = 0, 1, 2, \cdots, p)$$

are continuous then there is a decomposition of the determinant $\det|K(s_i, t_j)|$ of n-th order into at least 2np factors

$$\det|K(s_i, t_j)| = \mathfrak{K}_1 \mathfrak{K}_2 \cdots \mathfrak{K}_{2np} [S, T], \tag{4}$$

where each $\mathfrak{K}_i (S,T)$ is a L^2 function.

II. Proof of the Inequality (1)

Let $v(x)$ be a normed transpose characteristic function of $K(x,y)$ belonging to the characteristic value μ_1 so that

$$v(x) = \mu_1 K^* v\,[x]\,, \qquad \| v(x) \| = 1\,,$$

where

$$K^* v\,[x] = \int_a^b \overline{K(y,x)}\, v(y)\, dy\,,$$

then, since $KK^*[x,y]$ is a positive definite Hermitian kernel, we have, by a theorem of Courant[1)]

$$\frac{1}{\lambda_1^2} \geqslant \int_a^b \int_a^b K K^*\,[x,y]\, v(y)\, \bar{v}(x)\, dx\, dy = \int_a^b |\, K^* v[\xi]\, |^2\, d\xi$$

$$= \int_a^b \left| \frac{v(\xi)}{\mu_1} \right|^2 d\xi = \frac{1}{|\mu_1|^2}\,.$$

Whence we get (1).

III. Second Proof of the Inequality (1)

By Schwarz's inequality, we have

$$|\, K^2\,[x,y]\, |^2 \leqslant KK^*\,[x,x] \cdot K^*K\,[y,y]\,, \qquad \begin{pmatrix} a \leqslant x \leqslant b \\ a \leqslant y \leqslant b \end{pmatrix}.$$

i.e.,

$$\left| \int_a^b K(x,\xi)\, K(\xi,y)\, d\xi \right|^2 \leqslant \int_a^b \left| K(x,\xi) \right|^2 d\xi \cdot \int_a^b \left| K(\xi,y) \right|^2 d\xi\,.$$

If $K(x,y) \neq 0$, p.p. then evidently the two functions $KK^*[x,x]$ and $K^*K[y,y]$ are positive almost everywhere, and one of these two functions must be greater than $|K^2[x,y]|$. Without loss of generality, we may suppose that

$$KK^*\,[x,x] \geqslant |\, K^2\,[x,y]\, |\,, \qquad (a \leqslant x \leqslant b, \quad a \leqslant y \leqslant b)\,,$$

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1) See [6], p. 107 and pp. 112-113, or [9] p. 195.

then

$$|K^4[x,y]| = |K^2 \cdot K^2[x,y]| \leqslant KK^* \cdot KK^*[x,x] = (KK^*)^2[x,x].$$

By induction, we can get

$$|K^{2n}[x,y]| \leqslant (KK^*)^n[x,x], \quad (n = 1, 2, 3, \cdots). \tag{5}$$

Now, since the resolvent kernel of $K^2[x,y]$ and $KK^*[x,y]$ are respectively[2)]

$$H_{K^2}(\lambda; x, y) = K^2[x,y] + \lambda K^4[x,y] + \cdots + \lambda^n K^{2n+2}[x,y] + \cdots \tag{6}$$

and

$$H_{KK^*}(\lambda; x, y) = KK^*[x,y] + \\ + \lambda(KK^*)^2[x,y] + \cdots + \lambda^n (KK^*)^{n+1}[x,y] + \cdots. \tag{7}$$

These two series represent two meromorphic functions of λ. It is known[3)] that by considering x, y as constants, then each characteristic value of a kernel is a pole of the resolvent kernel of this kernel, and conversely, each pole of the resolvent kernel is a characteristic value of the original kernel. We also know[4)] that the circle of convergence of each power series passes through the singular point or the singular points of the function represented by the series, which are nearest to the origin. Now since the radius of convergence of the series (6) is $\underline{\lim}_{n\to\infty} |K^{2n+2}[x,y]|^{-1/n}$, and the radius of convergence of the series

$$H_{KK^*}(\lambda; x, x) = KK^*[x,x] + \\ + \lambda(KK^*)^2[x,x] + \cdots + \lambda^n(KK^*)^{n+1}[x,x] + \cdots \tag{7'}$$

which is a special case of (7), is $\underline{\lim}_{n\to\infty} |(KK^*)^{n+1}[x,x]|^{-1/n}$. By (5) we get

$$\underline{\lim}_{n\to\infty} \left| K^{2n+2}[x,y] \right|^{-1/n} \geqslant \underline{\lim}_{n\to\infty} \left| (KK^*)^{n+1}[x,x] \right|^{-1/n},$$

therefore $|\mu_1| \geqslant \lambda_1$ as we wished to prove.

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2) See [6] p. 119, or [9] p. 277.

3) See [8] or [9] p. 289-290.

4) See [10] p. 144.

IV. Proof of the Inequality (2)

We shall prove that there exists a L^2 kernel, of which the characteristic value of the least absolute value is $\mu_1 \mu_2 \cdots \mu_n$ and of which the least singular value is $\lambda_1 \lambda_2 \cdots \lambda_n$, then by the inequality (1), the inequality (2) immediately follows.

Let $u_1(x), u_2(x), \cdots, u_n(x)$ be the characteristic functions of $K(x, y)$ belonging respectively to the characteristic values $\mu_1, \mu_2, \cdots, \mu_n$, so that

$$u_h(x) = \mu_h K u_h[x], \quad u_h(x) \neq 0, \quad u_h(x) \in L^2, \quad (h = 1, 2, \cdots, n).$$

Let $(\varphi_1(x), \psi_1(y)), (\varphi_2(x), \psi_2(y)), \cdots, (\varphi_n(x), \psi_n(y))$ be the pairs of adjoint singular functions of $K(x, y)$ belonging respectively to the singular values $\lambda_1, \lambda_2, \cdots, \lambda_n$, so that

$$\varphi_h(x) = \lambda_h K \psi_h[x], \quad \psi_h(x) = \lambda_h K^* \varphi_h[x], \quad \varphi_h(x) \neq 0,$$

$$\varphi_h(x) \in L^2, \quad \psi_h(x) \neq 0, \quad \psi_h(x) \in L^2, \quad (h = 1, 2, \cdots, n).$$

We denote by $S = (s_1, s_2, \cdots, s_n)$ and $T = (t_1, t_2, \cdots, t_n)$ as two points in the space R_n of n dimensions, and by Ω the interval

$$a \leqslant t_i \leqslant b \quad (i = 1, 2, \cdots, n)$$

in the same space. Let $dT = dt_1 \, dt_2 \cdots dt_n$ be the volume element in R_n and

$$\mathfrak{K}(S, T) = \frac{1}{n!} \det |K(s_i, t_j)|, \quad \mathfrak{U}(T) = \det |u_i(t_j)|,$$

where

$$\det|K(s_i, t_j)| = \begin{vmatrix} K(s_1, t_1), \cdots, K(s_1, t_n) \\ \cdots\cdots\cdots\cdots\cdots \\ K(s_n, t_1), \cdots, K(s_n, t_n) \end{vmatrix}, \quad \det|u_i(t_j)| = \begin{vmatrix} u_1(t_1), \cdots, u_1(t_n) \\ \cdots\cdots\cdots\cdots \\ u_n(t_1), \cdots, u_n(t_n) \end{vmatrix}.$$

By a formula of Landsberg[4], for any $2n$ complex valued functions $\{f_h(x)\}$, $\{g_h(x)\}$, $(h = 1, 2, \cdots, n)$ defined and integrable in the interval $a \leqslant x \leqslant b$, we have

$$\frac{1}{n!} \int_a^b \cdots \int_a^b \det | f_i(x_j) | \cdot \det | g_i(x_j) | \, dx_1 \, dx_2 \cdots dx_n = \det | (f_i, \bar{g}_j) | .$$

where $(f_i, g_j) = \int_a^b f_i(x)\, \bar{g}_j(x)\, dx$, $(i, j = 1, 2, \cdots, n)$. In this formula, we take

$$f_i(t_j) = K(s_i, t_j), \quad g_i(t_j) = u_i(t_j);$$

then we get

$$\int_\Omega \mathfrak{K}(S, T)\, \mathfrak{U}(T)\, dT = \det \left| \frac{u_i(s_j)}{\mu_i} \right| = \frac{\mathfrak{U}(S)}{\mu_1 \mu_2 \cdots \mu_n}. \tag{8}$$

On the other side, let

$$D_n(\lambda) = 1 + \sum_{h=1}^{n} (-1)^h A_h \lambda^h,$$

where

$$A_h = \frac{1}{h!} \int_a^b \cdots \int_a^b \begin{vmatrix} K(s_1, s_1), \cdots, K(s_1, s_h) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ K(s_h, s_1), \cdots, K(s_h, s_h) \end{vmatrix} ds_1\, ds_2 \cdots ds_h, \quad (h = 1, 2, \cdots)$$

then the sequence of entire functions $\{D_n(\lambda)\}$ converges uniformly to $D(\lambda) = 1 + \sum_{h=1}^{\infty} (-1)^h A_h \lambda^h$, in any bounded portion of the complex plane. Let $\mu_1^{(n)}, \mu_2^{(n)}, \cdots, \mu_n^{(n)}$ be the zeros of $D_n(\lambda)$, then by a theorem of Hurwitz[5], we get $\mu_h^{(n)} \to \mu_h$ as $n \to \infty$, if μ_h exists. But for $D_n(\lambda)$, we have

$$A_i = \sum_{h_1 < h_2 < \cdots < h_i} \frac{1}{\mu_{h_1}^{(n)} \mu_{h_2}^{(n)} \cdots \mu_{h_i}^{(n)}}, \quad (h_i = 1, 2, \cdots; i = 2, 3, \cdots, n).$$

Thus for $D(\lambda)$ we have

$$A_i = \sum_{h_1 < h_2 < \cdots < h_i} \frac{1}{\mu_{h_1} \mu_{h_2} \cdots \mu_{h_i}}, \quad (h_i = 1, 2, \cdots; i = 2, 3, \cdots)$$

since the series on the right-hand side is always convergent. From this result and the equation (8), it is easy to deduce that the complete set of characteristic values of the kernel $\mathfrak{K}(S, T)$ is given by

5) See [10] p. 119.

$$\left\{ \mu_{h_1} \mu_{h_2} \cdots \mu_{h_n} \right\} h_1 < h_2 < \cdots < h_n ,$$

$$(h_i = 1, 2, 3, \ldots\ldots; i = 1, 2, \ldots\ldots n)$$

among which one of the least absolute values is $\mu_1 \mu_2 \cdots \mu_n$. Similarly, since

$$\mathfrak{K}\mathfrak{K}^* [S, T] = \frac{1}{(n!)^2} \int_a^b \cdots \int_a^b \det | K(s_i, \xi_j) | \cdot \det | \overline{K(t_i, \xi_j)} | \, d\xi_1 \, d\xi_2 \cdots d\xi_n$$

$$= \frac{1}{n!} \det | KK^* [s_i, t_j] | ,$$

therefore the complete set of characteristic values of $\mathfrak{K}\mathfrak{K}^* [S, T]$ is

$$\left\{ \lambda_{h_1}^2 \lambda_{h_2}^2 \cdots \lambda_{h_n}^2 \right\} h_1 < h_2 < \cdots < h_n , \quad (h_i = 1, 2, \cdots; i = 1, 2, \cdots, n) ,$$

among which the least one is $\lambda_1^2 \lambda_2^2 \cdots \lambda_n^2$, i.e., the least singular value of $\mathfrak{K}(S, T)$ is $\lambda_1 \lambda_2 \cdots \lambda_n$. And thus the inequality (2) follows from (1).

V. Proof of the Formulas (3) and (4)

It is known[3] that in order that a L^2 kernel $G(S, T)$ is representable as a composite product of m L^2 factors:

$$G(S, T) = K_1 K_2 \cdots K_m [S, T], \quad (K_i \in L^2; i = 1, 2, \cdots, m) ,$$

it is necessary and sufficient that the series

$$\sum_{h=1}^{\infty} | \lambda_h [G] |^{-2/m}$$

should be convergent, where $\{\lambda_h [G]\}$ denotes the complete set of singular values of $G(S, \mathrm{T})$. Now since the series $\sum_{h=1}^{\infty} | \lambda_h^{-2}$ is always convergent, therefore by a generalization of Hölder's inequality[6], we have

$$\sum_{h_1 < h_2 < \cdots < h_n} \frac{1}{(\lambda_{h_1} \lambda_{h_2} \cdots \lambda_{h_n})^{2/n}} \leqslant \prod_{i=1}^{n} \left(\sum_{h_i=1}^{\infty} \frac{1}{\lambda_{h_i}^2} \right)^{1/n} = \sum_{h=1}^{\infty} \frac{1}{\lambda_{h_i}^2} < +\infty ,$$

6) See [7] p. 22.

i.e., $\sum_{h=1}^{\infty} |\lambda_h[\mathfrak{K}]|^{-2/n}$ is convergent, where $\mathfrak{K}$ (S, T) denotes the determinant of n-th order det $|K(s_i, t_i)|$ and therefore this determinant has a representation as given by (3).

Next, suppose $\frac{\partial^p}{\partial x^p} K(x, y)$ is continuous in $a \leqslant x \leqslant y$, $a \leqslant y \leqslant b$, then by a known result[7], the series $\sum_{h=1}^{\infty} |\lambda_h[K]|^{-1/p}$ is convergent. Whence

$$\sum_{h=1}^{\infty} \frac{1}{|\lambda_h[\mathfrak{K}]|^{2/(2np)}} = \sum_{h_1<h_2<\cdots<h_n} \frac{1}{|\lambda_{h_1}\lambda_{h_2}\cdots\lambda_{h_n}|^{2/(2np)}} \leqslant$$

$$\leqslant \prod_{i=1}^{n} \left(\sum_{h_i=1}^{\infty} \frac{1}{\lambda_{h_i}^{1/p}} \right)^{1/n} = \sum_{h=1}^{\infty} \frac{1}{\lambda_h^{1/p}} < +\infty,$$

and consequently there exists a representation as given by (4).

NOTE. From the second proof of (1) as given by § III, it is clear that the kernel K(x, y) may have no characteristic value but there must exist at least one singular value. For if there exists a positive integer n such that $K^n[x, y]=0$ p. p., then the radius of convergence of the series (6) is infinity. In such case, no characteristic value of the kernel $K(x, y)$ can exist. However $KK^*[x, y]$ is a positive definite Hermitian kernel, therefore if $K(x, y) \neq 0$ p. p., then whatever the positive interger n may be, we have always $(KK^*)^n[x, x] \neq 0$, p. p. In such case, the radius of convergence of the series (7') is finite, and consequently there exists at least one singular value of $K(x, y)$. Indeed, we can find kernels which have no characteristic values but at the same time have infinitely many singular values. e.g., the kernel

$$K(x, y) = \sum_{h=1}^{\infty} \frac{\cos hx \cdot \sin hy}{\pi h}, \quad (-\pi \leqslant x \leqslant \pi, \quad -\pi \leqslant y \leqslant \pi)$$

of which the complete set of singular values is the set of all natural numbers $h=1, 2, 3, \ldots$, but there exists no characteristic value.

In the contrary case, for any finite non-singular square matrix M, the number of characteristic values and that of singular values are always equal. It is a difference between the properties of matrices and those of integral equations.

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7) See [5] p. 28.

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