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L-Functions of Jacobi forms with Shimura type

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Abstract For every Jacobi form of Shimura type over $H \times \mathbb{C}$, a system of L-functions associated to it is given. These L-functions can be analytically continued to the whole complex plane and satisfy a kind of functional equation. As a consequence, Hecke's inverse theorem on modular forms is extended to the context of Jacobi forms with Shimura type.

Keywords: Jacobi form, L-function, functional equation.

1 Introduction and notations

As is well known, one can attach to a holomorphic modular form an L-function which has many good properties shared with Riemannian zeta function. This L-function can be also analytically continued to the whole complex plane and satisfies a functional equation. It was in 1936 when Hecke[1] gave a brilliant theorem which states an equivalent relation between a cusp modular form and its L-function. This started the study of the so-called converse theorems. Generally, one wants to know how to find a criterion for an admissible irredudible representation of an algebraic reductive group to be an automorphic representation. Many authors, including Weil[2], Langlands[3], Shapiro[4], contributed their elegant work to this research field.

However, little is known concerning similar problems for an arbitrary algebraic non-reductive group, although there exist some investigations for Jacobi groups, a typical case of non-reductive groups. T. Shintani (unpublished notes) first introduced a standard zeta function associated to a cuspidal Jacobi form of degree n. Murase[5] established the analytic continuation and the functional equation of the zeta function. Unfortunately he imposed an extra stronger condition for the index to exclude the effect from the Heisenberg group into the Hecke algebra. Recently, Dulinski[6] extended Murase's results to the case of square free index in some special cases following an idea of Böcherer, and Martin[7] also proved that each L-function has an integral representation, and therefore admits a continuation to the whole space $\mathbb{C}^n$ and satisfies a functional equation of a particular type. T. Sugano and W. Kohnen also discussed this subject of L-function from their points of view.

The goal of this paper is to do the same thing for the Jacobi forms of Shimura type over $H \times \mathbb{C}$. For every level N, we can associate a system of L-functions to the Jacobi forms of index m and weight k with respect to a lattice. For the particular case $N = 1$, a kind of inverse theorem will be given, which extends the known results. For the general level, we cannot get the correspondence of Weil's inverse theorem in the present work, but we hope to return to it in the future.

In the rest of this section, some notations will be fixed. Throughout this paper m, l, k are positive integers, H denotes the upper-half plane. $\mathbb{C}, \mathbb{R}$ and $\mathbb{Z}$ denote the complex number field, real number field and the ring of integers respectively. Γ denotes the full modular group as usual, and $\Gamma(N)$ denotes a congruence subgroup of level N. For convenience, let $e(z)$ denote the exponential $\mathrm{e}^{2\pi \mathrm{i} z}$ for $z \in \mathbb{C}$. Write the complex variables $\tau = x + \mathrm{i}y,\ z = p\tau + q,\ x, y \in \mathbb{R}$. Finally, denote the lattice $l\mathbb{Z} \times l\mathbb{Z}$ of $\mathbb{Z}^2$ by $L(l)$.

Now we can define precisely the terminology of a Jacobi form of Shimura type. The standard reference for Jacobi forms is ref. [8].

Definition. A Jacobi form of weight k and index m with Shimura type on $(\Gamma(N), L(l))$ is a holomorphic function f mapping $H \times \mathbb{C}$ into $\mathbb{C}$ satisfying the following conditions:

(1) $f|_{k,m}\gamma = (c\tau + d)^{-k} e\Big(-\frac{mcz^2}{c\tau+d}\Big) f\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = f$ for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$.

(2) $f|_m X = e(m(\lambda^2\tau + 2\lambda z)) f(\tau, z + \lambda\tau + \mu) = f$ for every $X = [\lambda, \mu] \in L(l)$.

(3) For each $\gamma \in \Gamma(1)$, $f|_{k,m}\gamma$ has a Fourier expansion

$$\sum_{\substack{n, r \\ 4mn - r^2 \geqslant 0}} c(n, r) e(\tau)^n e(z)^r.$$

In fact, we could replace $L(l)$ by any lattice of $\mathbb{Z}^2$ invariant under Γ. But in this work, we are only interested in the special case $L(l)$. In principle, the method here can be used to deal with the general case. Now the space of all such functions f is denoted by $J_{k,m}(\Gamma(N), L(l))$, or simply written as $J_{k,m}(N, l)$ or $J_{k,m}(\Gamma(N), l)$, while the set of all Jacobi cusp forms in $J_{k,m}(\Gamma(N), l)$ is denoted by $J^0_{k,m}(\Gamma(N), l)$.

We next define a set, denoted by $S(m, l)$, of holomorphic functions f on $H \times \mathbb{C}$ satisfying

(H_1) $f(\tau, z) = \sum\limits_{\substack{n, r \\ 4mn - r^2 > 0}} c(n, r) e(\tau)^n e(z)^r,\ r \in \frac{1}{l}\mathbb{Z}.$

(H_2) For some $\nu > 0$,

$$f(\tau, z) e(mpz) = o(y^{-\nu}), \quad \text{when} \quad y \to 0.$$

(H_3) For each $\lambda \in l\mathbb{Z}$,

$$c(n, r) = c(n + \lambda r + \lambda^2 m, r + 2m\lambda).$$

It is obvious that $J^0_{k,m}(N, l) \subseteq S(m, l)$. A natural question is: Which one of $S(m, l)$ is the element of $J^0_{k,m}(\Gamma(N), l)$? An answer will be given in the next section as a consequence of the main results of this paper.

2 Main theorems

Let $f \in S(m, l)$. From the condition (H_3) and the Shimura correspondence, we have the following formula

$$f(\tau, z) = \sum_{a \in T} f_a(\tau) \theta_{2m,a}(\tau, z),$$

where $\frac{1}{2ml^2}T$ is a complete system of representatives of the cosets $(2ml^2)^{-1}\mathbb{Z}/\mathbb{Z}$, and

$$\theta_{2m,a}(\tau, z) = \sum_{\lambda \in \mathbb{Z}} e\Big(\frac{4m^2l^4 + 4aml^2 + a^2)\tau + (4m^2l^3 + 2aml)z}{4ml^4}\Big).$$

Furthermore we have
$$f_a(\tau)=\sum_{n=0}^{\infty}c_a(f,n)e\Big(\frac{\tau}{4ml^2}\Big)\quad(a\in T),$$
where $c_a(f,n)$ is nothing but $c(t,r)$ satisfying $n=4ml^2t-r^2$. For every $f_a(\tau)$, one can attach to it an L-function
$$L_a(f,s)=\sum_{n=1}^{\infty}\frac{c_a(f,n)}{n^s},$$
and extended L-functions
$$\begin{aligned}\Lambda_a(f,s)&=(2ml^2)^{s-\frac12}\pi^{-s}\Gamma(s)L_a(f,s),\\ \Lambda_a(f,N,s)&=(2\sqrt{N}ml^2)^{s-\frac12}\pi^{-s}\Gamma(s)L_a(f,s),\\ \Lambda_a^N(f,s)&=(2\sqrt{N}ml^2)^{s-\frac12}\pi^{-s}\Gamma(s)\sum_{n>1}\frac{4mnl^2-a^2}{(4mnl^2N-a^2)^s}.\end{aligned}$$
By the same argument of Hecke[1], one can prove that these L-functions converge absolutely and uniformly on any compact set of the complex half-plane $\mathrm{Re}(s)>\nu+1$. The first one of our main results is

Theorem 2.1. Suppose k and N are positive integers, and $f,g\in S(m,l)$ satisfy the transformation formula

(H_4) $g(\tau,z)=(N\tau)^{-k}e(-mz^2\tau^{-1})f(-(N\tau)^{-1},z(N\tau)^{-1})$.

Then for every $a\in T$, $\Lambda_a(f,N,s)$ can be holomorphically continued to the whole complex plane and satisfy the following functional equation
$$\Lambda_a\Big(g,N,k-s-\frac12\Big)=N^{-\frac{k}{2}}i^{-k}(2ml^2\sqrt{N})^{-\frac12}\sum_{a\in T}e\Big(-\frac{ra}{2ml^2\sqrt{N}}\Big)\Lambda_r^N(f,s).$$
To prove Theorem 2.1, we need the following lemmas.

Lemma 2.1. Keep the assumption of Theorem 2.1. If $\mathrm{Re}(s)<k-\frac12$, then
$$\begin{aligned}&i^k(N^{k+\frac12}2ml^2)^{\frac12}\Lambda_a\Big(g,N,k-s-\frac12\Big)\\ &=\int_0^{\infty}\int_0^1 f\Big(\frac{iy}{\sqrt{N}},\frac{pliy}{\sqrt{N}}-\frac{a}{2mlN}\Big)e(p^2ml^2\sqrt{N}iy)y^{s-\frac12}\mathrm{d}p\mathrm{d}y.\end{aligned}$$

Proof. By I_a, I_1 and I_2 we denote the integrations $\int_0^{\infty}\int_0^1$, $\int_0^1\int_0^1$ and $\int_1^{\infty}\int_0^1$ respectively. Exchange the variables by $y=x^{-1}$, we see
$$\begin{aligned}I_1&=\int_1^{\infty}\int_0^1 f\Big(\frac{iy}{\sqrt{N}},\frac{pliy}{\sqrt{N}}-\frac{a}{2mlN}\Big)e(p^2ml^2\sqrt{N}iy)y^{s-\frac12}\mathrm{d}p\mathrm{d}y\\ &=\int_0^1\int_0^1 f\Big(\frac{ix^{-1}}{\sqrt{N}},\frac{plix^{-1}}{\sqrt{N}}-\frac{a}{2mlN}\Big)e(p^2ml^2\sqrt{N}iy)x^{-s-\frac32}\mathrm{d}p\mathrm{d}x.\end{aligned}$$
Then applying the condition (H_4) and taking $\tau=\frac{i}{\sqrt{N}}x$, we have
$$I_1=\int_0^1\int_0^1 i^kN^{\frac{k}{2}}g\Big(\frac{1}{\sqrt{N}}ix,-pl-\frac{aix}{2ml\sqrt{N}}\Big)e\Big(pa+\frac{a^2ix}{4ml^2\sqrt{N}}\Big)x^{k-s-\frac32}\mathrm{d}p\mathrm{d}x.$$
Now we compute the integral I_2. For the same argument above, we have
$$\begin{aligned}I_2&=\int_0^1\int_0^1 f\Big(\frac{ix^{-1}}{\sqrt{N}},\frac{plix}{\sqrt{N}}-\frac{a}{2mlN}\Big)e(p^2ml^2\sqrt{N}ix)x^{s-\frac12}\mathrm{d}p\mathrm{d}x\\ &=i^kN^{\frac{k}{2}}\int_1^{\infty}\int_0^1 g\Big(ix\sqrt{N},-pl-\frac{aix}{2ml\sqrt{N}}\Big)e(p^2ml^2\sqrt{N}ix)x^{k-s-\frac32}\mathrm{d}p\mathrm{d}x.\end{aligned}$$

In other words, we obtain

$$I_a = \int_0^\infty \int_0^1 i^k N^{\frac{k}{2}} g\Big(\frac{ix}{\sqrt{N}}, -pl - \frac{aix}{2ml\sqrt{N}}\Big) e\Big(pa + \frac{a^2 ix}{4ml^2\sqrt{N}}\Big).$$

We denote the Fourier coefficients of g and g_a as $d(n,r)$ and $c_a(g,M)$ respectively. Thus

$$\begin{aligned} I_a &= i^k N^{\frac{k}{2}} \int_0^\infty \sum_{n,r} d(n,r) e\Big(\frac{nix}{\sqrt{N}} - \frac{raix}{2ml\sqrt{N}}\Big) e\Big(\frac{a^2 ix}{4ml^2\sqrt{N}}\Big) x^{k-s-\frac{3}{2}} \int_0^1 e(pa - rpl)\mathrm{d}p\mathrm{d}x \\ &= i^k N^{\frac{k}{2}} \sum_{M=1}^{\infty} c_a(g,M) \int_0^\infty e\Big(\frac{Mix}{4ml^2\sqrt{N}}\Big) x^{k-s-\frac{3}{2}} \mathrm{d}x. \end{aligned}$$

By the condition (H_2), we can interchange the order of the integration and the summation above. Also note that we have substituted $c_a(g,M)$ for $d(n,r)$, and have used the orthogonality of the exponential functions in the integration. So that we have shown that

$$I_a = i^k N^{\frac{k}{2}} (2ml^2\sqrt{N})^{\frac{1}{2}} \Lambda_a\Big(g, N, k-s-\frac{1}{2}\Big)$$

for $\mathrm{Re}(s) < k - \dfrac{1}{2}$. This finishes the proof of Lemma 2.1.

Lemma 2.2. For $\mathrm{Re}(s) > 0$, one has

$$I_a = \sum_{r\in T} e\Big(\frac{-ar}{2ml^2N}\Big) \Lambda_r^N(f,s).$$

Proof. We now give another computation for I_a, in order to deduce Lemma 2.2. Making a variable change, let $q = (l\sqrt{Nm})p + \frac{r}{2\sqrt{Nm}}$, we have

$$\begin{aligned} I_a &= \int_0^\infty \int_0^1 \sum_{n,r} c(n,r) e\Big(\Big(\frac{n+plr}{\sqrt{N}} + p^2 ml^2\sqrt{N}\Big) iy\Big) \cdot e\Big(-\frac{ar}{2mlN}\Big) y^{s-\frac{1}{2}} \mathrm{d}p\mathrm{d}y \\ &= (l\sqrt{Nm})^{-1} \int_0^\infty \sum_{N',r_1} c_{r_1}(N') e\Big(-\frac{ar_1}{2ml^2N}\Big) \cdot \int_{\frac{r_1}{2l\sqrt{Nm}}}^{l\sqrt{Nm}+\frac{r_1}{2l\sqrt{Nm}}} e\Big(\frac{q+n-\frac{r_1^2}{4ml^2N}}{\sqrt{N}} iy\Big) y^{s-\frac{1}{2}} \mathrm{d}q\mathrm{d}y \\ &= (l\sqrt{Nm})^{-1} \sum_{M,r} c_r(M) e\Big(-\frac{ar}{2mlN}\Big) \Big(\frac{\sqrt{N}}{2}\Big)^{\frac{1}{2}} \int_0^\infty e\Big(\frac{4nml^2N - r^2}{4Nml^2\sqrt{N}} iy\Big) y^{s-1} \mathrm{d}y \\ &= (l\sqrt{Nm})^{-1} \sum_{M,r} c_r(M) e\Big(-\frac{ar}{2mlN}\Big) \Big(\frac{\sqrt{N}}{2}\Big)^{\frac{1}{2}} \pi^{-s}\Gamma(s) \Big(\frac{4nml^2N - r^2}{2Nml^2\sqrt{N}}\Big)^{-s} \\ &= (l\sqrt{Nm})^{-1} (2Nml^2\sqrt{N})^s \Big(\frac{\sqrt{N}}{2}\Big)^{\frac{1}{2}} \pi^{-s}\Gamma(s) \sum_r e\Big(-\frac{ar}{2ml^2N}\Big) \sum_M \frac{c_r(M)}{(4mnl^2N - r^2)^s} \\ &= (2m\sqrt{N}l^2)^{s-\frac{1}{2}} \pi^{-s}\Gamma(s) \sum_r e\Big(-\frac{ar}{2ml^2N}\Big) \sum_M \frac{c_r(M)}{(4mnl^2N - r^2)^s}. \end{aligned}$$

This completes the proof of Lemma 2.2.

Proof of Theorem 2.1. Checking the process of the proof of Lemmas 2.1 and 2.2, one can find that I_1 and I_2 define two holomorphic functions on $\mathbb{C}$, since they are absolutely and uniformly convergent on any vertical strip domain of the complex plane. So that I_a is a holomorphic function on the complex plane, and therefore $\Lambda_a(g,N,s)$ can be holomorphically continued to the whole plane. As for the function equation, one can get it by comparing the expressions of the integrations in the lemmas. Thus we have finished the proof of Theorem 2.1.

In particular, the following statement is true.

Theorem 2.2. Let f and g be in $S(m,l)$, and satisfy the transformation formula (H_4). Then $\Lambda_a(f,s)$ can be analytically continued to a holomorphic function on the whole s-plane for every $a \in T$, and satisfies a functional equation

$$\Lambda_a\Big(g, k-s-\frac{1}{2}\Big) = i^{-k}(2ml^2)^{-\frac{1}{2}} \sum_{r\in T} e\Big(-\frac{ra}{2ml^2}\Big)\Lambda_r(f,s).$$

Since any element $f(\tau,z)$ of $J^0_{k,m}(\Gamma(1),l)$ satisfies the condition (H_1)—(H_4) (taking $g=f$ in (H_4)), we obtain the following

Corollary. If $f \in J^0_{k,m}(\Gamma(1),l)$, then

$$\Lambda_a\Big(f, k-s-\frac{1}{2}\Big) = i^{-k}(2ml^2)^{-\frac{1}{2}} \sum_{r\in T} e\Big(-\frac{ra}{2ml^2}\Big)\Lambda_r(f,s).$$

Following the classical argument from Hecke[1], or a version of Martin[7], we can prove the following

Theorem 2.3. Let $f \in S(m,l)$, the following two conditions are equivalent

(1) $f(\tau,z) \in J^0_{k,m}(\Gamma,l)$;

(2) $\Lambda_r(f,s)$ $(r\in T)$ can be analytically continued to whole complex plane, and are bounded on any vertical strip, and satisfy the functional equation

$$\Lambda_a\Big(f, k-s-\frac{1}{2}\Big) = i^{-k}(2ml^2)^{-\frac{1}{2}} \sum_{r\in T} e\Big(-\frac{ra}{2ml^2}\Big)\Lambda_r(f,s)$$

for every $a\in T$.

Proof. (1)⇒(2), by Theorem 2.1. For (2)⇒(1), we give here an outline of the proof.

Since $\Gamma(1)$ is generated by two elements $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, it is sufficient to show that (2) has to imply the following relation

$$f(\tau,z) = \tau^{-k} e\Big(-\frac{mz^2}{\tau}\Big) f\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big).$$

Recalling that

$$f(\tau,z) = \sum_{a\in T} f_a(\tau)\theta_{2m,a}(\tau,z),$$

we get

$$f\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) = \sum_{a\in T}(2ml^2)^{-\frac{1}{2}}\Big(\frac{\tau}{i}\Big)^{\frac{1}{2}} e\Big(\frac{mz^2}{\tau}\Big)\cdot \sum_{b\in T} e\Big(-\frac{ab}{2ml^2}\Big) f_b\Big(-\frac{1}{\tau}\Big)\theta_{2m,a}(\tau,z).$$

Thus the problem is reduced to prove

$$f_a(\tau) = \tau^{-k+\frac{1}{2}}(2ml^2 i)^{-\frac{1}{2}} \sum_{b\in T} e\Big(-\frac{ab}{2ml^2}\Big) f_b\Big(-\frac{1}{\tau}\Big). \qquad (*)$$

Because f_a is a holomorphic function on s-plane, it is enough to verify $(*)$ for the special line $\{\tau = iy:\ y\in\mathbb{R},\ y>0\}$.

First note a fact, for $\alpha>0$,

$$e^{-t} = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\alpha} \Gamma(s) t^{-s}\,\mathrm{d}s.$$

From this fact and the following expansion

$$f_a(iy) = \sum_N c_a(N) e\Big(\frac{Niy}{4ml^2}\Big),$$

we obtain

$$f_a(iy) = \sum_N c_a(N)\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\alpha}\Gamma(s)\Big(\frac{2\pi y N}{4ml^2}\Big)^{-s}\mathrm{d}s$$

for any $\alpha > 0$. If $\alpha > \nu + 1$, then $L_a(f, s)$ is uniformly convergent and bounded on $\mathrm{Re}(s) = \alpha$. So that $\Lambda_a(f, s)$ is absolutely integrable. Hence the exchange of the order of integration and summation is justice, and

$$f_a(iy) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\alpha}(2ml^2)^{\frac{1}{2}}y^{-s}\Gamma_a(f,s)\mathrm{d}s. \qquad (**)$$

Taking β such that $k - \frac{1}{2} - \beta > \nu + 1$, we can show that

$$\begin{aligned} f_a(iy) &= \frac{1}{2\pi i}\int_{\mathrm{Re}(t)=k-\frac{1}{2}-\beta}(2ml^2)^{\frac{1}{2}}y^{t-k+\frac{1}{2}}\Lambda_a\Big(f, k-t-\frac{1}{2}\Big)\mathrm{d}t \\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(t)=k-\frac{1}{2}-\beta} i^{-k}y^{t-k+\frac{1}{2}}\sum_{b\in T}e\Big(-\frac{ab}{2ml^2}\Big)\Lambda_b(f,t)\mathrm{d}t \\ &= (iy)^{-k+\frac{1}{2}}(2ml^2 i)^{-\frac{1}{2}}\sum_{b\in T}e\Big(-\frac{ab}{2ml^2}\Big)f_b\Big(-\frac{1}{iy}\Big). \end{aligned}$$

The last step uses the formula $(**)$. This finishes the proof of Theorem 2.3.

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References

1. Hecke, E., Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung, Math. Ann., 1936, 112: 664—699.
2. Weil, A., Über die Bestimmung Dirichlitscher Reihen durch Funktionalgleichungen, Math. Ann., 1967, 168: 149—156.
3. Jacquet, H., Langlands, R., Automorphic forms on $GL(2)$, Lecture Notes in Math., 109, Berlin: Springer, 1979.
4. Cogdell, J., Shapiro, I. P., Converse theorems for GL_n, Pub. Math. IHBS, 1994, 79: 157—214.
5. Murase, A., L-functions attached to Jacobi forms of degree n, II, Math. Ann., 1991, 290: 247—276.
6. Dulinski, J., L-functions for Jacobi forms of $H \times \mathbb{C}$, Results in Math., 1997, 31: 75—94.
7. Martin, Y., L-functions for Jacobi forms of arbitrary degree, Abh. Math. Sem. Univ. Hamburg, 1998, 68: 45—63.
8. Eichler, M., Zagier, D., The theorem of Jocabi forms, Progr. in Math., 55, Boston: Birkhäuser, 1985.