On bi-symmetric algebras

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LEFT-symmetric algebra is a new kind of algebra system obtained from the studying of Lie algebra, Lie group and differential geometry. It is very useful for many topics in geometry and algebra[1—3]. In this note, we discuss a special kind of left-symmetric algebra which is very meaningful——bi-symmetric algebra.

In this note, we let the base field be the algebraically closed field of characteristic 0. And the algebras which we discuss are of finite dimension.

1 Basic concepts and properties

Definition 1.1. Let A be a vector space. We define a bilinear product in A by denoting $(x, y) \rightarrow xy$. Set

$$(x, y, z) = (xy)z - x(yz), \quad \forall x, y, z \in A. \tag{1.1}$$

If A satisfies $(x, y, z) = (y, x, z)$ ($(x, y, z) = (x, z, y)$), then A is called a left-symmetric algebra (right-symmetric algebra). If A is not only a left-symmetric algebra, but also a right-symmetric algebra, then A is called a bi-symmetric algebra.

Theorem 1.1[4] . *Let A be a left-symmetric algebra with the product* $(\cdot)$. *If we* *define a product in A by* $(a, b) \to a \circ b$ *such that*

$$a \circ b = b \cdot a, \tag{1.2}$$

then A is a right-symmetric algebra with the product $(\circ)$. *We let* A' *denote this algebra.*

Theorem 1.2[5]. *Let A be a left-symmetric algebra. If we define a bracket product in A by*

$$[x, y] = xy - yx, \ \forall x, y \in A, \tag{1.3}$$

then A is a Lie algebra with the bracket product. The Lie algebra is said to be sub-adjacent to the left-symmetric algebra, and on the other hand, the left-symmetric algebra is said to be compatible with the Lie algebra.

Lemma 1.1. *Let A be a left-symmetric algebra with the product* $(\cdot)$. *Then A is abi-symmetric algebra if and only if* A' *is still a left-symmetric algebra with the product* $(\circ)$, *where* A' *is defined in Theorem* 1.1.

Definition 1.2[6]. Let A, B be two left-symmetric algebras (or right-symmetric algebras, or bi-symmetric algebras). A linear map $\varphi: A \to B$ is called an anti-homomorphism if

$$\varphi(xy) = \varphi(y)\varphi(x), \ \forall x, y \in A. \tag{1.4}$$

If in addition, φ is a linear isomorphism, then φ is called an anti-isomorphism.

Theorem 1.3. *Let A be a left-symmetric algebra with the product* $(\cdot)$, *and* A' *be the algebra with the product* $(\circ)$ *which is defined in Theorem* 1.1. *Then the following conditions are equivalent:*

(i) *A is a bi-symmetric algebra with the product* $(\cdot)$.

(ii) A' *is a bi-symmetric algebra with the product* $(\circ)$, *and* A' *is anti-isomorphic to A.*

(iii) *There exists an anti-isomorphism of A; that is, there exists a left-symmetric algebra B and a linear map* $\varphi: B \to A$ *such that* φ *is an anti-isomorphism from B onto A.*

Corollary 1.1. *In the setting of Theorem* 1.3, *if A is commutative, then* A' *is isomorphic to A. If* A' *is isomorphic and anti-isomorphic to A under the same map, then A is commutative, and at the moment, A is associative.*

Definition 1.3. Let A be a bi-symmetric algebra, A' be the algebra which is defined in Theorem 1.1. If A is isomorphic to A', then A is said to be of type Ⅰ; otherwise, A is said to be of type Ⅱ. Obviously, the bi-symmetric algebras of type Ⅱ appear in (non-isomorphic) pairs. We let $\sim$ denote the pair.

By the classification of left-symmetric algebras in dimension 2 given in ref. [7], we have:

Theorem 1.4 (Classification Theorem). *The classification of bi-symmetric algebras in dimension* 2 *is given as follows: (symbols as in ref.* [7])

(i) *Commutative algebras (hence they are associative and of type* Ⅰ), *i.e.* (*A* Ⅰ), (*A* Ⅱ), (*A* Ⅲ), (*A* Ⅳ), (*A* Ⅴ).

(ii) *There are just two pairs of bi-symmetric algebras of type* Ⅱ, *i.e.* A_1 *of* (*N* Ⅳ) — A_{-1} *of* (*N* Ⅱ), (*N* Ⅰ)—(*N* Ⅲ).

In the next, for a left-symmetric algebra A, we let L_x, R_x ($x \in A$) denote the left multiplication and the right multiplication respectively, i.e. $L_x(y) = xy$, $R_x(y) = yx$, $\forall y \in A$.

Proposition 1.1. *Let A be a vector space with a product. Then A is a bi-symmetric algebra if and only if* $\forall x, y \in A$,

$$[L_x, R_y] = R_{xy} - R_y R_x; \quad (1.5)$$
$$[L_x, L_y] = L_{[x, y]}; \quad (1.6)$$
$$[R_x, R_y] = R_{[x, y]}. \quad (1.7)$$

Corollary 1.2. *Let A be a bi-symmetric algebra. Then L: $x \to L_x$ and R: $x \to R_x$ are the (Lie algebra) homomorphisms of its sub-adjacent Lie algebra respectively. Conversely, if a left-symmetric algebra satisfies (1.7), then A is bi-symmetric.*

Theorem 1.5. *Let A be a bi-symmetric algebra.*

(i) *Both $N(A) = \{x \in A \mid L_x = 0\}$ and $R(A) = \{x \in A \mid R_x = 0\}$ are the ideals of A.*

(ii) *Let I be an ideal of A. Then $C_A(I) = \{x \in A \mid xy = yx = 0, \forall x \in I\}$ is an ideal of A. $C_A(I)$ is called the centralizer of I in A.*

(iii) *Let J be a subalgebra. Then $N_A(J) = \{x \in A \mid xy \in J, yx \in J, \forall y \in J\}$ is a subalgebra of A. $N_A(J)$ is called the normalizer of J in A.*

(iv) *If I is an ideal of A, then $A \cdot I$ and $I \cdot A$ are also the ideals of A, where $B \cdot C = \{bc \mid b \in B, c \in C\}$, B and C are two arbitrary subalgebras of A.*

2 Transitive bi-symmetric algebras and bi-symmetric derivation algebras

Definition 2.1[5]. Let A be a left-symmetric algebra. If $\forall a \in A$, $x \to x + xa$ is a linear isomorphism, then A is called a transitive left-symmetric algebra. If $\forall x \in A$, R_x is a nilpotent linear transformation, then A is called a nilpotent left-symmetric algebra.

Lemma 2.1. *Let A be a left-symmetric algebra. Then we have*

(i) *A is nilpotent if and only if A is transitive*[8];

(ii) *if $\forall x \in A$, L_x is nilpotent, then A is nilpotent and its sub-adjacent Lie algebra is nilpotent*[9];

(iii) *(Scheuneman, cf. ref. [9]) if A is transitive, and its sub-adjacent Lie algebra is nilpotent, then $\forall x \in A$, L_x is nilpotent.*

Remark. In ref. [8], a transitive left-symmetric algebra is also called a complete left-symmetric algebra.

Theorem 2.1. *Let A be a bi-symmetric algebra and A be transitive (we call A a transitive bi-symmetric algebra). Then*

(i) *$\forall x \in A$, L_x, R_x are nilpotent;*

(ii) *the sub-adjacent Lie algebra of A is nilpotent;*

(iii) *the ideals $N(A)$, $R(A)$ are non-zero.*

Corollary 2.1. *Let K be a Lie group. If K admits a complete, locally flat, left-invariant connection ∇ such that*

$$\nabla_{\nabla_Z Y} X - \nabla_{\nabla_Z X} Y = \nabla_Z [X, Y], \ \forall X, Y, Z \in \mathscr{X} = \Gamma(K), \quad (2.1)$$

then K is nilpotent.

Corollary 2.2. *Let $\mathscr{L}$ be a Lie algebra. If $\mathscr{L}$ has an etale affine representation ρ (cf. ref. [5]) such that ρ defines a transitive bi-symmetric structure, then $\mathscr{L}$ must contain non-trivial one-parameter subgroups of translations.*

Lemma 2.2. *Let A be a bi-symmetric algebra. Set*

$$A^1 = A, \ A^{i+1} = A \cdot A^i, \ i \geqslant 1; \ A_1 = A, \ A_{i+1} = A_i \cdot A, \ i \geqslant 1. \quad (2.2)$$

Then

(i) *$\forall i$, A^i and A_i are ideals of A;*

(ii) *there exists n such that* $A^{n+k} = A_n$, $\forall\, k \geqslant 1$;

(iii) *there exists m such that* $A_{m+k} = A_m$, $\forall\, k \geqslant 1$.

Proposition 2.1. *Let A be a bi-symmetric algebra. Then the following conditions are equivalent*:

(i) *A is transitive*;

(ii) *there exists n such that* $A^n = \{0\}$;

(iii) *there exists m such that* $A_m = \{0\}$.

Definition 2.2[5]. Let A be a left-symmetric algebra. If $\forall\, x \in A$, L_x is a derivation of its sub-adjacent Lie algebra, then A is called a left-symmetric derivation algebra.

Lemma 2.3[5]. *Let A be a left-symmetric algebra. Then A is a left-symmetric derivation algebra if and only if* $\forall\, x, y \in A$, $R_x R_y = L_{xy}$.

Theorem 2.2. *Let A be a left-symmetric derivation algebra. Then A is bi-symmetric if and only if the sub-adjacent Lie algebra of A is 2-nilpotent* (*i.e.* $[x, [y, z]] = 0$, $\forall\, x, y, z \in A$).

Corollary 2.3. *The left-symmetric derivation algebra which is compatible with Heisenberg algebra must be bi-symmetric.*

Corollary 2.4. *Let K be a Lie group. Let K possess a locally flat, left-invariant connection adapted to the adjoint structure (the existence of such structure is obtained in ref.* [5]). *Then K is 2-nilpotent if and only if the connection satisfies* (2.1).

3 Classifications of transitive bi-symmetric algebras in dimension ≤4

Definition 3.1. Let A be a left-symmetric algebra, $e_1, \cdots, e_n$ be a basis. Set

$$A_{ij} = e_i e_j = \sum_{k=1}^{n} a_{ij}^k e_k.$$

Then the (formal) matrix $\mathscr{A} = (A_{ij})$, i.e.

$$\mathscr{A} = \begin{pmatrix} \sum_{k=1}^{n} a_{11}^k e_k & \cdots & \sum_{k=1}^{n} a_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} a_{n1}^k e_k & \cdots & \sum_{k=1}^{n} a_{nn}^k e_k \end{pmatrix}$$

is called the (formal) characteristic matrix of A.

Theorem 3.1. *Let A be a left-symmetric algebra. Then A is bi-symmetric if and only if the transposed matrix* $\mathscr{A}' = (A_{ji})$ *of the characteristic matrix* $\mathscr{A}$ *is also the characteristic matrix of some left-symmetric algebra.*

Corollary 3.1. *Let A be a left-symmetric algebra,* $\mathscr{A}$ *be its characteristic matrix.*

(i) *If* $\mathscr{A}$ *is symmetric, i.e.* $A_{ji} = A_{ij}$, *then A is commutative. Hence A is a bi-symmetric algebra of type* I;

(ii) *If* $\mathscr{A}$ *is anti-symmetric, i.e.* $A_{ji} = -A_{ij}$, *then A is a bi-symmetric algebra of type* I.

Proposition 3.1. (i) *The transitive bi-symmetric algebra in dimension* 1 *is just the trivial left-symmetric algebra.*

(ii) *The transitive bi-symmetric algebra in dimension* 2 *must be commutative. Hence it must be isomorphic to the left-symmetric algebras of type* (A IV) *or* (A V) *in Theorem*

1.4.

In refs. [9, 10], Kim gave the classifications of transitive left-symmetric algebras whose sub-adjacent Lie algebras are nilpotent in dimensions 3 and 4 using the extensions of left-symmetric algebras. From these results and Theorem 2.1, we have

Theorem 3.2. *The transitive bi-symmetric algebra in dimension* 3 *must be isomorphic to one of the following types of left-symmetric algebras* (*symbols as in Proposition* 4.2 *in ref.* [10]):

(i) 1, 2, 3, 4_λ, $5_{\pm 1}$, 7 *and* 8 *are bi-symmetric algebras of type* I ;

(ii) $5_0 \sim 6$; $5_\mu \sim 5_{\frac{1}{\mu}}$, $\mu \neq 0, \pm 1$.

Theorem 3.3. *The transitive bi-symmetric algebra in dimension* 4 *must be isomorphic to one of the following types of left-symmetric algebras* (*symbols as in Theorem* 5.1 *in ref.* [9]):

(i) 3 — 8_t *type* I ;

(ii) 18_3, $20_{3,0}$ *type* I ;

(iii) $30_{\pm 1}$, $31_{\pm 1}$ *type* I ; $28 \sim 30_0$; $29 \sim 31_0$; $30_t \sim 30_{\frac{1}{t}} (t \neq 0, \pm 1)$; $31_t \sim 31_{\frac{1}{t}} (t \neq 0, \pm 1)$;

(iv) 40_1 *type* I ; $37_{-1} \sim 44_{-2}$; $39_{-2} \sim 41_{\frac{1}{2},\frac{1}{2}}$; $40_0 \sim 45_{-2}$;

(v) $41_{\mu, t}$ *is bi-symmetric if* μ, t *satisfy*

$$\mu^2 + \mu(t-1) + (-2t+1) = 0, \quad t \neq (1+\mu)/2.$$

Hence $\mu \neq 0$, 1 *and* $\frac{1}{2}$, $t = \frac{1}{2}$ *when* $\mu = (1-t)$ *for the above equation. Therefore*, $41_{\mu, t}$—$41_{\mu', t'}$, *where* $\mu' = \frac{1}{\mu}$, $t' = \frac{t}{\mu}(t-1+\mu)$, $\mu \neq \pm 1$, 0 *and* $\mu \neq 1-t$; $41_{-1,1}$ *type* I ;

(vi) 46 — 56, 57_0 *type* I ; $57_t \sim 57_{-t} (t \neq 0)$;

(vii) 59, $60_{\pm 1}$, 61, 62 *type* I ; $58 \sim 60_0$; $60_t \sim 60_{\frac{1}{t}} (t \neq 0, \pm 1)$.

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