Science in China Series F: Information Sciences

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On the quantum master equation under feedback control

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The nature of the quantum trajectories, described by stochastic master equations, may be jump-like or diffusive, depending upon different measurement processes. There are many different unravelings corresponding to different types of stochastic master equations for a given master equation. In this paper, we study the relationship between the quantum stochastic master equations and the quantum master equations in the Markovian case under feedback control. We show that the corresponding unraveling no longer exists when we further consider feedback control besides measurement. It is due to the fact that the information gained by the measurement plays an important role in the control process. The master equation governing the evolution of ensemble average cannot be restored simply by eliminating the noise term unlike the case without a control term. By establishing a fundamental limit on performance of the master equation with feedback control, we demonstrate the differences between the stochastic master equation and the master equation via theoretical proof and simulation, and show the superiority of the stochastic master equation for feedback control.

stochastic master equation, master equation, unraveling, control master equation, feedback control

1 Introduction

The quantum master equation governing the evolution of a density matrix plays an important role in relaxation and decoherence theory[1,2]. In contrast to closed quantum systems, we cannot describe a system interacting with its environment by a state vector, since the system state is still entangled with its environment (or bath)[3]. Hence, for an open quantum system, we use a density matrix to describe the system state which can be considered as an ensemble of the state vector. Here we have assumed that other systems interacting with the system of interest can be well treated as a bath. This is a good approximation in many cases[3]. In many important situations a complete solution to the total system's dynamics is rather complicated. Even though the solution is known, one is confronted with the task of determining the dynamics of the system of interest by averaging irrelevant degrees. Even worse, sometimes we do not know exactly all the degrees interacting with the system of interest. According to whether or not the bath can quickly dissipate the information of the system, one can derive a Markovian or non-Markovian evolution equation. The non-Markovian equation,

Received February 12, 2009; accepted June 6, 2009
doi: 10.1007/s11432-009-0206-6
Supported by the National Natural Science Foundation of China (Grant No. 60821091)

Citation: Qi B. On the quantum master equation under feedback control. Sci China Ser F-Inf Sci, 2009, 52(11): 2133–2139, doi: 10.1007/s11432-009-0206-6

which is characterized by strong memory effects, finite revival times and non-exponential damping or decoherence time, plays an important role in many fields of physics, such as quantum optics[4], solid state physics[5], and quantum information[6]. However, it may be a difficult task to deal with the non-Markovian master equation theoretically[7]. Even the numerical simulation of such a process turns out to be very difficult and time-consuming[8,9]. In the following we will focus on the Markovian evolution equation. Quantum Markovian process represents the simplest evolution of open systems. It can be derived from various microscopic models[3], such as weak-coupling interaction of radiation with matter, the Caldeira-Leggett model, nonlinear master equation in open many-body systems. We will only focus on the master equation of Lindblad form in the following.

Given a master equation, its unraveling is not unique. Different unravelings corresponding to different types of stochastic master equations may depend upon different measurement processes. For example, in quantum optics, when a direct or homodyne detection is performed, the trajectories may be jump-like or diffusive-like[10]. To the best of the authors' knowledge, in all of the unravelings, the stochastic master equations and master equations do not involve any control term[10−15]. We will show that even when a stochastic master equation is a specific unraveling of a master equation without a control term, it is no longer the unraveling of the corresponding master equation in the case with a control term. This is due to the important effect of the information gained from the measurement in the control process. By establishing a fundamental limit on performance of a control master equation, we will demonstrate via theoretical proof and simulation differences between the control master equation and the control stochastic master equation.

The paper is organized as follows. In section 2, we first introduce the coherent control model. Then we demonstrate that we cannot easily unravel a master equation into stochastic master equations when considering feedback control. In section 3, we further compare differences between the control master equation and control stochastic master equation. Section 4 concludes the paper with some remarks.

2 The unraveling problem

We will use a model system in the field of quantum optics (see e.g., ref. [16] for more details). A cold atomic ensemble consisting of N atoms is put into a cavity. We assume that the atomic transitions are far detuned from the cavity resonance. We further consider the atomic Hamiltonian $H_A = \hbar\Delta F_z + \hbar u(t) F_y$, where Δ is the atomic detuning and $u(t)$ is the strength of a magnetic field in the y-direction, F_z and F_y are the spin-$N/2$ collective dipole moments of the ensemble. In the feedback control case, the spins interact with an optical mode in the z-direction, which is ultimately detected by an optical detector. The cavity is used to increase the interaction strength between the light and the atoms. We further consider the decoherence due to the spontaneous emission which is included phenomenologically.

Suppose that the initial state of the system is $\rho_0 = \sum_{i=1}^{n} p_i \rho_i$, where the state ρ_i has the corresponding probability p_i, $\sum_{i=1}^{n} p_i = 1$, $n \geqslant 2$. In the control case, our goal is to prepare a desired eigenstate ρ_f of F_z with a high fidelity.

In this section we first consider the unraveling problem in the case without a control term. In this case there are many different unravelings depending upon different types of measurement processes. We will give two typical kinds of unravelings: jump-like and diffusive corresponding to direct and homodyne detection, respectively. In contrast to the case without a control term, we will demonstrate that the unraveling form is no longer true if we further consider feedback control besides measurement. In this case, the master equation governing the evolution of the average ensemble cannot simply be restored by averaging the noise term.

2.1 Case without a control term

If we neglect the spontaneous emission and do not perform any measurement and control (i.e., $u(t)$

$\equiv 0$ in the atomic Hamiltonian H_A), the dynamics of the system is described by the Schrödinger equation

$$\frac{d\rho_t}{dt} = -i[\Delta F_z, \rho_t]. \tag{1}$$

Now we consider the spontaneous emission. Since we have no access to measure all the emission photons, this phenomenon describes the effect of noise of the bath, and can be described phenomenologically[16]. As a consequence, a Lindblad term is added to (1):

$$\frac{d\rho_t}{dt} = -i[\Delta F_z, \rho_t] + \gamma \mathcal{D}[\sigma]\rho_t, \tag{2}$$

where γ is the decoherence strength, σ is the atomic decay operator and the superoperator $\mathcal{D}$ is defined by

$$\mathcal{D}[\Lambda]\rho = -\frac{1}{2}[\Lambda, [\Lambda, \rho]].$$

If we perform a continuous measurement on the optical mode which interacts with the atoms in the z-direction but throw away the information (the measurement records), the effect of the measurement may be included by adding another Lindblad term to the master equation (2):

$$\frac{d\rho_t}{dt} = -i[sF_z, \rho_t] + \gamma \mathcal{D}[\sigma]\rho_t + M\mathcal{D}[F_z]\rho_t, \tag{3}$$

where s is determined by some experimental parameters, such as the coupling strength between the atoms and the cavity, Δ and so on; M is the effective interaction strength[16].

Now we consider the unraveling of the master equation (3). We will give two specific kinds of unravelings.

If we perform a direct photodetection on the optical mode, the conditioned state matrix obeys the explicit stochastic master equation[10]

$$d\rho_c(t) = \gamma \mathcal{D}[\sigma]\rho_c(t)dt + \Big\{ dN_c(t)\mathcal{G}[F_z] - dt\mathcal{H}\Big[isF_z + \frac{M}{2}F_z^2\Big]\Big\}\rho_c(t). \tag{4}$$

Here the nonlinear superoperators $\mathcal{G}$ and $\mathcal{H}$ are defined by

$$\mathcal{G}[r]\rho = \frac{r\rho r^\dagger}{\mathrm{Tr}[r\rho r^\dagger]} - \rho,$$
$$\mathcal{H}[r]\rho = r\rho + \rho r^\dagger - \mathrm{Tr}[r\rho + \rho r^\dagger]\rho.$$

The infinitesimal process $dN_c(t)$ represents the increment in the photon count in the time interval $[t, t+dt]$, and is defined by

$$E[dN_c(t)] = M\mathrm{Tr}[F_z\rho_c(t)F_z^\dagger]dt.$$

We can obtain the master equation (3) from $\rho(t) = E[\rho_c(t)]$ simply by replacing $dN_c(t)$ by its ensemble average value $E[dN_c(t)]$. Actually, as we have mentioned, in the master equation (3), we have thrown away the information gained from the measurement, and $\rho(t+dt)$ is derived by averaging all possible evolutions during the time $[t, t+dt]$ from $\rho(t)$.

If we perform a homodyne photodetection on the optical mode, the conditioned density matrix obeys a diffusive type of stochastic master equation[16−18]

$$\begin{aligned} d\rho_c(t) = & -is[F_z, \rho_c(t)]dt + \gamma\mathcal{D}[\sigma]\rho_c(t)dt \\ & + M\mathcal{D}[F_z]\rho_c(t)dt \\ & + \sqrt{M\eta}\mathcal{H}[F_z]\rho_c(t)dW_t, \end{aligned} \tag{5}$$

where η is the detection efficiency and the innovation process W_t satisfies

$$dW_t = dY_t - 2\sqrt{M\eta}\ \mathrm{Tr}\ (F_z\rho_c(t))dt,$$

where Y_t is the observation process. An important result is that the innovation process W_t is in fact a Wiener process[19,20]. It is clear that the ensemble average evolution equation (3) can be restored by eliminating the noise term.

2.2 Case with a control term

However, if we add a control term in the Hamiltonian H_A, where $u(t) \neq 0$, and $u(\cdot)$ may depend upon $\rho_c(t)$ in the evolution equation (4) or (5), the stochastic master equations (4) and (5) are no longer the unravelings of the master equation (3) with an additional control term. This is due to the fact that the term $u(\rho_c)\cdot[F_y, \rho_c]$ is essentially nonlinear about ρ_c in this case. As a result, the product operation and average operation are not interchangeable. We will give a detailed explanation by focusing on the example of the diffusive type. Notice that the master equation and the stochastic master equation corresponding to (3) and (5) become

$$\begin{aligned} \frac{d\rho_t}{dt} = & -i[sF_z + u(t)F_y, \rho_t] \\ & + \gamma\mathcal{D}[\sigma]\rho_t + M\mathcal{D}[F_z]\rho_t, \end{aligned} \tag{6}$$

$$
\begin{aligned}
d\rho_c(t) = & -i[sF_z + u(t)F_y, \rho_c(t)]dt \\
& + \gamma\mathcal{D}[\sigma]\rho_c(t)dt + M\mathcal{D}[F_z]\rho_c(t)dt \\
& + \sqrt{M\eta}\mathcal{H}[F_z]\rho_c(t)dW_t.
\end{aligned} \tag{7}
$$

Without loss of generality, we consider the case of a spin which may be used as a qubit for simplicity, i.e., the two-dimensional case. Denote by $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$ the two eigenvectors of F_z. Under this vector representation, we have

$$
\begin{aligned}
F_z &= \frac{1}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|) = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
F_y &= \frac{1}{2}(i|0\rangle\langle 1| - i|1\rangle\langle 0|) = -\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma &= |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Suppose $\rho = \frac{1}{2}\left(\begin{smallmatrix} 1+z & x-iy \\ x+iy & 1-z \end{smallmatrix}\right)$. Corresponding to (6) and (7) respectively, we have the following evolution equations:

$$
\begin{aligned}
\frac{dx_t}{dt} &= -\frac{\gamma+M}{2}x_t - u_t z_t + s y_t, \\
\frac{dy_t}{dt} &= -\frac{\gamma+M}{2}y_t - s x_t, \\
\frac{dz_t}{dt} &= \gamma(1-z_t) + u_t x_t,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
dx_c(t) = & -\frac{\gamma+M}{2}x_c(t)dt - u_t z_c(t)dt \\
& + s y_c(t)dt + \sqrt{M\eta}x_c(t)z_c(t)dW_t, \\
dy_c(t) = & -\frac{\gamma+M}{2}y_c(t)dt - s x_c(t)dt \\
& + \sqrt{M\eta}y_c(t)z_c(t)dW_t, \\
dz_c(t) = & \gamma(1-z_c(t))dt + u_t x_c(t)dt \\
& - \sqrt{M\eta}(1-z_c^2(t))dW_t.
\end{aligned} \tag{9}
$$

From (9), it is clear that the density matrix $\rho(t) = E\rho_c(t)$ of the ensemble average obeys the following equation:

$$
\begin{aligned}
\frac{dx_t}{dt} &= -\frac{\gamma+M}{2}x_t - Eu_t z_c(t) + s y_t, \\
\frac{dy_t}{dt} &= -\frac{\gamma+M}{2}y_t - s x_t, \\
\frac{dz_t}{dt} &= \gamma(1-z_t) + Eu_t x_c(t).
\end{aligned} \tag{10}
$$

Here we notice that, generally, $Eu_t z_c(t) \neq z_t Eu_t$ and $Eu_t x_c(t) \neq x_t Eu_t$ since u may be a function of x_c, y_c and z_c. Hence, the stochastic master equation (7) is not an unraveling of the master equation (6) in the case with a feedback control term. We argue that if $u(t)$ in the Hamiltonian H_A is only time-varying but not based on the measurement information, the stochastic master equation (7) is still an unraveling of the master equation (6).

3 The control problem

In this section, we further emphasize the importance of the information gained from the measurement by comparing performance of (8) and (9) through a control task. Our task is to prepare a desired eigenstate $|\psi_f\rangle$ ($|\psi_f\rangle = |0\rangle$ or $|1\rangle$) of F_z with a high fidelity[1)]. We use $F(\rho) = \langle\psi_f|\rho|\psi_f\rangle$ as the fidelity of the state ρ with the target state $\rho_f = |\psi_f\rangle\langle\psi_f|$.

3.1 Case without decoherence

First we consider the simplest case: $\gamma = 0$; i.e., the effect of decoherence can be neglected. We have the following evolution equations corresponding to (8) and (9), respectively

$$
\begin{aligned}
\frac{dx_t}{dt} &= -\frac{M}{2}x_t - u_t z_t + s y_t, \\
\frac{dy_t}{dt} &= -\frac{M}{2}y_t - s x_t, \\
\frac{dz_t}{dt} &= u_t x_t,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
dx_c(t) &= -\frac{M}{2}x_c(t)dt - u_t z_c(t)dt + s y_c(t)dt \\
& \quad + \sqrt{M\eta}x_c(t)z_c(t)dW_t, \\
dy_c(t) &= -\frac{M}{2}y_c(t)dt - s x_c(t)dt \\
& \quad + \sqrt{M\eta}y_c(t)z_c(t)dW_t, \\
dz_c(t) &= u_t x_c(t)dt - \sqrt{M\eta}(1-z_c^2(t))dW_t.
\end{aligned} \tag{12}
$$

A straightforward calculation shows that $x, y, z = 0$ is an equilibrium point of eq. (11), and once the state is $x, y, z = 0$ (corresponding to a completely mixed state), it will be stuck at this point no matter what admissible control law is performed. Hence we cannot prepare the target state (corresponding to $z = 1$ or -1) from an arbitrary initial state by using the ensemble control model (11). In contrast to this, Theorem 4.2 of ref. [21]

1) This control task may be considered as robustly preparing a qubit.

proposes an explicit control law which can globally stabilize (12) around ρ_f; i.e., in this case, we can approximately prepare the target state from an arbitrary initial state with probability 1. Hence, in this sense, we have shown the superiority of the trajectory control model where the measurement information is used for feedback control.

3.2 Case with decoherence

In contrast to the case without decoherence, it is easy to see that when $u \equiv 0$, the state of (8) and (9) will approximate to the eigenstate $|0\rangle$ of F_z corresponding to $z = 1$ (with probability 1 for (9)). Hence, we only consider the case of $\rho_f = |1\rangle\langle 1|$ in the following. In this case, $F(\rho) = \langle 1|\rho|1\rangle = \frac{1-z}{2}$.

First, we obtain the following proposition.

Proposition 3.1. For model (8), for an arbitrary initial state and an arbitrary admissible control law (such that model (8) has a unique solution), we have $\limsup_{t\to\infty} z_t \geqslant 0$; i.e., $\liminf_{t\to\infty} F_t \leqslant 50\%$.

Proof. Denote $V_t = x_t^2 + y_t^2 + z_t^2$. From (8), it is easy to get

$$\frac{dV_t}{dt} = -\gamma V_t - M(x_t^2 + y_t^2) - \gamma z_t^2 + 2\gamma z_t.$$

Hence

$$V_t = \mathrm{e}^{-\gamma t}V_0 - M\mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{\gamma s}(x_s^2 + y_s^2)ds + \mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{\gamma s}(-\gamma z_s^2 + 2\gamma z_s)ds.$$

We use a contradiction argument. If there exists $\varepsilon > 0$ and an admissible control law $u(\cdot)$, together with $T(\varepsilon, u)$, such that $z_t < -\varepsilon$ whenever $t > T(\varepsilon, u)$, we have

$$\begin{aligned} V_t &= \mathrm{e}^{-\gamma t}V_0 - M\mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{\gamma s}(x_s^2 + y_s^2)ds + \mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{\gamma s}(-\gamma z_s^2 + 2\gamma z_s)ds, \\ &= \mathrm{e}^{-\gamma t}V_0 - M\mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{\gamma s}(x_s^2 + y_s^2)ds + \mathrm{e}^{-\gamma t}\int_0^T \mathrm{e}^{\gamma s}(-\gamma z_s^2 + 2\gamma z_s)ds + \mathrm{e}^{-\gamma t}\int_T^t \mathrm{e}^{\gamma s}(-\gamma z_s^2 + 2\gamma z_s)ds \\ &< \varepsilon^2 \quad \text{as} \quad t \to \infty. \end{aligned}$$

However, when $t > T(\varepsilon, u)$, we have

$$V(t) = x_t^2 + y_t^2 + z_t^2 > \varepsilon^2.$$

This is a contradiction.

This proposition shows that we cannot always prepare the target state with a high fidelity (at least greater than 50%) after some limited time no matter what admissible control law is performed if we only use the information of the ensemble average for feedback control.

In the following we turn to (9). We will demonstrate by simulation that in this case we can always prepare the target state with a high fidelity after some limited time by a simple control law with appropriate parameters.

We consider a simple control law $u = -B(x_c + \delta)$, where B is the control strength and δ is an adjustable parameter which avoids the trajectory of z_c sticking at 1[2]. The simulation results are shown in Figures 1–3. Here we choose the parameters $s = 0$, $\delta = 0.01$, the step size of the simulations is 5×10^{-4}, and we average 300 sample paths for every curve in Figures 1–3. In order to compare the results of different parameters, we choose $\gamma = 1$ in Figures 1 and 2, $\gamma = 0.1$ in Figure 3; $\eta = 1$ in Figures 1 and 3; $M = 10$, $B = 25$ in Figure 1, $M = 5$, $B = 15$ in Figure 3. The other parameters for the curves are shown in the legend, where eta means η.

In contrast to the case of (8), by the simulations we learn that $Ez_c(t)$ decreases as the control strength B increases or the decoherence strength γ decreases, and we can prepare the target state always with a high fidelity after some limited time. This is because in this case the feedback acts immediately after a detection unlike the case in (8), where the control law is based on the information of the ensemble average since the measurement records have been thrown away.

4 Concluding remarks

In this paper, we have discussed the question of unraveling a master equation into stochastic mas-

2) It is not difficult to see that, the "worst" case $z = 1$ implies $x = 0$ and hence $u = 0$ if $\delta = 0$. Thus, z_c will stick at 1 since it is a stable point of eq. (9).

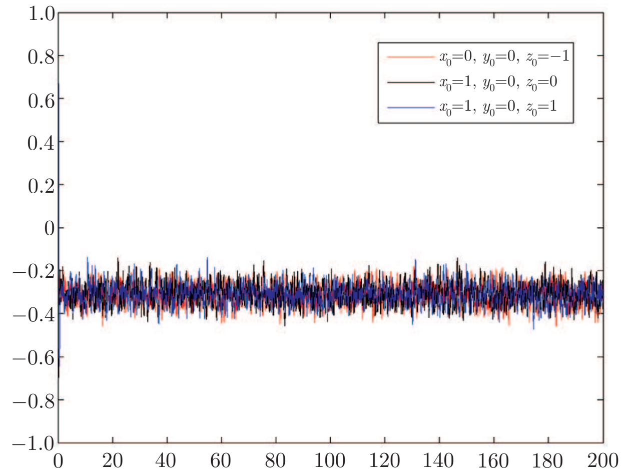

Figure 1 Function $Ez_c(t)$ of t with different initial states.

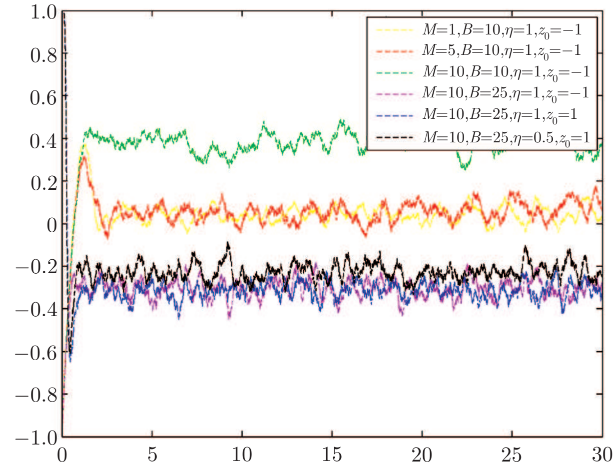

Figure 2 Function $Ez_c(t)$ of t with different parameters.

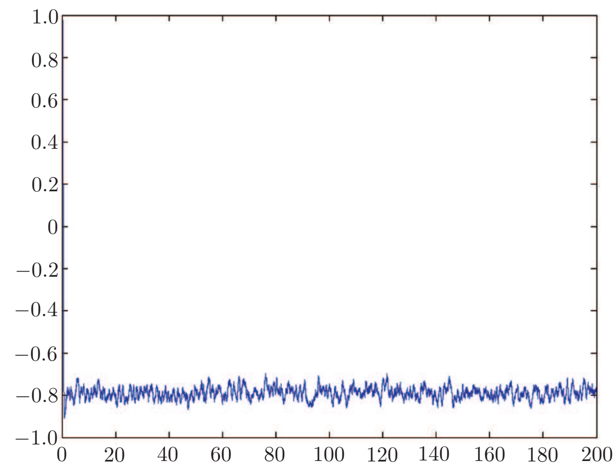

Figure 3 Function $Ez_c(t)$ of t.

ter equations in the Markovian case. We explicitly show that the stochastic master equation cannot be replaced by a master equation when we consider feedback control. This is because information plays an important role in the feedback control process. We explicitly demonstrate that the ensemble average and feedback control are not interchangeable using a coherent control model. This result clarifies the differences between the two types of control modes: ensemble control and trajectory control corresponding to the control master equation and the control stochastic master equation, respectively, and shows the superiority of the trajectory control mode in feedback control.

The author thanks Lei Guo, Zairong Xi, Daoyi Dong and Hao Pan for helpful discussion.

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