SCIENCE CHINA
Mathematics

Special Issue on Representation Theory
• ARTICLES •

February 2018 Vol. 61 No. 2: 191–206
https://doi.org/10.1007/s11425-017-9161-7

Coset vertex operator algebras and $\mathcal{W}$-algebras of A-type

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Received January 3, 2017; accepted September 11, 2017; published online November 8, 2017

Abstract We give an explicit description for a weight three generator of the coset vertex operator algebra $C_{L_{\widehat{\mathfrak{sl}_n}}(l,0)\otimes L_{\widehat{\mathfrak{sl}_n}}(1,0)}(L_{\widehat{\mathfrak{sl}_n}}(l+1,0))$, for $n \geqslant 2$, $l \geqslant 1$. Furthermore, we prove that the commutant $C_{L_{\widehat{\mathfrak{sl}_3}}(l,0)\otimes L_{\widehat{\mathfrak{sl}_3}}(1,0)}(L_{\widehat{\mathfrak{sl}_3}}(l+1,0))$ is isomorphic to the $\mathcal{W}$-algebra $\mathcal{W}_{-3+\frac{l+3}{l+4}}(\mathfrak{sl}_3)$, which confirms the conjecture for the $\mathfrak{sl}_3$ case that $C_{L_{\widehat{\mathfrak{g}}}(l,0)\otimes L_{\widehat{\mathfrak{g}}}(1,0)}(L_{\widehat{\mathfrak{g}}}(l+1,0))$ is isomorphic to $\mathcal{W}_{-h+\frac{l+h}{l+h+1}}(\mathfrak{g})$ for simply-laced Lie algebras $\mathfrak{g}$ with its Coxeter number h for a positive integer l.

Keywords $\mathcal{W}$-algebra, coset vertex operator algebra, rationality

MSC(2010) 17B69

Citation: Arakawa T, Jiang C P. Coset vertex operator algebras and $\mathcal{W}$-algebras of A-type. Sci China Math, 2018, 61: 191–206, https://doi.org/10.1007/s11425-017-9161-7

1 Introduction

Given a vertex operator algebra V and a vertex operator subalgebra $U \subseteq V$, $C_V(U)$ which is called the commutant of U in V or coset construction, is the subalgebra of V which commutes with U. The coset vertex algebra construction, initiated in [46], was introduced by Frenkel and Zhu [44]. Coset construction is one of the major ways to construct new vertex operator algebras from given ones. It is widely believed that the commutant $C_V(U)$ inherits properties from V and U in many ways. In particular, it is expected that $C_V(U)$ is rational if both V and U are rational. No general results of this kind have been known so far. Nevertheless, many interesting examples, especially coset vertex operator algebras related to affine vertex operator algebras, have been extensively studied both in the physics and mathematics literature. For coset parafermion vertex operator algebras one can refer to [8, 9, 15, 17, 26, 27, 29, 30], etc. For coset vertex operator algebras related to tensor decompositions of affine vertex operator algebras one can refer to [46–48, 56–58], etc. See [2, 19, 21, 22, 45], etc, for other examples.

Let $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}}$ be the affine Kac-Moody Lie algebras associated to a complex simple Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$, respectively. For $k \in \mathbb{C}$ such that $k \neq -h^\vee$, where $h^\vee$ is the dual Coxter number of $\mathfrak{g}$, let $L_{\widehat{\mathfrak{g}}}(k,0)$ and $L_{\widehat{\mathfrak{h}}}(k,0)$ be the associated simple vertex operator algebras with level k, respectively (see [44, 59]). The commutant of the Heisenberg vertex operator algebra $L_{\widehat{\mathfrak{h}}}(k,0)$ in $L_{\widehat{\mathfrak{g}}}(k,0)$, denoted

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by $K(\mathfrak{g},k)$, is the so-called parafermion vertex operator algebra (see [65]). Parafermion vertex operator algebras have been studied extensively (see [8, 9, 20, 26, 27, 29, 30, 58]). Among other things, for a positive integer k, the C_2-cofiniteness of $K(\mathfrak{g},k)$ was established in [8, 30], the generators of $K(\mathfrak{g},k)$ were given in [29], and the rationalities of $K(\mathfrak{sl}_2,k)$ and $K(\mathfrak{sl}_k,2)$ were established in [9] and [47, 48], respectively.

Let $L_{\widehat{\mathfrak{g}}}(1,0)$ be the simple affine vertex operator algebra associated to the simple Lie algebra $\mathfrak{g}$ with level 1. For $l\in\mathbb{Z}_{\geqslant 2}$, $L_{\widehat{\mathfrak{g}}}(1,0)^{\otimes l}$ has a tensor product vertex operator algebra structure with $L_{\widehat{\mathfrak{g}}}(l,0)$ being a vertex operator subalgebra. The commutant of $L_{\widehat{\mathfrak{g}}}(l,0)$ in $L_{\widehat{\mathfrak{g}}}(1,0)^{\otimes l}$ is a simple vertex operator subalgebra of $L_{\widehat{\mathfrak{g}}}(1,0)^{\otimes l}$. It was proved in [47] and in [56] independently that $C_{L_{\widehat{\mathfrak{sl}_n}}(1,0)^{\otimes l}}(L_{\widehat{\mathfrak{sl}_n}}(l,0))\cong K(\mathfrak{sl}_l,n)$ as vertex operator algebras, presenting a version of level-rank duality. The classification of irreducible modules and the rationality of $C_{L_{\widehat{\mathfrak{sl}_2}}(1,0)^{\otimes l}}(L_{\widehat{\mathfrak{sl}_2}}(l,0))$ were established in [48]. Then by the level-rank duality, the parafermion vertex operator algebra $K(\mathfrak{sl}_l,2)$ is rational for any $l\in\mathbb{Z}_{\geqslant 2}$.

More generally, given a sequence of positive integers $\underline{\ell}=(l_1,\ldots,l_s)$, the tensor product vertex operator algebra $L_{\widehat{\mathfrak{g}}}(\underline{\ell},0)=L_{\widehat{\mathfrak{g}}}(l_1,0)\otimes L_{\widehat{\mathfrak{g}}}(l_2,0)\otimes\cdots\otimes L_{\widehat{\mathfrak{g}}}(l_s,0)$ contains a vertex operator subalgebra isomorphic to $L_{\widehat{\mathfrak{g}}}(|\underline{\ell}|,0)$ with $|\underline{\ell}|=l_1+\cdots+l_s$. On the other hand, the sequence $\underline{\ell}$ defines a Levi subalgebra $\mathfrak{l}_{\underline{\ell}}$ of $\mathfrak{sl}_{|\underline{\ell}|}$. Denote by $L_{\widehat{\mathfrak{l}_{\underline{\ell}}}}(n,0)$ the vertex operator subalgebra of $L_{\widehat{\mathfrak{sl}_{|\underline{\ell}|}}}(n,0)$ generated by $\mathfrak{l}_{\underline{\ell}}$. Set $K(\mathfrak{sl}_{|\underline{\ell}|},\mathfrak{l}_{\underline{\ell}},n)=C_{L_{\widehat{\mathfrak{sl}_{|\underline{\ell}|}}}(n,0)}(L_{\widehat{\mathfrak{l}_{\underline{\ell}}}}(n,0))$. It was established in [47] that $C_{L_{\widehat{\mathfrak{sl}_n}}(\underline{\ell},0)}(L_{\widehat{\mathfrak{sl}_n}}(|\underline{\ell}|,0))\cong K(\mathfrak{sl}_{|\underline{\ell}|},\mathfrak{l}_{\underline{\ell}},n)$, which is a more general version of rank-level duality. In particular, we have

$$C_{L_{\widehat{\mathfrak{sl}_n}}(l,0)\otimes L_{\widehat{\mathfrak{sl}_n}}(1,0)}(L_{\widehat{\mathfrak{sl}_n}}(l+1,0))\cong C_{L_{\widehat{\mathfrak{sl}_{l+1}}}(n,0)}(L_{\widehat{\mathfrak{sl}_l}}(n,0)\otimes L_{\widehat{\mathfrak{h}_{\underline{\ell}}}}(n,0)),$$

where $\mathfrak{h}_{\underline{\ell}}$ is the subalgebra of the Cartan subalgebra of $\mathfrak{sl}_{l+1}(\mathbb{C})$ commuting with $\mathfrak{sl}_l(\mathbb{C})$, and $\widehat{\mathfrak{h}_{\underline{\ell}}}$ is the associated affine Lie algebra.

(The principal) $\mathcal{W}$-algebras are one-parameter families of vertex algebras associated to simple Lie algebras. First example of a $\mathcal{W}$-algebra was introduced by Zamolodchikov [64] in an attempt to classify extended conformal algebras with two generating fields. Since then there have been several approaches to the construction of $\mathcal{W}$-algebras (see [16, 19, 32, 33, 36, 37, 60, 62]). $\mathcal{W}$-algebras have also been studied as extended Virasoro algebras (see [11–13, 18, 61]).

For $k\in\mathbb{C}$, let $V_{\widehat{\mathfrak{g}}}(k,0)$ be the universal affine vertex operator algebra associated to the simple Lie algebra $\mathfrak{g}$ with level k. The associated (principal) affine $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g})$, which is a vertex algebra, can be realized as the cohomology of the BRST (Becchi, Rouet, Stora and Tyutin) complex of the quantum Drinfeld-Sokolov reduction (see [35, 37]). From this point of view, $\mathcal{W}$-algebras have been studied deeply and extensively, and many remarkable results have been achieved recently (see [3–6, 51–54]). For some earlier results, one can also refer to [10, 14] and [34–41], etc. Among other things, it was established in [4] that the character of each irreducible highest weight representation of $\mathcal{W}^k(\mathfrak{g})$ is completely determined by that of the corresponding irreducible highest weight representations of the affine Lie algebra $\widehat{\mathfrak{g}}$. The C_2-cofiniteness and the rationality of the minimal series principal $\mathcal{W}$-algebras were established in [6] and [7], respectively.

Recall that [41] a rational number k with the denominator $u\in\mathbb{N}$ is called principal admissible if

$$u(k+h^\vee)\geqslant h^\vee,\quad (u,r^\vee)=1,$$

where h is the Coxeter number of $\mathfrak{g}$, $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, and $r^\vee$ is the maximal number of the edges in the Dykin diagram of $\mathfrak{g}$. A principal admissible number k is called non-degenerate if the denominator u is equal or greater than the Coxeter number h. For a non-degenerate admissible principal number k, denote by $\mathcal{W}_k(\mathfrak{g})$ the simple quotient of $\mathcal{W}^k(\mathfrak{g})$, which are also called minimal series principal $\mathcal{W}$-algebras (see [6, 7, 41]).

One conjecture (see [18, 41]) about minimal series principal $\mathcal{W}$-algebras asserts that $\mathcal{W}_k(\mathfrak{g})$ is isomorphic to the commutant $C_{L_{\widehat{\mathfrak{g}}}(p,0)\otimes L_{\widehat{\mathfrak{g}}}(1,0)}(L_{\widehat{\mathfrak{g}}}(p+1,0))$ of $L_{\widehat{\mathfrak{g}}}(p+1,0)$ in $L_{\widehat{\mathfrak{g}}}(p,0)\otimes L_{\widehat{\mathfrak{g}}}(1,0)$ for $\mathfrak{g}$ being simply-laced and $k=-h^\vee+\frac{p+h^\vee}{p+h^\vee+1}$, $p\in\mathbb{Z}_+$. This conjecture comes partially from the fact that $\mathcal{W}_k(\mathfrak{g})$ and $C_{L_{\widehat{\mathfrak{g}}}(p,0)\otimes L_{\widehat{\mathfrak{g}}}(1,0)}(L_{\widehat{\mathfrak{g}}}(p+1,0))$ share the same normalized characters (see [41, 53]). If $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$, then $\mathcal{W}_k(\mathfrak{g})$ is the simple Virasoro vertex operator algebra $L(c_p,0)$, where $c_p=1-\frac{6}{(p+2)(p+3)}$. For this case, the conjecture follows from the Goddard-Kent-Olive construction (see [46]). If $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ $(n \geqslant 2)$ and $p = 1$, it was proved in [9] that $\mathcal{W}_k(\mathfrak{g})$ is isomorphic to the parafermion vertex operator algebra $K(\mathfrak{sl}_2, n)$, with which the rationality of $K(\mathfrak{sl}_2, n)$ is established. Then the conjecture in this case follows from the fact that $K(\mathfrak{sl}_2, n) \cong C_{L_{\widehat{\mathfrak{sl}_n}}(1,0)^{\otimes 2}}(L_{\widehat{\mathfrak{sl}_n}}(2,0))$ (see [47,56]). For general $l \in \mathbb{Z}_{\geqslant 1}$ and $n \in \mathbb{Z}_{\geqslant 2}$, by the level-rank duality and the reciprocity law given in [47],

$$C_{L_{\widehat{\mathfrak{sl}_n}}(l,0)\otimes L_{\widehat{\mathfrak{sl}_n}}(1,0)}(L_{\widehat{\mathfrak{sl}}_n}(l+1,0)) \cong C_{K(\mathfrak{sl}_{l+1},n)}(K(\mathfrak{sl}_l, n)).$$

In this paper, we first give explicitly a generator of $C_{K(\mathfrak{sl}_{l+1},n)}(K(\mathfrak{sl}_l, n))$ of weight three. Then we prove that the conjecture is true for the case that $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$.

The paper is organized as follows. In Section 2, we briefly review some basics on vertex operator algebras. In Section 3, we recall some notation and facts about principal affine $\mathcal{W}$-algebras. In Section 4, we review parafermion vertex operator algebras and the level-rank duality for tensor power decompositions of rational vertex operator algebras of type A. In Section 5, we study the coset vertex operator algebra $C_{K(\mathfrak{sl}_{l+1},n)}(K(\mathfrak{sl}_l, n))$. The main results of this paper are stated in this part.

2 Preliminaries

In this section, we recall some notation and basic facts on vertex operator algebras (see [17,28,42,43,59,66]).

Definition 2.1. A vertex operator algebra $V = (V, Y, \mathbf{1}, \omega)$ is defined as follows:

(1) $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is a $\mathbb{Z}$-graded vector space over $\mathbb{C}$ equipped with a linear map Y:

$$\begin{aligned} &V \to (\operatorname{End} V)[[z, z^{-1}]],\\ &v \mapsto Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}, \quad v_n \in \operatorname{End} V, \end{aligned}$$

such that $\dim V_n < \infty$, $V_m = 0$ if $m << 0$, and for any $u, v \in V$, $v_n u = 0$ for sufficiently large n.

(2) There exist two distinguished vectors: the vacuum $\mathbf{1} \in V_0$, and the conformal vector $\omega \in V_2$ such that

$$\begin{aligned} &Y(\mathbf{1}, z) = \mathrm{id}, \quad \lim_{x\to 0} Y(u, x)\mathbf{1} = u,\\ &Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},\\ &[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}c. \end{aligned}$$

The complex number c is called the central charge of V. Moreover, $L(0) = n$ on V_n, and for $u \in V$,

$$[L(-1), Y(u, z)] = \frac{d}{dz} Y(u, z).$$

(3) For any $u, v \in V$, there exists $n \geqslant 0$ such that

$$(z_1 - z_2)^n Y(u, z_1) Y(v, z_2) = (z_1 - z_2)^n Y(v, z_2) Y(u, z_1).$$

Definition 2.2. Let $(V, \mathbf{1}, \omega, Y)$ be a vertex operator algebra. A weak V-module is a vector space M equipped with a linear map

$$\begin{aligned} &Y_M : V \to \operatorname{End}(M)[[z, z^{-1}]],\\ &v \mapsto Y_M(v, z) = \textstyle\sum_{n\in\mathbb{Z}} v_n z^{-n-1}, \quad v_n \in \operatorname{End} M \end{aligned}$$

satisfying the following:

(1) $v_n w = 0$ for $n \gg 0$, where $v \in V$ and $w \in M$.

(2) $Y_M(\mathbf{1}, z) = \mathrm{id}_M$.

(3) The Jacobi identity holds

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y_M(v, z_2)Y_M(u, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_M(Y(u, z_0)v, z_2). \tag{2.1}$$

Definition 2.3. An admissible V-module is a weak V-module which carries a $\mathbb{Z}_+$-grading $M = \bigoplus_{n\in\mathbb{Z}_+} M(n)$, such that if $v \in V_r$ then $v_m M(n) \subseteq M(n + r - m - 1)$.

Definition 2.4. An ordinary V-module is a weak V-module which carries a $\mathbb{C}$-grading $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$, such that

(1) $\dim(M_\lambda) < \infty$.

(2) $M_{\lambda+n} = 0$ for fixed λ and $n \ll 0$.

(3) $L(0)w = \lambda w = \mathrm{wt}(w)w$ for $w \in M_\lambda$, where $L(0)$ is the component operator of $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.

It is easy to see that an ordinary V-module is an admissible one. If W is an ordinary V-module, we simply call W a V-module.

We call a vertex operator algebra C_2-cofinite if $V/C_2(V)$ is finite-dimensional, where

$$C_2(V) = \langle u_{-2}v \,|\, u, v \in V\rangle$$

(see [66]). A vertex operator algebra is called rational if the admissible module category is semisimple (see [28, 66]). We have the following result from [1, 28, 66].

Theorem 2.5. *If V is a vertex operator algebra satisfying the C_2-cofinite property, V has only finitely many irreducible admissible modules up to isomorphism. The rationality of of V also implies the same result.*

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and $(U, Y, \mathbf{1}, \omega')$ a vertex operator subalgebra of V. Set

$$C_V(U) = \{v \in V \,|\, [Y(u, z_1), Y(v, z_2)] = 0, u \in U\}.$$

Recall from [44, 59] that if $\omega' \in U \cap V_2$ and $L(1)\omega' = 0$, then

$$C_V(U) = \{v \in V \,|\, u_m v = 0,\ u \in U,\ m \geqslant 0\}$$

is a vertex operator subalgebra of V with the conformal vector $\omega - \omega'$. We shall write

$$Y(\omega', z) = \sum_{n\in\mathbb{Z}} L'(n)z^{-n-2},$$

where we view the operators $L'(n)$ as acting on V. We have the following result from [59].

Proposition 2.6. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and $(U, Y, \mathbf{1}, \omega')$ a vertex operator subalgebra of V. Then*

$$C_V(U) = \mathrm{Ker}_V\, L'(-1).$$

3 $\mathcal{W}$-algebras for principle case

In this section, we recall some notation and facts on the $\mathcal{W}$-algebras. $\mathcal{W}$-algebras may be defined in several ways. The definition here was given by using the quantum Drinfeld-Sokolov reduction (see [4, 34, 36, 40, 41, 51]). Throughout this section, k is a complex number with no restriction unless otherwise stated.

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l. Let $(\cdot\,|\,\cdot)$ be the normalized non-degenerate bilinear form on $\mathfrak{g}$, i.e., $(\cdot\,|\,\cdot) = \frac{1}{2h^\vee}\cdot$Killing form, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

Let e be a principal nilpotent element of $\mathfrak{g}$ so that $\dim \mathfrak{g}^e = l$, where $\mathfrak{g}^e = \{x \in \mathfrak{g} \,|\, [x, e] = 0\}$. By the Jacobson-Morozov theorem, there exists an $\mathfrak{sl}_2$-triple $\{e, f, h_0\}$ associated to e satisfying

$$[h_0, e] = 2e, \quad [h_0, f] = -2f, \quad [e, f] = h_0.$$

Set

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \,|\, [h_0, x] = 2jx\}, \quad \text{for} \quad j \in \mathbb{Z}.$$

This gives a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where

$$\mathfrak{h} = \mathfrak{g}_0, \quad \mathfrak{n}_+ = \bigoplus_{j \geqslant 1} \mathfrak{g}_j, \quad \mathfrak{n}_- = \bigoplus_{j \geqslant 1} \mathfrak{g}_{-j}.$$

Denote by $\Delta_+ \subset \mathfrak{h}^*$ the set of positive roots of $\mathfrak{g}$ and by $\{\alpha_1, \dots, \alpha_l\}$ the subset of simple roots. Let $\mathfrak{n}_+^*$ be the dual of $\mathfrak{n}_+$. Define $\bar{\chi}_+ \in \mathfrak{n}_+^*$ by

$$\bar{\chi}_+(x) = (f \,|\, x), \quad \text{for} \quad x \in \mathfrak{n}_+.$$

Then $\bar{\chi}_+$ is a character of n_+, i.e., $\bar{\chi}_+([n_+, n_+]) = 0$. Let $\widehat{\mathfrak{g}}$ be the non-twisted affine Lie algebra associated to $\mathfrak{g}$ (see [49] for details), i.e.,

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

with the commutation relations

$$\begin{aligned}
&[X(m), Y(n)] = [X, Y](m+n) + m\delta_{m+n,0}(X \,|\, Y)K, \\
&[D, X(m)] = mX(m), \quad [K, \mathfrak{g}] = 0
\end{aligned}$$

for $X, Y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, where $X(n) = X \otimes t^n$. The invariant symmetric bilinear form $(\cdot \,|\, \cdot)$ is extended from $\mathfrak{g}$ to $\widehat{\mathfrak{g}}$ as follows:

$$\begin{aligned}
&(X(m) \,|\, Y(n)) = (X \,|\, Y)\delta_{m+n,0}, \quad (D \,|\, K) = 1, \\
&(X(m) \,|\, D) = (X(m) \,|\, K) = (D \,|\, D) = (K \,|\, K) = 0.
\end{aligned}$$

Denote

$$\begin{aligned}
&\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D, \\
&\widehat{\mathfrak{g}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t] \oplus (\mathfrak{n}_- \oplus \mathfrak{h}) \otimes \mathbb{C}[t]t, \\
&\widehat{\mathfrak{g}}_- = \mathfrak{n}_- \otimes \mathbb{C}[t^{-1}] \oplus (\mathfrak{n}_+ + \mathfrak{h})\mathbb{C}[t^{-1}]t^{-1}.
\end{aligned}$$

Then $\widehat{\mathfrak{g}}$ has the triangular decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{g}}_-$. Let $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ be the dual of $\widehat{\mathfrak{h}}$, where Λ_0 and δ are the dual elements of K and D, respectively. For $\lambda \in \widehat{\mathfrak{h}}^*$, the number $\lambda(K)$ is called the level of λ. Let $\widehat{\Delta}$ be the set of roots of $\widehat{\mathfrak{g}}$, $\widehat{\Delta}_+$ the set of positive roots, and $\widehat{\Delta}_- = -\widehat{\Delta}_+$. Denote by $\widehat{\Delta}^{\mathrm{re}}$ and $\widehat{\Delta}^{\mathrm{im}}$ the set of real roots and the set of imaginary roots, respectively. Then

$$\widehat{\Delta}^{\mathrm{im}} = \{n\delta \,|\, n \in \mathbb{Z}\}, \quad \widehat{\Delta}_+^{\mathrm{re}} = \{\alpha + n\delta, -\alpha + m\delta \,|\, \alpha \in \Delta_+, n \in \mathbb{Z}_{\geqslant 0}, m \in \mathbb{Z}_{\geqslant 1}\}.$$

Let $V_{\widehat{\mathfrak{g}}}(k, 0)$ be the universal vertex operator algebra associated to $\mathfrak{g}$ with level k (see [44, 49, 59] for details). Let $\mathcal{C}l$ be the superalgebra generated by odd generators: $\psi_\alpha(n)$, $\alpha \in \Delta, n \in \mathbb{Z}$ with the following super Lie relations:

$$[\psi_\alpha(m), \psi_\beta(n)]_+ = \delta_{\alpha+\beta,0}\delta_{m+n,0},$$

for $\alpha, \beta \in \Delta$, $m, n \in \mathbb{Z}$. Here, ψ_α is regarded as the element of $\mathcal{C}l$ corresponding to the root vector $e_\alpha(n) \in \widehat{\mathfrak{g}}_\alpha$. Let $\mathcal{F}$ be the irreducible $\mathcal{C}l$-module generated by the cycle vector $\mathbf{1}$ such that

$$\psi_\alpha(n)\mathbf{1} = 0, \quad \text{if} \quad \alpha + n\delta \in \widehat{\Delta}_+^{\mathrm{re}}.$$

$\mathcal{F}$ is naturally a vertex operator superalgebra with the vacuum vector $\mathbf{1}$ and the fields defined by

$$Y(\psi_\alpha(-1)\mathbf{1},z)=\psi_\alpha(z):=\sum_{n\in\mathbb{Z}}\psi_\alpha(n)z^{-n-1},\quad \text{for}\quad \alpha\in\Delta_+,$$
$$Y(\psi_\alpha(0)\mathbf{1},z)=\psi_\alpha(z):=\sum_{n\in\mathbb{Z}}\psi_\alpha(n)z^{-n},\quad \text{for}\quad \alpha\in\Delta_-.$$

The conformal vector is chosen as

$$\omega=\sum_{\alpha\in\Delta_+}\psi_{-\alpha}(-1)\psi_\alpha(-1)\mathbf{1}.$$

Let

$$C_k(\mathfrak{g})=V_{\widehat{\mathfrak{g}}}(k,0)\otimes\mathcal{F}$$

be the tensor product of the vertex operator algebra $V_{\widehat{\mathfrak{g}}}(k,0)$ and the vertex superalgebra $\mathcal{F}$. Then $C_k(\mathfrak{g})$ is a vertex algebra. Define vertex operators $Q_+^{st}(z)$ and $\psi_+(z)$ as follows:

$$Q_+^{st}(z)=\sum_{n\in\mathbb{Z}}Q_+^{st}(n)z^{-n-1}:=\sum_{\alpha\in\Delta_+}e_\alpha(z)\psi_{-\alpha}(z)-\frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_+}c_{\alpha,\beta}^{\gamma}\psi_{-\alpha}(z)\psi_{-\beta}(z)\psi_\gamma(z),$$
$$\psi_+(z)=\sum_{n\in\mathbb{Z}}\psi_+(n)z^{-n}:=\sum_{\alpha\in\Delta_+}\bar{\chi}_+(e_\alpha)\psi_{-\alpha}(z),$$

where $\bar{\chi}_+$ is defined as above, and $c_{\alpha,\beta}^{\gamma}$ is the structure constant of $\mathfrak{g}$, i.e., for $\alpha,\beta\in\Delta_+$, $e_\alpha\in\mathfrak{g}_\alpha$, $e_\beta\in\mathfrak{g}_\beta$,

$$[e_\alpha,e_\beta]=\sum_{\gamma\in\Delta_+}c_{\alpha,\beta}^{\gamma}e_\gamma.$$

As in [4], by abuse notation, we set

$$Q_+^{st}:=Q_+^{st}(0)=\sum_{\alpha\in\Delta_+,n\in\mathbb{Z}}e_\alpha(-n)\psi_{-\alpha}(n)-\frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_+,s+r+m=0}c_{\alpha,\beta}^{\gamma}\psi_{-\alpha}(s)\psi_{-\beta}(r)\psi_\gamma(m),$$
$$\chi_+:=\chi_+(1)=\sum_{\alpha\in\Delta_+}\bar{\chi}_+(e_\alpha)\psi_{-\alpha}(1),$$
$$Q_+:=Q_+^{st}+\chi_+.$$

We have the following lemma (see [4, 40]).

Lemma 3.1. *It holds that* $(Q_+^{st})^2=\chi_+^2=[Q_+^{st},\chi_+]=0$, $Q_+^2=0$.

Let $\mathcal{F}=\bigoplus_{i\in\mathbb{Z}}\mathcal{F}^i$ be an additional $\mathbb{Z}$-gradation of the vertex algebra $\mathcal{F}$ defined by

$$\deg\mathbf{1}=0,\quad \deg\psi_\alpha(n)=\begin{cases}1, & \text{for}\quad \alpha\in\Delta_-,\\ -1, & \text{for}\quad \alpha\in\Delta_+.\end{cases}$$

For $i\in\mathbb{Z}$, set

$$C_k^i(\mathfrak{g})=V_{\widehat{\mathfrak{g}}}(k,0)\otimes\mathcal{F}^i.$$

This gives a $\mathbb{Z}$-gradation of $C_k(\mathfrak{g})$:

$$C_k(\mathfrak{g})=\bigoplus_{i\in\mathbb{Z}}C_k^i(\mathfrak{g}).$$

By definition,

$$Q_+\cdot C_k^i(\mathfrak{g})\subset C_k^{i+1}(\mathfrak{g}).$$

Then by Lemma 3.1, $(C_k(\mathfrak{g}),Q_+)$ is a BRST complex of vertex algebras in the sense of [4, Subsection 3.15] (see also [40]). This complex is called the BRST complex of the quantized Drinfeld-Sokolov ("+") reduction (see [4, 37, 40]). The following assertion was proved by Feigin and Frenkel [37] for generic k, by de Boer and Tjin [23] for k in the case that $\mathfrak{g}=\mathfrak{sl}_n$, and by Frenkel and Ben-Zvi [40] for the general case.

Theorem 3.2. *The cohomology $H^i(C_k(\mathfrak{g}))$ is zero for all $i \neq 0$.*

Set

$$\mathcal{W}^k(\mathfrak{g}) := H^0(C_k(\mathfrak{g})).$$

Then $\mathcal{W}^k(\mathfrak{g})$ is a vertex operator algebra. We have the following result from [40].

Theorem 3.3. *The vertex operator algebra $H^0(C_k(\mathfrak{g}))$ is strongly generated by elements of degrees d_i+1, $i=1,2,\ldots,l$, where d_i is the i-th exponent of $\mathfrak{g}$ and l is the rank of $\mathfrak{g}$.*

Denote by $\mathcal{W}_k(\mathfrak{g})$ the unique simple quotient of $\mathcal{W}^k(\mathfrak{g})$ at a non-critical level k. The following theorem has been proved in [63] in the case that $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$[1)] and in [25] in the case that $\mathfrak{g}=\mathfrak{sl}_3(\mathbb{C})$ and in [7] for the general case.

Theorem 3.4. *The simple $\mathcal{W}$-algebra $\mathcal{W}_k(\mathfrak{g})$ is rational (and C_2-cofinite [6]), and the set of isomorphism classes of minimal series representations of $\mathcal{W}^k(\mathfrak{g})$ forms the complete set of the isomorphism classes of simple modules over $\mathcal{W}_k(\mathfrak{g})$, if k satisfies $k+h_{\mathfrak{g}}^{\vee}=\frac{p}{q}\in\mathbb{Q}_{>0}$, $(p,q)=1$ and*

$$\begin{cases} p\geqslant h_{\mathfrak{g}}^{\vee}, \quad q\geqslant h_{\mathfrak{g}}, \quad \text{if} \quad (q,r^{\vee})=1,\\ p\geqslant h_{\mathfrak{g}}, \quad q\geqslant r^{\vee}h_{{}^L\mathfrak{g}}^{\vee}, \quad \text{if} \quad (q,r^{\vee})=r^{\vee},\end{cases}$$

where $h_{\mathfrak{g}}$ is the Coxeter number of $\mathfrak{g}$, $h_{\mathfrak{g}}^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$, ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra of $\mathfrak{g}$ and $r^{\vee}$ is the maximal number of the edges in the Dykin diagram of $\mathfrak{g}$.

The following conjecture is well known (see [18, 41, 55]).

Conjecture 3.5. *Let $\mathfrak{g}$ be a simply-laced simple Lie algebra over $\mathbb{C}$ and h its Coxeter number. Then for $l\in\mathbb{Z}_+$ and $k=-h+\frac{l+h}{l+h+1}$,*

$$\mathcal{W}_k(\mathfrak{g})\cong C_{L_{\widehat{\mathfrak{g}}}(l,0)\otimes L_{\widehat{\mathfrak{g}}}(1,0)}(L_{\widehat{\mathfrak{g}}}(l+1,0)).$$

4 Level-rank duality

For $k\in\mathbb{Z}_+$ and a complex finite-dimensional simple Lie algebra $\mathfrak{g}$ with normalized non-degenerate bilinear form, let $\widehat{\mathfrak{g}}$ be the corresponding affine Lie algebra and $L_{\widehat{\mathfrak{g}}}(k,0)$ be the simple vertex operator algebra associated with the integrable highest weight module of $\widehat{\mathfrak{g}}$ with level k. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $L_{\widehat{\mathfrak{h}}}(k,0)$ the associated Heisenberg vertex operator subalgebra of $L_{\widehat{\mathfrak{g}}}(k,0)$. Let

$$K(\mathfrak{g},k)=\{v\in L_{\widehat{\mathfrak{g}}}(k,0)\,|\,[Y(u,z_1),Y(v,z_2)]=0, u\in L_{\widehat{\mathfrak{h}}}(k,0)\}.$$

Then $K(\mathfrak{g},k)$ is the so-called parafermion vertex operator algebra (see [15, 27]).

Let $s\in\mathbb{Z}_{\geqslant 2}$ and $\underline{\ell}=(l_1,\ldots,l_s)$ such that $l_1,\ldots,l_s\in\mathbb{Z}_+$. Let $L_{\widehat{\mathfrak{g}}}(l_i,0)$ be the simple vertex operator algebra associated with the integrable highest weight module of $\widehat{\mathfrak{g}}$ with level l_i, $i=1,2,\ldots,s$. Then we have the tensor product vertex operator algebra

$$V=L_{\widehat{\mathfrak{g}}}(l_1,0)\otimes L_{\widehat{\mathfrak{g}}}(l_2,0)\otimes\cdots\otimes L_{\widehat{\mathfrak{g}}}(l_s,0).$$

Denote

$$l=|\underline{\ell}|=\sum_{i=1}^{s}l_i.$$

$\mathfrak{g}$ can be naturally imbedded into the weight one subspace of V diagonally as follows:

$$\begin{aligned}&\mathfrak{g}\hookrightarrow V_1\subseteq V,\\ &a\mapsto a(-1)\mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}+\mathbf{1}\otimes a(-1)\mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}+\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes a(-1)\mathbf{1}.\end{aligned}$$

It is known that the vertex operator subalgebra U of V generated by $\mathfrak{g}$ is isomorphic to the simple vertex operator algebra $L_{\widehat{\mathfrak{g}}}(l,0)$ (see [44, 49, 59]). Let $C_V(U)$ be the commutant of U in V. We have the following lemma (see [47]).

1) Beilinson A, Feigin B, Mazur B. Introduction to algebraic field theory on curves. 1993

Lemma 4.1. *$C_V(U)$ is a simple vertex operator subalgebra of V.*

Denote

$$s_0 = 0, \quad s_j = l_1 + l_2 + \cdots + l_j, \quad 1 \leqslant j \leqslant s.$$

For $l_k \geqslant 2$, let $\mathfrak{sl}_{l_k}(\mathbb{C})$ be the simple Lie subalgebra of $\mathfrak{sl}_l(\mathbb{C})$ consisting of matrices $A = (a_{ij})_{l\times l} \in \mathfrak{sl}_l(\mathbb{C})$ such that $a_{ij} = 0$ for all the pairs (i,j) such that at least one of i and j is not in the set $\{s_{k-1}+1, s_{k-1}+2, \ldots, s_k\}$. Let $\mathfrak{h}_{\underline{\ell}}$ be the abelian subalgebra of $\mathfrak{sl}_l(\mathbb{C})$ consisting of diagonal matrices $A \in \mathfrak{sl}_l(\mathbb{C})$ such that $[A,B]=0$, for all $B \in \mathfrak{sl}_{l_k}(\mathbb{C})$ such that $l_k \geqslant 2$. Then

$$\left[\mathfrak{h}_{\underline{\ell}}, \bigoplus_{k=1,l_k\geqslant 2}^{s} \mathfrak{sl}_{l_k}(\mathbb{C})\right] = 0.$$

Set

$$\mathfrak{l}_{\underline{\ell}} = \mathfrak{h}_{\underline{\ell}} \bigoplus \left(\bigoplus_{k=1,l_k\geqslant 2}^{s} \mathfrak{sl}_{l_k}(\mathbb{C}) \right).$$

Then $\mathfrak{l}_{\underline{\ell}}$ is a Levi subalgebra of $\mathfrak{sl}_l(\mathbb{C})$ and $\mathfrak{h}_{\underline{\ell}}$ is the center of $\mathfrak{l}_{\underline{\ell}}$ which is contained in the (fixed) Cartan subalgebra of $\mathfrak{sl}_l$. Denote by $L_{\widehat{\mathfrak{l}_{\underline{\ell}}}}(n,0)$ the vertex operator subalgebra of $L_{\widehat{\mathfrak{sl}_l}}(n,0)$ generated by $\mathfrak{l}_{\underline{\ell}}$. It is easy to see that

$$L_{\widehat{\mathfrak{l}_{\underline{\ell}}}}(n,0) \cong \left(\bigotimes_{k=1,l_k\geqslant 2}^{s} L_{\widehat{\mathfrak{sl}_{l_k}}}(n,0) \right) \bigotimes L_{\widehat{\mathfrak{h}_{\underline{\ell}}}}(n,0), \tag{4.1}$$

where $L_{\widehat{\mathfrak{h}_{\underline{\ell}}}}(n,0)$ is the Heisenberg vertex operator subalgebra of $L_{\widehat{\mathfrak{sl}_l}}(n,0)$ generated by $\mathfrak{h}_{\underline{\ell}}$. We denote

$$K(\mathfrak{sl}_l, \mathfrak{l}_{\underline{\ell}}, n) = C_{L_{\widehat{\mathfrak{sl}_l}}(n,0)}(L_{\widehat{\mathfrak{l}_{\underline{\ell}}}}(n,0)).$$

The following theorem comes from [47].

Theorem 4.2. *We have the following level-rank duality and the reciprocity law*:

$$C_{L_{\widehat{\mathfrak{sl}_n}}(l_1,0)\otimes\cdots\otimes L_{\widehat{\mathfrak{sl}_n}}(l_m,0)}(L_{\widehat{\mathfrak{sl}_n}}(l,0)) \cong K(\mathfrak{sl}_l, \mathfrak{l}_{\underline{\ell}}, n),$$
$$K(\mathfrak{sl}_l, \mathfrak{l}_{\underline{\ell}}, n) \cong C_{L_{\widehat{\mathfrak{sl}_n}}(l_1,0)\otimes\cdots\otimes L_{\widehat{\mathfrak{sl}_n}}(l_s,0)}(L_{\widehat{\mathfrak{sl}_n}}(l,0)) \cong C_{K(\mathfrak{sl}_l,n)}(K(\mathfrak{sl}_{l_1},n)\otimes\cdots\otimes K(\mathfrak{sl}_{l_s},n)).$$

Remark 4.3. If $l_1 = \cdots = l_m = 1$, then by Theorem 4.2,

$$C_{L_{\widehat{\mathfrak{sl}_n}}(1,0)^{\otimes l}}(L_{\widehat{\mathfrak{sl}_n}}(l,0)) \cong K(\mathfrak{sl}_l, n),$$

which was also established independently by Lam [56].

5 The commutant of $L_{\widehat{\mathfrak{sl}_n}}(l+1,0)$ in $L_{\widehat{\mathfrak{sl}_n}}(l,0) \otimes L_{\widehat{\mathfrak{sl}_n}}(1,0)$

Recall from [9,47,56] that $C_{L_{\widehat{\mathfrak{sl}_n}}(l,0)\otimes L_{\widehat{\mathfrak{sl}_n}}(1,0)}(L_{\widehat{\mathfrak{sl}_n}}(l+1,0)) \cong \mathcal{W}_k(\mathfrak{sl}_n)$ for $l=1$ and $k = -n + \frac{n+1}{n+2}$. So in the following we always assume that $l \geqslant 2$.

By Theorem 4.2, we have

$$\begin{aligned} C_{L_{\widehat{\mathfrak{sl}_n}}(l,0)\otimes L_{\widehat{\mathfrak{sl}_n}}(1,0)}(L_{\widehat{\mathfrak{sl}_n}}(l+1,0)) &\cong C_{L_{\widehat{\mathfrak{sl}_{l+1}}}(n,0)}(L_{\widehat{\mathfrak{sl}_l}}(n,0) \otimes L_{\widehat{\mathfrak{h}_l}}(n,0)) \\ &\cong C_{K(\mathfrak{sl}_{l+1},n)}(K(\mathfrak{sl}_l,n)), \end{aligned} \tag{5.1}$$

where $L_{\widehat{\mathfrak{sl}_l}}(n,0)$ is the vertex operator subalgebra of $L_{\widehat{\mathfrak{sl}_{l+1}}}(n,0)$ associated to the simple root system $\{\alpha_i, 1 \leqslant i \leqslant l-1\}$ and $L_{\widehat{\mathfrak{h}_l}}(n,0)$ is the Heisenberg vertex operator subalgebra of $L_{\widehat{\mathfrak{sl}_{l+1}}}(n,0)$ generated by

$$\mathfrak{h}_l = \mathbb{C}\left(\sum_{i=1}^{l} i h_{\alpha_i} \right).$$

Let $\{e_\alpha, f_\alpha, h_{\alpha_i} \,|\, \alpha \in \Delta_+, 1 \leqslant i \leqslant l\}$ be a Chevalley basis of $\mathfrak{sl}_{l+1}(\mathbb{C})$, and let e_α, f_α and h_α be a standard basis of the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ associated to $\alpha \in \Delta_+$. For $\alpha \in \Delta_+$, denote by W^α and ω^α the weight 3 generator and the Virasoro vector of $K(\mathfrak{sl}_{l+1}, n)$ associated to α introduced in [27], respectively. Then

$$\begin{aligned}
\omega^\alpha &= \frac{1}{2n(n+2)}[-nh_\alpha(-2)\mathbf{1} - h_\alpha(-1)^2\mathbf{1} + 2ne_\alpha(-1)f_\alpha(-1)\mathbf{1}],\\
W^\alpha &= n^2 h_\alpha(-3)\mathbf{1} + 3nh_\alpha(-2)h_\alpha(-1)\mathbf{1} + 2h_\alpha(-1)^3\mathbf{1}\\
&\quad - 6nh_\alpha(-1)e_\alpha(-1)f_\alpha(-1)\mathbf{1} + 3n^2(e_\alpha(-2)f_\alpha(-1)\mathbf{1} - e_\alpha(-1)f_\alpha(-2)\mathbf{1}).
\end{aligned}$$

For $\alpha \in \Delta_+$, we denote by $W^{4,\alpha}$ and $W^{5,\alpha}$ the primary vectors of weight 4 and weight 5 of $K(\mathfrak{sl}_{l+1}, n)$ associated to α introduced in [27], respectively. From [27], if $n = 3$, then both $W^{4,\alpha}$ and $W^{5,\alpha}$ are zeros for every $\alpha \in \Delta_+$, and $K(\mathfrak{sl}_{l+1}, n)$ is generated by $\{W^\alpha, \alpha \in \Delta_+\}$.

Denote

$$V = C_{K(\mathfrak{sl}_{l+1},n)}(K(\mathfrak{sl}_l, n)).$$

Then V has the conformal vector ω with the central charge c_V as follows:

$$\begin{aligned}
\omega &= \frac{n+2}{n+l+1}\Bigg(\sum_{1\leqslant i\leqslant l} \omega^{\alpha_i+\cdots+\alpha_l} - \sum_{1\leqslant i\leqslant j\leqslant l-1} \frac{1}{(n+l)}\omega^{\alpha_i+\cdots+\alpha_j}\Bigg),\\
c_V &= \frac{l(n-1)(2n+l+1)}{(n+l)(n+l+1)}.
\end{aligned} \tag{5.2}$$

For $\alpha, \beta \in \Delta$, denote by $c_{\alpha,\beta} \in \mathbb{C}$ the structure constant, i.e.,

$$[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}, \quad \alpha, \beta, \alpha+\beta \in \Delta.$$

It is easy to check that for $\alpha, \beta \in \Delta_+$ such that $\alpha + \beta \in \Delta_+$,

$$\begin{aligned}
\omega_1^\alpha W^\beta \;&= \frac{1}{n+2}[-3nh_\beta(-2)h_\alpha(-1) - 6h_\alpha(-1)h_\beta(-1)^2 + 6n(h_\alpha(-1) - h_\beta(-1))e_\beta(-1)f_\beta(-1)\\
&\quad + 6nh_\beta(-1)(e_{\alpha+\beta}(-1)f_{\alpha+\beta}(-1) - e_\alpha(-1)f_\alpha(-1)) + 3n^2(e_\beta(-2)f_\beta(-1) - e_\beta(-1)f_\beta(-2))\\
&\quad - 3n^2(e_\alpha(-2)f_\alpha(-1) - e_\alpha(-1)f_\alpha(-2) + e_{\alpha+\beta}(-2)f_{\alpha+\beta}(-1) - e_{\alpha+\beta}(-1)f_{\alpha+\beta}(-2))\\
&\quad + 3n^2 c_{\alpha,\beta}(f_\alpha(-1)e_{\alpha+\beta}(-1)f_\beta(-1) + e_\alpha(-1)f_{\alpha+\beta}(-1)e_\beta(-1))]\mathbf{1}.
\end{aligned}$$

The following lemmas can be checked directly.

Lemma 5.1. *For $\alpha, \beta \in \Delta_+$ such that $\alpha + \beta \in \Delta$, we have*

$$\begin{aligned}
\omega_1^\alpha W^{\alpha+\beta} &= \omega_1^\alpha W^\beta + \frac{1}{n+2}(2W^{\alpha+\beta} + W^\alpha - 2W^\beta),\\
\omega_1^\alpha W^\beta + \omega_1^\beta W^\alpha &= \frac{1}{n+2}(W^\alpha + W^\beta - W^{\alpha+\beta}).
\end{aligned}$$

Lemma 5.2. *For $1 \leqslant p \leqslant l-1$, we have*

$$\begin{aligned}
&e_{\alpha_p}(0)\Bigg(\sum_{1\leqslant i\leqslant j<l} 2(n+2)w_1^{\alpha_i+\cdots+\alpha_j} W^{\alpha_{j+1}+\cdots+\alpha_l} - \sum_{i=1}^{l}(n+4i-2)W^{\alpha_i+\cdots+\alpha_l}\Bigg)\\
&\quad = (2n^2 - 12np + 16)e_{\alpha_p}(-3)\mathbf{1} + (-3n^2 + 6np - 12p - 12n - 12)h_{\alpha_p}(-1)e_{\alpha_p}(-2)\mathbf{1}\\
&\qquad + (3n^2 + 6np)h_{\alpha_p}(-2)e_{\alpha_p}(-1)\mathbf{1} + (6n + 12p)h_{\alpha_p}(-1)^2 e_{\alpha_p}(-1)\mathbf{1}\\
&\qquad - 12n \sum_{1\leqslant i\leqslant p-1} c_{\alpha_p,\alpha_i+\cdots+\alpha_{p-1}} h_{\alpha_p}(-1)e_{\alpha_i+\cdots+\alpha_p}(-1)f_{\alpha_i+\cdots+\alpha_{p-1}}(-1)\mathbf{1}\\
&\qquad - 12n \sum_{1\leqslant i\leqslant p} e_{\alpha_p}(-1)e_{\alpha_i+\cdots+\alpha_p}(-1)f_{\alpha_i+\cdots+\alpha_p}(-1)\mathbf{1}
\end{aligned}$$

$$+ 12n \sum_{1\leqslant i\leqslant p-1} e_{\alpha_p}(-1)e_{\alpha_i+\cdots+\alpha_{p-1}}(-1)f_{\alpha_i+\cdots+\alpha_{p-1}}(-1)\mathbf{1}$$
$$- 24 \sum_{1\leqslant i\leqslant p-1} h_{\alpha_i+\cdots+\alpha_{p-1}}(-1)(e_{\alpha_p}(-2)\mathbf{1} - h_{\alpha_p}(-1)e_{\alpha_p}(-1)\mathbf{1}).$$

Lemma 5.3. *For $1 \leqslant p \leqslant l-1$, we have*

$$\begin{aligned}
&e_{\alpha_p}(0)\Bigg(\sum_{1\leqslant i\leqslant j<q\leqslant l-1} 2(n+2)w_1^{\alpha_i+\cdots+\alpha_j}W^{\alpha_{j+1}+\cdots+\alpha_q} - \sum_{1\leqslant i\leqslant q\leqslant l-1}(n+4i-2)W^{\alpha_i+\cdots+\alpha_q}\Bigg)\\
&= (n+l)\Bigg[(2n^2-12np+16)e_{\alpha_p}(-3)\mathbf{1} + (-3n^2+6np-12p-12n-12)h_{\alpha_p}(-1)e_{\alpha_p}(-2)\mathbf{1}\\
&\quad + (3n^2+6np)h_{\alpha_p}(-2)e_{\alpha_p}(-1)\mathbf{1} + (6n+12p)h_{\alpha_p}(-1)^2e_{\alpha_p}(-1)\mathbf{1}\\
&\quad - 12n\sum_{1\leqslant i\leqslant p-1} c_{\alpha_p,\alpha_i+\cdots+\alpha_{p-1}}h_{\alpha_p}(-1)e_{\alpha_i+\cdots+\alpha_p}(-1)f_{\alpha_i+\cdots+\alpha_{p-1}}(-1)\mathbf{1}\\
&\quad - 12n\sum_{1\leqslant i\leqslant p} e_{\alpha_p}(-1)e_{\alpha_i+\cdots+\alpha_p}(-1)f_{\alpha_i+\cdots+\alpha_p}(-1)\mathbf{1}\\
&\quad + 12n\sum_{1\leqslant i\leqslant p-1} e_{\alpha_p}(-1)e_{\alpha_i+\cdots+\alpha_{p-1}}(-1)f_{\alpha_i+\cdots+\alpha_{p-1}}(-1)\mathbf{1}\\
&\quad - 24\sum_{1\leqslant i\leqslant p-1} h_{\alpha_i+\cdots+\alpha_{p-1}}(-1)(e_{\alpha_p}(-2)\mathbf{1} - h_{\alpha_p}(-1)e_{\alpha_p}(-1)\mathbf{1})\Bigg].
\end{aligned}$$

Lemma 5.4. *For $\alpha,\beta,\gamma\in\Delta_+$, we have*

$$\omega_2^\gamma W^\alpha = 0, \quad \omega_2^\gamma\omega_1^\alpha W^\beta = 0.$$

Denote

$$\begin{aligned}
&X^{(0)} = 0, \quad X^{(1)} = -(n+2)W^{\alpha_1},\\
&X^{(p)} = 2(n+2)\sum_{1\leqslant i\leqslant j<q\leqslant p}\omega_1^{\alpha_i+\cdots+\alpha_j}W^{\alpha_{j+1}+\cdots+\alpha_q} - \sum_{1\leqslant i\leqslant q\leqslant p}(n+4i-2)W^{\alpha_i+\cdots+\alpha_q},
\end{aligned}$$

for $p\geqslant 2$. Set

$$W = \frac{1}{n+l}X^{(l-1)} - \frac{1}{n+l+1}X^{(l)}.$$

We have the following result.

Theorem 5.5. *For $l\geqslant 1$, W is a primary vector of $V = C_{K(\mathfrak{sl}_{l+1},n)}(K(\mathfrak{sl}_l,n))$ of weight 3.*

Proof. It follows from Lemmas 5.2 and 5.3 that for $1\leqslant p\leqslant l-1$,

$$e_{\alpha_p}(0)W = 0. \tag{5.3}$$

Consider $L_{\widehat{\mathfrak{sl}_{l+1}}}(n,0)$ as a module of the simple Lie algebra generated by $e_{\alpha_p}, f_{\alpha_p}$ and h_{α_p}, for $1\leqslant p\leqslant l-1$. Then (5.3) implies that W is a highest weight vector. Since $h_{\alpha_p}(0)W = 0$ for $1\leqslant p\leqslant l$, we have $f_{\alpha_p}(0)W = 0$, $1\leqslant p\leqslant l-1$. Notice that $W\in K(\mathfrak{sl}_{l+1},n)$. It follows that $W\in C_{K(\mathfrak{sl}_{l+1},n)}(K(\mathfrak{sl}_l,n))$. Lemma 5.4 implies that W is a primary element. □

For $\alpha\in\Delta_+$, from [27] we have the following lemma.

Lemma 5.6. *It holds that*

$$\begin{aligned}
W_5^\alpha W^\alpha &= 12n^3(n-2)(n-1)(3n+4)\mathbf{1},\\
W_3^\alpha W^\alpha &= 36n^3(n-2)(n+2)(3n+4)\omega^\alpha,\\
W_2^\alpha W^\alpha &= 18n^3(n-2)(n+2)(3n+4)\omega_{-2}^\alpha\mathbf{1}.
\end{aligned}$$

The following lemmas can be checked directly.

Lemma 5.7. *For $\alpha, \beta \in \Delta_+$ such that $\alpha \neq \pm\beta$*

$$W_5^\alpha W^\beta = 6(\alpha\,|\,\beta)n^3(n-1)(n-2)\mathbf{1},$$
$$W_2^\alpha W^\beta = 18(\alpha\,|\,\beta)n^3(n-2)(n+2)^2\omega_0^\alpha\omega^\beta,$$
$$W_3^\alpha W^\beta = \begin{cases} -18n^3(n-2)(n+2)(\omega^\alpha+\omega^\beta-\omega^{\alpha+\beta}), & \text{if} \quad (\alpha\,|\,\beta)=-1,\\ 18n^3(n-2)(n+2)(\omega^\alpha+\omega^\beta-\omega^{\alpha-\beta}), & \text{if} \quad (\alpha|\beta)=1,\\ 0, & \text{if} \quad (\alpha\,|\,\beta)=0. \end{cases}$$

Lemma 5.8. *For $\alpha, \beta, \gamma \in \Delta_+$ such that $(\alpha|\beta) = -1$, we have*

$$W_3^\alpha\omega_1^\alpha W^\beta = 18n^3(n-2)[-(n+4)\omega^\beta+(n+4)\omega^{\alpha+\beta}-3(n+2)\omega^\alpha],$$
$$W_3^\beta\omega_1^\alpha W^\beta = 18n^3(n-2)[(n+4)\omega^\alpha+3(3n+4)\omega^\beta-(n+4)\omega^{\alpha+\beta}],$$
$$W_3^{\alpha+\beta}\omega_1^\alpha W^\beta = 18n^3(n-2)(n+1)(\omega^\alpha-\omega^\beta-3\omega^{\alpha+\beta}),$$
$$W_3^\gamma\omega_1^\alpha W^\beta = 18n^3(n-2)(\omega^{\alpha+\beta-\gamma}+\omega^\beta-\omega^{\alpha+\beta}-\omega^{\beta-\gamma}), \quad \text{if} \quad (\gamma\,|\,\alpha)=0, \quad (\gamma\,|\,\beta)=1,$$
$$W_3^\gamma\omega_1^\alpha W^\beta = 18n^3(n-2)(\omega^{\alpha+\beta}+\omega^{\beta+\gamma}-\omega^{\alpha+\beta+\gamma}-\omega^\beta), \quad \text{if} \quad (\gamma\,|\,\alpha)=0, \quad (\gamma\,|\,\beta)=-1,$$
$$W_3^\gamma\omega_1^\alpha W^\beta = 18n^3(n-2)(\omega^\alpha+2\omega^\beta+3\omega^\gamma-\omega^{\alpha+\gamma}-2\omega^{\gamma-\beta}), \quad \text{if} \quad (\gamma\,|\,\alpha)=-1, \quad (\gamma\,|\,\beta)=1,$$
$$W_3^\gamma\omega_1^\alpha W^\beta = 0, \quad \text{if} \quad (\alpha\,|\,\gamma)=1, \quad (\gamma\,|\,\beta)=0.$$

By Lemmas 5.6–5.8, we can deduce the following result.

Lemma 5.9. *It holds that*

$$W_5 W = \frac{6n^3 l(n-1)(n-2)(n+2l)(2n+l+1)(3n+2l+2)}{(n+l+1)(n+l)}\mathbf{1},$$
$$W_3 W = 36n^3(n-2)(n+2l)(3n+2l+2)\omega.$$

Notice that

$$W_2 W = -W_2 W + \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j!}L(-1)^j W_{j+2}W.$$

So by Lemma 5.9, we have

$$W_2 W \in \langle\omega\rangle, \tag{5.4}$$

where $\langle\omega\rangle$ is the Virasoro vertex operator algebra generated by the conformal vector ω of V.

For $k+n=\frac{n+l}{n+l+1}$, let $\widetilde{V}=\mathcal{W}_k(\mathfrak{sl}_n)$ be the $\mathcal{W}$-algebra associated to $\mathfrak{sl}_n$. Denote by $\widetilde{\omega}$ its conformal vector. Let $\widetilde{W}$ be a weight-three primary vector of $\mathcal{W}_k(\mathfrak{sl}_n)$ such that

$$(\widetilde{W},\widetilde{W}) = \frac{6n^3 l(n-1)(n-2)(n+2l)(2n+l+1)(3n+2l+2)}{(n+l+1)(n+l)}\mathbf{1}. \tag{5.5}$$

Then by the fusion rules of the Virasoro algebra, we have

$$\widetilde{W}_3\widetilde{W} = 36n^3(n-2)(n+2l)(3n+2l+2)\widetilde{\omega}. \tag{5.6}$$

Note that $c_V = c_{\widetilde{V}}$. We denote $c = c_V = c_{\widetilde{V}}$.

In view of (5.1), the character of V has been given in [53, Theorem 3.1]. This coincides with the character of $\widetilde{V}$ that was conjectured in [41, Conjecture 3.4$_-$ and Proposition 3.4] and proved in [4, Main Theorem 1]. Therefore we have the following.

Lemma 5.10. *V and $\widetilde{V}$ share the same characters.*

Recall from Theorem 3.4 that $\mathcal{W}_k(\mathfrak{g})$ is rational if $k=-h+\frac{p+h}{p+h+1}$, for some $p\in\mathbb{Z}_+$. Then by [31] the Virasoro vertex operator subalgebra of $\mathcal{W}_k(\mathfrak{g})$ generated by $\widetilde{\omega}$ is simple, denoted by $L(c,0)$. The following lemma follows from the OPE given in [40, Subsection 15.3.2].

Lemma 5.11. *Let $\mathfrak{g}=\mathfrak{sl}_3$ and $k=-h+\frac{l+h}{l+h+1}$. Then $\widetilde{W}_i\widetilde{W}\in L(c,0)$, for all $i\geqslant 0$.*

Recall from (5.2) that for $n=3$, $c=\frac{2l(l+7)}{(l+3)(l+4)}$. Since $l\geqslant 2$, we have $c>1$. By [50] the Virasoro vertex operator subalgebra of V generated by ω is simple. We identify the simple Virasoro vertex operator subalgebra $L(c,0)$ inside V and $\widetilde{V}$. By [24, Proposition 3.3], both V and $\widetilde{V}$ are completely reducible $L(c,0)$-modules. We have the following lemma.

Lemma 5.12. *Assume that $n=3$. Then $W_iW=\widetilde{W}_i\widetilde{W}\in L(c,0)$, for $i\geqslant 0$.*

Proof. By Lemmas 5.11 and 5.9, (5.5), and the fusion rules of Virasoro vertex operator algebras, it is enough to prove that $W_iW\in L(c,0)$. By Lemma 5.9 and (5.4), $W_iW\in L(c,0)$, $i\geqslant 2$. Then by the skew-symmetry, we only need to show that $W_1W\in L(c,0)$. Recall from Theorem 3.3 that $\widetilde{V}=\mathcal{W}_k(\mathfrak{sl}_3)$ is strongly generated by $\widetilde{\omega}$ and $\widetilde{W}$. Thus, $(\mathcal{W}_k(\mathfrak{sl}_3))_4=(L(c,0))_4+\mathbb{C}L(-1)\widetilde{W}$. Moreover, the character formula of $\mathcal{W}_k(\mathfrak{sl}_3)$ obtained in [4] shows that

$$(\mathcal{W}_k(\mathfrak{sl}_3))_4=(L(c,0))_4\oplus\mathbb{C}L(-1)\widetilde{W}.$$

Then by Lemma 5.10, we may assume that

$$W_1W=aL(-1)W+u,$$

for some $u\in L(c,0)$ and $a\in\mathbb{C}$. If $a\neq 0$, then

$$L(1)(W_1W-u)=aL(1)L(-1)W=6aW. \tag{5.7}$$

On the other hand, we have

$$L(1)W_1W=\sum_{i=0}^{2}\binom{2}{i}(\omega_iW)_{3-i}W=3W_2W\in L(c,0),$$

contradicting (5.7). We deduce that $a=0$, which implies that $W_1W\in L(c,0)$. □

Denote by U the vertex operator subalgebra of V generated by W. Then by Lemma 5.12, for $n=3$, U is linearly spanned by

$$\begin{aligned}\mathcal{S}=\{&L(-m_s)\cdots L(-m_1)W_{-k_p}\cdots W_{-k_1}\mathbf{1},k_p\geqslant\cdots\geqslant k_1\geqslant 1,\\ &n\in\mathbb{Z},\ m_s\geqslant\cdots\geqslant m_1\geqslant 1,\ s,p\geqslant 0\}.\end{aligned}$$

From Theorem 3.3, for $n=3$, $\widetilde{V}=\mathcal{W}_k(\mathfrak{sl}_n)$ is linearly spanned by

$$\begin{aligned}\widetilde{\mathcal{S}}=\{&L(-m_s)\cdots L(-m_1)\widetilde{W}_{-k_p}\cdots\widetilde{W}_{-k_1}\mathbf{1},\ k_p\geqslant\cdots\geqslant k_1\geqslant 1,\\ &n\in\mathbb{Z},\ m_s\geqslant\cdots\geqslant m_1\geqslant 1,\ s,p\geqslant 0\}.\end{aligned}$$

For any

$$u=\sum_{j=1}^{q}b_jL(-n_{j1})\cdots L(-n_{jt_j})W_{-r_{j1}}\cdots W_{-r_{jp_j}}\mathbf{1},$$

where $q\geqslant 1,t_j,p_j\geqslant 0$, $n_{j1},\dots,n_{jt_j},r_{j1},\dots,r_{jp_j}\in\mathbb{Z}_+$, $j=1,2,\dots,q$, we always denote

$$\widetilde{u}=\sum_{j=1}^{q}b_jL(-n_{j1})\cdots L(-n_{jt_j})\widetilde{W}_{-r_{j1}}\cdots\widetilde{W}_{-r_{jp_j}}\mathbf{1}.$$

We have the following lemma.

Lemma 5.13. *For $m\in\mathbb{Z}$ and*

$$u=\sum_{j=1}^{q}b_jL(-n_{j1})\cdots L(-n_{jt_j})W_{-r_{j1}}\cdots W_{-r_{jp_j}}\mathbf{1},$$

where $q \geqslant 1, t_j, p_j \geqslant 0$, $n_{j1}, \ldots, n_{jt_j}, r_{j1}, \ldots, r_{jp_j} \in \mathbb{Z}_+$, $j = 1, 2, \ldots, q$, *if*

$$W_m u = v$$

is a linear combination of vectors from $\mathcal{S}$, *then*

$$\widetilde{W}_m \widetilde{u} = \widetilde{v}.$$

Proof. We may assume that u is homogeneous. We prove the lemma by induction on the weight of u. If $\mathrm{wt} u \leqslant 3$, the lemma is true by the fact that $(W, W) = (\widetilde{W}, \widetilde{W})$ and the fusion rules of the Virasoro algebra $L(c, 0)$. Suppose that the lemma holds for homogeneous u such that $\mathrm{wt} u < N$. We now assume that $\mathrm{wt} u = N$. For each monomial $u_j = L(-n_{j1}) \cdots L(-n_{jt_j}) W_{-r_{j1}} \cdots W_{-r_{jp_j}} \mathbf{1}$, let $W_m u_j = v_j$ be a linear combination of elements in $\mathcal{S}$. It is obvious that we may assume that $t_j = 0$. If $m \leqslant -r_{j1}$, then we have $\widetilde{W}_m \widetilde{u}_j = \widetilde{v}_j$. If $m > -r_{j1}$, we have

$$\begin{aligned} W_m u_j &= W_m W_{-r_{j1}} \cdots W_{-r_{jp_j}} \mathbf{1} \\ &= \sum_{s=0}^{\infty} \binom{m}{s} (W_s W)_{m - r_{j1} - s} W_{-r_{j2}} \cdots W_{-r_{jp_j}} \mathbf{1} + W_{-r_{j1}} W_m W_{-r_{j2}} \cdots W_{-r_{jp_j}} \mathbf{1}, \\ \widetilde{W}_m u_j &= \widetilde{W}_m \widetilde{W}_{-r_{j1}} \cdots \widetilde{W}_{-r_{jp_j}} \mathbf{1} \\ &= \sum_{s=0}^{\infty} \binom{m}{s} (\widetilde{W}_s \widetilde{W})_{m - r_{j1} - s} \widetilde{W}_{-r_{j2}} \cdots \widetilde{W}_{-r_{jp_j}} \mathbf{1} + \widetilde{W}_{-r_{j1}} \widetilde{W}_m \widetilde{W}_{-r_{j2}} \cdots \widetilde{W}_{-r_{jp_j}} \mathbf{1}. \end{aligned}$$

Then the lemma follows from Lemma 5.12 and the inductive assumption. □

Lemma 5.14. *Suppose that* $n = 3$. *For any* $q \geqslant 1, t_j, p_j \geqslant 0$, $n_{j1}, \ldots, n_{jt_j}, r_{j1}, \ldots, r_{jp_j} \in \mathbb{Z}_+$, $j = 1, 2, \ldots, q$, *if*

$$u = \sum_{j=1}^{q} b_j L(-n_{j1}) \cdots L(-n_{jt_j}) W_{-r_{j1}} \cdots W_{-r_{jp_j}} \mathbf{1} = 0$$

for some $b_j \in \mathbb{C}$, *then*

$$\widetilde{u} = \sum_{j=1}^{q} b_j L(-n_{j1}) \cdots L(-n_{jt_j}) \widetilde{W}_{-r_{j1}} \cdots \widetilde{W}_{-r_{jp_j}} \mathbf{1} = 0.$$

Proof. We may assume that u is a linear combination of homogeneous elements having the same weight. Suppose that $\tilde{u} \neq 0$. Since $\mathcal{W}_k(\mathfrak{sl}_n)$ is self-dual and generated by $\widetilde{W}$, there is $\widetilde{W}_{r_1} \widetilde{W}_{r_2} \cdots \widetilde{W}_{r_q} \mathbf{1} \in \mathcal{W}_k(\mathfrak{sl}_n)$ such that

$$(\widetilde{u}, \widetilde{W}_{r_1} \widetilde{W}_{r_2} \cdots \widetilde{W}_{r_q} \mathbf{1}) \neq 0. \tag{5.8}$$

Claim. For any $k_p \geqslant \cdots \geqslant k_1 \geqslant 1$, $q_1, q_2, \ldots, q_t \in \mathbb{Z}$, $m_s \geqslant \cdots \geqslant m_1 \geqslant 1$, $p, t \geqslant 0$,

$$\begin{aligned} &(L(-m_s) \cdots L(-m_1) W_{-k_p} \cdots W_{-k_1} \mathbf{1}, W_{q_1} W_{q_2} \cdots W_{q_t} \mathbf{1}) \\ &\quad = (L(-m_s) \cdots L(-m_1) \widetilde{W}_{-k_p} \cdots \widetilde{W}_{-k_1} \mathbf{1}, \widetilde{W}_{q_1} \widetilde{W}_{q_2} \cdots \widetilde{W}_{q_t} \mathbf{1}). \end{aligned} \tag{5.9}$$

We may assume that

$$\mathrm{wt}(L(-m_s) \cdots L(-m_1) \widetilde{W}_{-k_p} \cdots \widetilde{W}_{-k_1} \mathbf{1}) = \mathrm{wt}(\widetilde{W}_{q_1} \widetilde{W}_{q_2} \cdots \widetilde{W}_{q_t} \mathbf{1}).$$

We prove (5.9) by induction on $\mathrm{wt}(L(-m_s) \cdots L(-m_1) \widetilde{W}_{-k_p} \cdots \widetilde{W}_{-k_1} \mathbf{1})$. By Lemma 5.9 and (5.5), (5.9) holds if $\mathrm{wt}(L(-m_s) \cdots L(-m_1) \widetilde{W}_{-k_p} \cdots \widetilde{W}_{-k_1} \mathbf{1}) \leqslant 3$. Now assume that the claim holds for

$$\mathrm{wt}(L(-m_s) \cdots L(-m_1) \widetilde{W}_{-k_p} \cdots \widetilde{W}_{-k_1} \mathbf{1}) < N.$$

Then by Lemma 5.12, inductive assumption, and the invariance of the bilinear form, the claim holds for $L(-m_s)\cdots L(-m_1)\widetilde{W}_{-k_p}\cdots\widetilde{W}_{-k_1}\mathbf{1}$ such that $\mathrm{wt}(L(-m_s)\cdots L(-m_1)\widetilde{W}_{-k_p}\cdots\widetilde{W}_{-k_1}\mathbf{1})=N$.

By the claim and (5.8), we have

$$(u, W_{q_1}W_{q_2}\cdots W_{q_t}\mathbf{1})\neq 0,$$

which contradicts the assumption that $u=0$. □

Theorem 5.15. *For $n=3$ and $k=-n+\frac{n+l}{n+l+1}$, we have $C_{K(\mathfrak{sl}_{l+1},n)}(K(\mathfrak{sl}_l,n))\cong\mathcal{W}_k(\mathfrak{sl}_n)$.*

Proof. Define $\varphi: U\to\widetilde{V}$ as follows: for any

$$u=\sum_{j=1}^{q}b_jL(-n_{j1})\cdots L(-n_{jt_j})W_{-r_{j1}}\cdots W_{-r_{jp_j}}\mathbf{1},$$

where $q\geqslant 1, t_j, p_j\geqslant 0$, $n_{j1},\ldots,n_{jt_j},r_{j1},\ldots,r_{jp_j}\in\mathbb{Z}_+$, $j=1,2,\ldots,q$,

$$\varphi(u)=\widetilde{u}=\sum_{j=1}^{q}b_jL(-n_{j1})\cdots L(-n_{jt_j})\widetilde{W}_{-r_{j1}}\cdots\widetilde{W}_{-r_{jp_j}}\mathbf{1}.$$

By Lemmas 5.14 and 5.12, φ is a surjective vertex operator algebra homomorphism from U to $\widetilde{V}$. Since $U\subseteq V$, and V and $\widetilde{V}$ share the same characters, we deduce that $U=V$ and φ is an isomorphism. □

Acknowledgements This work was supported by Japan Society for the Promotion of Science Grants (Grant Nos. 25287004 and 26610006) and National Natural Science Foundation of China (Grant Nos. 11371245 and 11531004).

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