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• **ARTICLES** •

Multiplicity one theorems, *S*-version

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Abstract It is well known by the strong multiplicity one that π is uniquely determined by the Satake parameter $c(\pi, v)$ for almost all v. Also, it suffices for us to test only finitely many v. We proved some S-effective version of multiplicity one theorems. Roughly speaking, if π and π' are not equivalent, then there is also a bound $\mathfrak{N}(S)$ which is some expression in terms of K, d and $\max(N(\pi), N(\pi'))$, which are analytic conductor of π and π', respectively (will be defined soon), such that there is a $v \notin S$ with $\pi_v \cong \pi'_v$ and $N\mathfrak{p}_v < \mathfrak{N}$. We also proved S-effective multiplicity one for the Chebotarev Density Theorem, and for GL(1).

Keywords multiplicity one theorem, S-version, automorphic form, L-function

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1 Introduction

Let K be a number field, and π a cuspidal automorphic representation of $\mathrm{GL}_d(\mathbb{A}_K)$. It is well known that π is uniquely determined by the Satake parameter $c(\pi, v)$ for almost all v, and even more precisely, almost all v of degree 1. Also, it suffices for us to test only finitely many v, namely, there is a bound $\mathfrak{N}$ which is some expression in terms of K, d, $N(\pi)$ such that there is a v with $\pi_v \cong \pi'_v$ and $N\mathfrak{p}_v < \mathfrak{N}$ (see [2, 12, 13, 16, 17, 19, 24, 31]).

In this paper, we are going to prove some S-effective version of multiplicity one theorems. Roughly speaking, all situation as above, if π and π' are not equivalent, then there is also a bound $\mathfrak{N}(S)$ which is some expression in terms of K, d and $\max(N(\pi), N(\pi'))$ which are analytic conductor of π and π', respectively (will be defined soon), such that there is a $v \notin S$ with $\pi_v \cong \pi'_v$ and $N\mathfrak{p}_v < \mathfrak{N}$. We encounter such S-versions in a series work [29, 30].

Now we state our first two theorems. One is the S-effective version of the Chebotarev density theorem, and the other one is the multiplicity one for $GL(1)$. Throughout, for any place v of K, denote $\mathfrak{p}_v$ to be the formal prime corresponding to v which denotes also the prime ideal of $\mathcal{O}_K$ or $\mathcal{O}_{K_v}$. d_K and d_L denote the discriminants of K and L, respectively, and $\zeta_K(s)$ and $\zeta_L(s)$ denote the Dedekind zeta functions of K and L, respectively. If χ is a global character of $\mathbb{A}_K^\times$, $N(\chi)$ denotes the conductor of χ (for explicit definitions see Section 2. Readers may also refer to [9, 19, 25–27]). For a finite set S of places of K, N_S denotes the product of norms of the places in S.

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Theorem 1.1 (S-effective version of the Chebotarev density theorem). *Let L/K be a finite Galois extension of number fields and $L \neq \mathbb{Q}$. Then there is an effectively computable absolute constant C satisfying the following: For each finite set S of places of K, a conjugacy class $[\sigma]$ in $\mathrm{Gal}(L/K)$, there is a place v of K such that*

(1) *$v \notin S$ and is of degree 1.*

(2) *The Artin symbol $(\frac{L/K}{\mathfrak{p}_v}) = [\sigma]$.*

(3) $N\mathfrak{p}_v \leqslant (d_L N_S^{[L:K]})^C$.

Theorem 1.2 (S-effective version of the multiplicity one for GL(1)). *Let K be a number field and χ be a global character of $\mathbb{A}_K^\times$. Then there is an effectively computable absolute constant C satisfying the following: For each finite set S of places of K, there is a place v of K such that*

(1) *$v \notin S$ and is of degree 1.*

(2) *χ_v is unramified and nontrivial.*

(3) $N\mathfrak{p}_v \leqslant (d_K N(\chi) N_S)^C$.

Now we make some definitions. Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_d(\mathbb{A}_K)$, and

$$L(s,\pi_\infty) = \pi^{-sdn_K/2} \prod_{j=1}^{dn_K} \Gamma((s+b_j(\pi))/2).$$

Moreover, $N(\pi)$ is the level of π so that $\Lambda(s,\pi) = L(s,\pi_\infty)L(s,\pi)$ satisfies

$$\Lambda(s,\pi) = W(\pi)N(\pi)^{1/2-s}\Lambda(1-s,\tilde{\pi}),$$

where $W(\pi)$ is the root number of π.

Now define the *extended analytic conductor* $C(\pi)$ (in the sense of [9, 27] etc.) to be

$$N(\pi)\prod_{j=1}^{dn_K}(1+|b_j(\pi)|).$$

Let π' be another unitary cuspidal automorphic representation of $\mathrm{GL}_{d'}(\mathbb{A}_K)$, and

$$L(s,\pi_\infty \times \pi'_\infty) = \pi^{-sdd'n_K/2} \prod_{j=1}^{dd'n_K} \Gamma((s+b_j(\pi\times\pi'))/2).$$

Moreover, $N(\pi\times\pi')$ is the level of the Rankin-Selberg $\pi\times\pi'$ so that

$$\Lambda(s,\pi\times\pi') = L(s,\pi_\infty\times\pi'_\infty)L(s,\pi_{\mathrm{f}}\times\pi'_{\mathrm{f}})$$

satisfies

$$\Lambda(s,\pi\times\pi') = W(\pi\times\pi')N(\pi\times\pi')^{1/2-s}\Lambda(1-s,\tilde{\pi}\times\tilde{\pi}'),$$

where $W(\pi\times\pi')$ is the root number of $\pi\times\pi'$.

Now define the *extended analytic conductor* $C(\pi\times\pi')$ (in the sense of [9, 27]) to be

$$N(\pi\times\pi')\prod_{j=1}^{dd'n_K}(1+|b_j(\pi\times\pi')|).$$

Moreover, define the *bound for Ramanujuan* for π be the upper log bounds of coefficients of the cusp form. Namely, for each $\pi \in \mathcal{A}_0(K)$, its bound for Ramanujuan $\mathrm{RB}(\pi) = \mathrm{Sup}_v \mathrm{Max}_{i=1,\dots,d} \log_{q_v} \alpha_{v,i}(\pi)$, where v runs through all finite places of K where π_v is unramified and $\alpha_{v,1},\dots,\alpha_{v,d}$ are Satake parameters of π_v.

Theorem 1.3. *Let π and π' be two unitary cuspidal automorphic representations of* $\mathrm{GL}_d(K)$*. Let S be a finite set of finite places of K, and $Q = \max(C(\pi), C(\pi'))$ and assume that the bound for Ramanujuan for π and π' are $< R$.*

Then if $\pi \not\cong \pi'$, there exists a place v of K such that $\pi_v \not\cong \pi'_v$, $v \notin S$ and

$$N\mathfrak{p}_v \leqslant \begin{cases} CQ^{1+\epsilon}N_S^{\epsilon}, & d = 1, \\ CQ^{2d+\frac{d(d-2)}{dH+1}+\epsilon}N_S^{\frac{d^3(2R+H)}{dH+1}+\epsilon}, & \text{general } d, \end{cases}$$

where C is some effectively computable constant only depending on arbitrarily chosen number $H > 2R$, $\epsilon > 0$, K and d.

Remark 1.4. When $d \geqslant 2$, the two ends of $H > 2R$ lead to the following two estimates:

$$N\mathfrak{p}_v \ll_\epsilon \min(Q^{2d+\frac{d(d-2)}{1+2dR}+\epsilon}N_S^{\frac{4Rd^3}{2dR+1}+\epsilon}, Q^{2d+\epsilon}N_S^{d^2+\epsilon}).$$

In particular, if $R = 0$ (i.e., Ramanujuan holds), then the two ends are

$$N\mathfrak{p}_v \ll_\epsilon \min(Q^{d^2+\epsilon}N_S^{\epsilon}, Q^{2d+\epsilon}N_S^{d^2+\epsilon}).$$

This theorem is an S-effective refinement of [17]. When take $H = 0$, and be aware that the unitary Grossencharacters are cusp forms of GL(1) with the bound for Ramanujuan as 0, we have the following corollary (see Theorem 4.1(B) and [30]).

Corollary 1.5. *Let K be a number field and χ a nontrivial unitary character of C_K. Then there is a place v of K such that*

(1) $\mathfrak{p}_v \notin S$.

(2) $\chi_v \neq 1$ *and is not ramified.*

(3) $N\mathfrak{p}_v \ll_{\epsilon,K} N(\chi)^{1/2+\epsilon}N_S^{\epsilon}$ *for every $\epsilon > 0$.*

Now we summarize the technique we used to proved the theorem. Theorems 1.1 and 1.2 use a refinement of arguments of [12] and Theorem 1.3 uses Landau's idea plus arguments with modification of series papers [2, 17, 31]. Results in [30] are just the special case.

This paper is a byproduct of the author's projects on the effective version of the Grunwald-Wang and here he expresses thanks to his advisor Dinakar Ramakrishnan for the introduction of the problem and the guidance and help during his student year and continuing years afterwards.

2 Notation and preliminaries

In this section, we recall certain notations and the preliminaries to be used in the proofs. Lots of standard results can be found in various textbooks. Experts can skip most parts of this section.

2.1 Hecke Characters and Hecke L-functions

First, we recall some basic concepts and facts. Throughout, K denotes a number field and $\mathbb{A}_K$, $\mathbb{A}_K^\times$ denote the Adele rings and the idele group of K. It is well known that K, $K^\times$ embed into $\mathbb{A}_K$ and $\mathbb{A}_K^\times$, respectively.

A Hecke character χ of K has two types of expression: It is a character of the group fractional ideals J_K of K, or a continuous character of $\mathbb{A}_K^\times/K^\times$. χ is also called a *global character*.

Let K_v denote the completion of K at a place v with $\mathcal{O}_{K_v}$, $\mathfrak{p}_v$, q_v the valuation ring, the prime ideal and the residue size $\#(\mathcal{O}_{K_v}/\mathfrak{p}_v)$ of K_v, respectively. A continuous character of K_v is called a *local character*. Given a global character χ, let χ_v be the restriction of χ to $K_v^\times$, then $\chi(x) = \prod_v \chi_v(x_v)$ if $x = (x_v) \in \mathbb{A}_K^\times$ and the product is finite for each x since all but finitely many χ_v are unramified.

Now define the conductor of χ_v and χ. For each finite place v, we define the *arithmetic conductor* $N(\chi_v)$ of χ_v as

$$N(\chi_v)=\begin{cases}1, & \text{if } \chi_v \text{ is unramified, i.e., } \chi_v(\mathcal{O}_{K_v})=1,\\ q_v^r, & \text{if } r \text{ is the smallest integer such that } \chi_v(1+\mathfrak{p}_v^r)=1\end{cases}$$

and $N(\chi)=\prod_v N(\chi_v)$.

Recall that the L-function $L(s,\chi)=\prod_{v<\infty}L(s,\chi_v)$ and the complete L-function,

$$\Lambda(s,\chi)=L_\infty(s,\chi)L(s,\chi)=\prod_v L(s,\chi_v),$$

where the Gamma factor is $L_\infty(s,\chi)=\prod_{v\,|\,\infty}L(s,\chi_v)$. Here the local L-factors are

$$\begin{aligned}L(s,\chi_v)&=\pi^{-(s+d)/2}\Gamma((s+d)/2)\quad (v \text{ is real, } \chi_v(x)=|x|^d \text{ or } |x|^{d-1}\mathrm{Sgn})\\ &=2(2\pi)^{-(s+d)}\Gamma(s+d)\quad (v \text{ is complex, } \chi_v(z)=z^a\bar{z}^b \text{ with } d=\max(a,b))\\ &=\frac{1}{1-\chi_v(\pi_v)q_v^{-s}}\quad (v \text{ is finite and } \chi_v \text{ is unramified})\\ &=1\quad (v \text{ is finite and } \chi_v \text{ is ramified}),\end{aligned}$$

where π_v is a uniformizer of K_v, i.e., a generator of the unique prime ideal (also denoted $\mathfrak{p}_v$) of $\mathcal{O}_{K_v}$.

When $\chi=1$ then $L(s,\chi)=\zeta_K(s)$. Moreover, it is well known that $L(s,\chi)$ is entire of order 1 if $\chi\neq 1$ is finite order. Moreover, $L(s,\chi)$ satisfies the following functional equation:

$$\Lambda(s,\chi)=W(\chi)A(\chi)^{1/2-s}\Lambda(1-s,\chi^{-1})$$

if χ is unitary. Here $A(\chi)=d_K N(\chi)$ is the *analytic conductor* of χ.

Let us list out some facts to be used in later parts of this paper. One is the Euler expression

$$-\frac{L'(s,\chi)}{L(s,\chi)}=\sum_{v<\infty,\chi_v\ \text{is unramified}}\ \sum_{k=1}^{\infty}\chi_v^k(\pi_v)q_v^{-ks}\log q_v$$

for Re$s>1$ which can be easily deduced from the Euler product. Moreover, by the class field theory, if χ is of finite order, then χ is associated to a cyclic extension L/K, and in fact, it is associated to a 1-dimensional representation of $\mathrm{Gal}(L/K)$ (see the preliminary on Galois representation). Moreover, Weil [32,33] showed that any Hecke character χ is an unramified twist of a character of finite order unless it is a CM character (see [6]).

2.2 Cuspidal and isobaric automorphic representations and automorphic L-functions

Now we recall some notation and facts the automorphic forms on $\mathrm{GL}(d)$. Most details can be found in various literatures (see [4,25], etc).

Denote $\mathcal{A}(d,K)$ (resp. $\mathcal{A}_0(d,K)$) the set of unitary irreducible automorphic (resp. cuspidal automorphic) representations of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathcal{A}(K)$ (resp. $\mathcal{A}_0(K)$) the set of unitary irreducible automorphic (resp. cuspidal automorphic) representations of $\mathrm{GL}_d(\mathbb{A}_K)$ for some d. Denote $L(s,\pi)=\prod_{v<\infty}L(s,\pi_v)$ be the (finite part) L-function associated to π, $L_\infty(s,\pi)=\prod_{v|\infty}L(s,\pi_v)$, $\Lambda(s,\pi)=L(s,\pi)L_\infty(s,\pi)$ be the infinite part, and the complete L-function of π, respectively.

For $\pi\in\mathcal{A}(K)$, It is well known that $\pi=\otimes\pi_v$ where π_v is an irreducible admissible representation of $\mathrm{GL}_d(K_v)$ for each v. Moreover, by the Langlands classification, for each $\pi\in\mathcal{A}(K)$, there are $\pi_1,\dots,\pi_r\in\mathcal{A}_0(K)$ such that we have the following isobaric sum decomposition (see [10,11]) $\pi=n_1\pi_1\boxplus\cdots\boxplus n_r\pi_m$. In particular, $L(s,\pi)=\prod_{i=1}^r L(s,\pi_i)^{n_i}$, $L_\infty(s,\pi)=\prod_{i=1}^r L_\infty(s,\pi_i)^{n_i}$.

For $\pi\in\mathcal{A}_0(K)$, it is also well known that, $L(s,\pi)$ and $\Lambda(s,\pi)$ are entire unless π is a trivial character so that $L(s,\pi)=\zeta_K(S)$ has a simple pole at $s=1$. In fact, we have $\Lambda(s,\pi)=W(\pi)N(\pi)^{1/2-s}\Lambda(1-s,\tilde{\pi})$

where $\tilde{\pi}$ is the contragredient of π. Here $N(\pi)$ is some positive integer called the *level* or the *analytic conductor* of π, and it is closedly related to the *arithmetic conductor* which is defined in an arithmetic way. In particular, if $\pi \in \mathcal{A}_0(1, K)$, then π is in fact a unitary continuous idele class character χ of $\mathbb{A}_K^\times/K^\times$, and the arithmetic conductor of χ is $N(\chi)$ and the analytic conductor is $A(\chi) = d_K N(\chi)$. Moreover $W(\pi)$ is a constant of complex number of absolute value 1.

Now we talk about the local L-factors. Let $\pi \in \mathcal{A}_0(d, K)$. For almost all finite place v, π_v is unramified. In fact, π_v is a constituent of the unramified (normalized) induced representation $\mathrm{Ind}_P^{\mathrm{GL}_n}(\chi_1 \otimes \cdots \otimes \chi_d)$ where $\chi_1, \ldots, \chi_d$ are unramified characters of $K_v^\times$. Let $a_{v,i} = \chi_i(\pi_p)$. Then we have

$$L(s, \pi_v) = \prod_{i=1}^{d} (1 - a_{v,i} q_v^{-s})^{-1} = \mathrm{Det}(1 - q^{-s} c(\pi, v))^{-1},$$

where *Satake Parameter* $c(\pi, v)$ is the conjugacy class of $\mathrm{diag}(a_{v,1}, \ldots, a_{v,d})$. Moreover, if v is an infinite place, then

$$L(s, \pi_v) = \prod_{j=1}^{d} \Gamma_v(s + b_{v,j}(\pi)),$$

for some $b_{v,j}(\pi)$ $(j = 1, \ldots, d)$. Here $\Gamma_v(s)$ is defined as $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. $b_{v,j}(\pi)$ are called the *infinite type constants.* Put $L(s, \pi_v)$ for $v \mid \infty$ together, we get $L_\infty(s, \pi)$ and also the infinite type constant $b_j(\pi)$ $j = 1, \ldots, dn_K$. In fact, $\{b_j(\pi)\}$ consists of those $b_{v,j}(\pi)$ for v real and $b_{v,j}(\pi)$ and $b_{v,j}(\pi) + 1$ for v complex.

Now let us talk about the "extended analytic conductor" $C(\pi)$. It is widely used in estimation. Unlike $N(\pi)$, it shows up quite differently in different literature. However, it is essentially the same for the purpose of the estimations. In fact, as $L_\infty(s, \pi) = \pi^{-dn_K/2} \prod_{j=1}^{dn_K} \Gamma((s + b_j(\pi))/2)$ we define

$$C(\pi) = N(\pi) \prod_{i=1}^{dn_K} (1 + |b_j(\pi)|).$$

In general, (at least for all cases we know) $\mathrm{Re} b_j(\pi) \geqslant 0$.

Now look at $\tilde{\pi}$. Note that $N(\tilde{\pi}) = N(\pi)$, As π is unitary, $\tilde{\pi} = \bar{\pi}$ so that all related quantities associated to $\tilde{\pi}$ are complex conjugate of those of π. In particular, the Satake parameter $c(\tilde{\pi}, v) = \overline{c(\pi, v)}$ when π_v is unramified.

Later on, we will see that $L(s, \pi)$ is standard.

2.3 Galois representations and Artin *L*-functions

By the class field theory, when χ is of finite order m, then χ is associated to a cyclic Galois extension L/K of degree m and $L(s, \chi) = L(s, \rho)$, where ρ is the 1 dimensional complex representation of $\mathrm{Gal}(L/K)$ associated to χ. In particular, v splits in L/K if and only if χ_v is trivial. So this is an example of Artin L-functions.

In general, let ρ be a Galois representation, namely, a finite dimensional representation of $\mathrm{Gal}(L/K)$ where L is a finite Galois extension of K. Recall that we can define the L-function as the following: Let V be the representation space of ρ and $n = \dim_{\mathbb{C}} V$. For each finite place v of K, w a place of L over v let $\sigma_{w/v} \in \mathrm{Gal}(L/K)$ be of the Frobenious element of L/K at w/v, i.e., $\sigma_{w/v} \in \mathrm{Gal}(L/K)$ such that (i) $\sigma_{w/v}(\mathfrak{p}_w) = \mathfrak{p}_w$ where $\mathfrak{p}_w$ the prime ideal of L associated to w, (ii) $\sigma_{w/v}(x) \equiv x^{q_w}$ where q_w be the size of $\mathcal{O}_{L_w}/\mathfrak{p}_w$. It is well known that, the conjugacy class of $\sigma_{w/v}$ is independent of the choice of w. Now define

$$L(s, \rho_v) = \det_{V^{I_{w/v}}}(I - q_v^{-s} \sigma_{w/v})^{-1},$$

where $I_{w/v}$ be the inertial group of L/K at w/v, and $V^{I_{w/v}}$ be the subspace of V fixed by $I_{w/v}$. It is easy to see that $L(s, \rho_v)$ is independent of the choice of w.

Define $L(s,\rho)=\prod_v L(s,\rho_v)$. We can also define the Artin conductor $A_{L/K}(\rho)$ (the explicit definition is a little bit complicated, and can be found in a lot of literatures.) It is well known that (1) $L(s,\rho)$ is meromorphic, (2) it allows a functional equation

$$\Lambda(s,\rho)=\pi^{-ns/2}\Gamma(s/2)^{a(\rho)}\Gamma((s+1)/2)^{b(\rho)}.$$

Recall that The Artin conjecture asserts that $L(s,\rho)$ is entire if ρ is a nontrivial irreducible representation. Moreover, the strong Artin conjecture asserts that, in this case there is a unitary cuspidal automorphic representation π of $\mathrm{GL}_d(\mathbb{A}_K)$ such that $L(s,\rho)=L(s,\pi)$. In this case, not only $L(s,\rho)$ has good analytic property but so do also a large families of L-functions, eg., the Rankin-Selberg L-functions (see later subsections). Moreover, by the standard argument which could be seen in a lot of literatures, the analytic Artin conductor $A_{L/K}(\rho)$ is the same as $N(\pi)$, the analytic conductor. In particular, this gives rises to internal relations of arithmetic features (for example, conductors) between ρ and π.

2.4 Rankin-Selberg *L*-functions

Now we recall the fact of Rankin-Selberg L-functions, which provide a large family of L-functions with good analytic properties. For details, see [3, 4, 10, 11].

Let π and π' be two isobaric automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathrm{GL}_{d'}(\mathbb{A}_K)$. Then we can define the Rankin-Selberg product $L(s,\pi\times\pi')$ which is an L-function of degree $dd'n_K$. We know there are various ways to define the Rankin-Selberg L-functions. One way is a formal one, namely, by the local Langlands. For each place v of K, let W'_{K_v} be the Weil-Delign group of the local field K_v, let $\sigma(\pi_v)$ be the d-dimensional admissible representations of W'_{K_v} associated to π_v, and $\sigma(\pi'_v)$ defined similarly. Define the local L-factors $L(s,\pi_v\times\pi'_v)=L(s,\sigma(\pi_v)\otimes\sigma(\pi'_v))$, where the local L-factors of Weil-Delign group representations are defined similarly as the L-functions of local Galois representations. In particular, if v is a finite place and π and π' are unramified at v, and

$$L(s,\pi_v)=\prod_{i=1}^{d}(1-a_{v,i}(\pi)q_v^{-s})^{-1},\quad L(s,\pi'_v)=\prod_{j=1}^{d'}(1-a_{v,j}(\pi')q_v^{-s})^{-1},$$

then

$$L(s,\pi_v\times\pi'_v)=\prod_{i=1}^{d}\prod_{j=1}^{d'}(1-a_{v,i}(\pi)a_{v,j}(\pi')q_v^{-s})^{-1}.$$

Like automorphic L-functions, we can define $L(s,\pi\times\pi')=\prod_{v<\infty}L(s,\pi_v\times\pi'_v)$, $L_\infty(s,\pi\times\pi')=\prod_v L(s,\pi_v\times\pi'_v)$ and $\Lambda(s,\pi\times\pi')=L(s,\pi\times\pi')L_\infty(s,\pi\times\pi')$.

We can also define these through certain zeta integrals.

It is well known that $L(s,\pi\times\pi')$ converges on some right half plane of s, and extends to a meromorphic function. There is a positive integer $N=N(\pi\times\pi')$ and a number $W=W(\pi\times\pi')$ of norm 1 such that

$$\Lambda(s,\pi\times\pi')=WN^{1/2-s}\Lambda(1-s,\tilde{\pi}\times\tilde{\pi}').$$

Using the language above, $L(s,\pi\times\pi')$ is standard when π and π' are isobaric, with its dual $L(s,\tilde{\pi}\times\tilde{\pi}')$.

Proposition 2.1 (See [3,10,11]). *Let π and π' be two isobaric automorphic representations of $\mathrm{GL}_d(\mathbb{A}_K)$ and $\mathrm{GL}_{d'}(\mathbb{A}_K)$, respectively.*

(1) *If π and π' are cuspidal, then $L(s,\pi\times\pi')$ is entire unless $\pi'\cong\tilde{\pi}\otimes||^{it_0}$, and in this case, $L(s,\pi\otimes\pi')$ has a simple pole at $s=1-it_0$, and is holomorphic else where.*

(2) *If $\pi=\boxplus m_i\pi_i$ and $\pi'=\boxplus n_j\pi'_j$ are the isobaric decompositions of π and π'. Then $L(s,\pi\times\pi')$ is meromorphic, and all of its poles lie on the line $\mathrm{Re}(s)=1$. Moreover, $s=1-it_0$ is a pole if and only if $\pi'_j\cong\tilde{\pi}_i\otimes||^{it_0}$ for some i and j, and the order of the pole at $s=1-it_0$ is $\sum m_jn_k$ where the sum is taken over (j,k) such that $\pi'_k\cong\tilde{\pi}_j\otimes||^{it_0}$.*

(3) *In particular, π is cuspidal if and only if $L(s,\pi\times\tilde{\pi})$ has a simple pole at $s=1$.*

Note that, the automorphic L-functions are the special case of the Rankin-Selberg L-functions.

Now we define the extended analytic conductor.

Write $L_\infty(s,\pi\times\pi') = \pi^{-dd'n_K/2}\prod_{j=1}^{dd'n_K}\Gamma((s+b_j(\pi\times\pi'))/2)$ we define

$$C(\pi\times\pi') = N(\pi\times\pi')\prod_{j=1}^{dd'n_K}(1+|b_j(\pi\times\pi')|).$$

In general, (at least for all cases we know) $\mathrm{Re}b_j(\pi)\geqslant 0$.

We quote the following estimation for (extended) analytic conductor Rankin-Selberg product:

Proposition 2.2. *Let $\pi\in\mathcal{A}(d,K)$ and $\pi'\in\mathcal{A}(d',K)$. Then there are constant $C'>0$ depending on n_K such that*

$$N(\pi\times\pi')\leqslant N(\pi)^{d'}N(\pi')^{d},\quad C(\pi\times\pi')\leqslant C'C(\pi)^{d'}C(\pi')^{d}.$$

The proof can be found in [5, 8] (by choosing $C'=2^{dd'n_K}$).

2.5 Bounds for Ramanujuan

In 1916, Ramanujuan conjectured that the Ramanujuan Δ function has its coefficients $\tau(n)\ll n^{11/2+\epsilon}$. More than half century later, Deligne proved this conjecture. In fact, the most general formulation of the Ramanujuan conjecture is formulated as, for each $\pi\in\mathcal{A}_0(d,K)$, and each place v of K, the Satake parameters $a_{v,j}$ are all purely imaginary numbers. (In this case, π_v is called *tempered*).

Unfortunately, this is still open, even for the case $\mathrm{GL}_2(\mathbb{Q})$. In fact, holomorphic modular forms satisfies the Ramanujuan conjecture but for the Maass wave forms, this is the Selberg conjecture, and is still open.

In fact, for each $\pi\in\mathcal{A}_0(K)$, we define the following bound called *bound for Ramanujuan* for π

$$\mathrm{RB}(\pi) = \mathrm{Max}_{v,j=1,\dots,d}\mathrm{Re}\log_{q_v}a_{v,j}(\pi)$$

and say π satisfies $H_d(\delta)$ if for (sufficiently large) v, $\mathrm{Re}\log_{q_v}a_{v,j}(\pi)\leqslant\delta$. See [1]. We say that $H_d(\delta)$ holds if each $\pi\in\mathcal{A}_0(K)$ satisfies $H_d(\delta)$.

Remark 2.3. We have, by [1], $H_n(R_n)$ holds for $R_2=7/64$, $R_3=5/14$, $R_4=9/22$ and $R_n=1/2-2/(n(n+1)+2)$ for general n.

Remark 2.4. One is easy to see that, if $\pi\in\mathcal{A}_0(K)$ has R as bound for Ramanujuan. Then $L(1+H+2R,\pi\times\tilde{\pi})\leqslant(\zeta_K(1+H))^{d^2}$ for $H>0$.

2.6 A lemma on coefficients of Rankin-Selberg L-functions of positive type

We quote a crucial lemma on coefficients of Rankin-Selberg L-functions of positive type (see [2, Lemma 2]).

Proposition 2.5. *Let $\pi\in\mathcal{A}_0(d,K)$, and $L(s,\pi\times\tilde{\pi})=\sum_{n=1}a_nn^{-s}$ be the Dirichlet expansion. For each place v of K, we have $a_{q_v^d}\geqslant 1$*

Remark 2.6. This is a corollary of certain properties of Schur polynomials.

3 L-O Method

In this section, we prove Theorems 1.1 and 1.2. We mainly follow the method of [12, 13]. Also see [29].

By the class field theory, idele class characters χ of finite order correspond in a canonical way to characters of $\mathrm{Gal}(\bar{K}/K)$ of finite order. By abuse of notation, we will still use the letter χ to denote this Galois character. Moreover, there is a canonically associated finite abelian extension L/K such that the kernel of χ as an idele class character is $N_{L/K}C_L\subset C_L$, the norm from C_L.

Throughout this section, for any number field L, d_L denotes the discriminant of L, and $d_{L/K}$ denotes the relative discriminant of L over a subfield K.

3.1 Preparations

First we introduce two kernel functions used in the classical analytic method, which was also used by Murty [20], Lagarias et al. [12,13] and Serre [28]. The use of these two different kernel functions is related to the *Explicit formulas* of Guinand [7] and Weil [32].

Let

$$k_1(s) = k_1(s; x, y) = \left(\frac{y^{s-1} - x^{s-1}}{s-1}\right)^2, \quad k_2(s) = k_2(s; x) = x^{s^2+s}.$$

Thus

$$k_1(1) = \left(\log \frac{y}{x}\right)^2, \quad k_2(1) = x^2.$$

For each smooth function $k(s)$, denote $\hat{k}(u)$ the inverse Mellin transform, defined as

$$\hat{k}(u) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} k(s) u^{-s} ds,$$

where a is a sufficiently large number.

Thus for $a > 1$, we have

$$\hat{k}_1(u) = \hat{k}_1(u; x, y) = \begin{cases} 0, & \text{if } u \geqslant y^2 \text{ or } u \leqslant x^2, \\ \dfrac{1}{u} \log \dfrac{y^2}{u}, & \text{if } xy \leqslant u \leqslant y^2, \\ \dfrac{1}{u} \log \dfrac{u}{x^2}, & \text{if } x^2 \leqslant u \leqslant xy, \end{cases}$$

$$\hat{k}_2(u) = \hat{k}_2(u; x) = (4\pi \log x)^{-\frac{1}{2}} \exp\left\{-\frac{(\log \frac{u}{x})^2}{4 \log x}\right\}.$$

Note that for each j and u, $\hat{k}_j(u) \geqslant 0$, and for large u, $\hat{k}_j(u)$ is small.

Lemma 3.1. *Assume that L/K is cyclic.*

(1) *Let Σ^R denote the summation over the prime ideals of K that ramify in L, then*

$$\sum_{\mathfrak{p}}^{R} \sum_{m \geqslant 1} \log(N\mathfrak{p}) \hat{k}_1(N\mathfrak{p}^m) \ll \frac{\log \frac{y}{x}}{x^2} \log d_L,$$

$$\sum_{\mathfrak{p}}^{R} \sum_{m \geqslant 1, N\mathfrak{p}^m \leqslant x^{10}} \log(N\mathfrak{p}) \hat{k}_2(N\mathfrak{p}^m) \ll (\log x)^{\frac{1}{2}} \log d_L.$$

(2) *Let Σ^S denote the summation over the prime ideals of K in S, then*

$$\sum_{\mathfrak{p}}^{S} \sum_{m \geqslant 1} \log(N\mathfrak{p}) \hat{k}_1(N\mathfrak{p}^m) \ll \frac{\log \frac{y}{x}}{x^2} \log N_S,$$

$$\sum_{\mathfrak{p}}^{S} \sum_{m \geqslant 1, N\mathfrak{p}^m \leqslant x^{10}} \log(N\mathfrak{p}) \hat{k}_2(N\mathfrak{p}^m) \ll (\log x)^{\frac{1}{2}} \log N_S.$$

(3) *Let Σ^P denote the summation over the pairs $(\mathfrak{p}, m)$ for which $N\mathfrak{p}^m$ is not a rational prime, then*

$$\sum_{\mathfrak{p}, m}^{P} \log(N\mathfrak{p}) \hat{k}_1(N\mathfrak{p}^m) \ll n_K \frac{(\log \frac{y}{x})(\log y)}{x(\log x)},$$

$$\sum_{\mathfrak{p}, m}^{P} \log(N\mathfrak{p}) \hat{k}_2(N\mathfrak{p}^m) \ll n_K x^{7/4}.$$

(4)

$$\sum_{\mathfrak{p}} \sum_{m\geqslant 1, N\mathfrak{p}^m > x^{3+\delta}} \log(N\mathfrak{p})\hat{k}_2(N\mathfrak{p}^m) \ll n_K x^{2-\frac{\delta^2}{4}}(\log x),$$

where δ is any positive number.

Proof. Also see [13] and [12].

(1-a) (See [12, Lemma 3.1])

$$\begin{aligned}\sum_{\mathfrak{p}}^{R} \sum_{m=1}^{\infty} \log(N\mathfrak{p})\hat{k}_1(N\mathfrak{p}^m) &\ll \sum_{\mathfrak{p}}^{R} \log(N\mathfrak{p}) \log(y/x) \sum_{m=1}^{\infty} (N\mathfrak{p})^{-m}\\ &\ll \frac{\log(y/x)}{x^2} \sum_{\mathfrak{p}}^{R} \log(N\mathfrak{p})\\ &\ll \frac{\log(y/x)}{x^2} \log(d_L).\end{aligned}$$

Here we use two facts: (i) $\hat{k}_1(u) \ll u^{-1}\log(y/x)$ and $\hat{k}_1 = 0$ if $u < x^2$; (ii) The conductor-discriminant formula $\sum_{\mathfrak{p}}^{R} \log(N\mathfrak{p}) \ll \log(d_L)$.

(1-b) (See [12, Lemma 3.1])

$$\begin{aligned}\sum_{\mathfrak{p}}^{R} \sum_{m\geqslant 1, N\mathfrak{p}^m \leqslant x^{10}} \log(N\mathfrak{p})\hat{k}_2(N\mathfrak{p}^m) &\ll \sum_{\mathfrak{p}}^{R} \log(N\mathfrak{p}) \sum_{m\geqslant 1, N\mathfrak{p}^m \leqslant x^{10}} (\log x)^{-1/2}\\ &\ll \sum_{\mathfrak{p}}^{R} \log(N\mathfrak{p})(\log x)^{1/2}\\ &\ll (\log x)^{1/2}\log(d_L).\end{aligned}$$

Here we use two trivial estimates $\hat{k}_2(u) \ll (\log x)^{-1/2}$ and $\#\{N\mathfrak{p}^m \leqslant x^{10}\} \ll \log x$.

(2) The proof is almost the same as (1) except that we need to estimate $\sum^{S} \log(N\mathfrak{p})$ instead of $\sum^{R} \log(N\mathfrak{p})$.

(3-a) (See [12, Lemma 3.2]) Use the fact that the number of pairs $(\mathfrak{p}, m)$ such that $N\mathfrak{p}^m = q$ is at most n_K, and we have

$$\begin{aligned}\sum_{\mathfrak{p},m}^{P} \log(N\mathfrak{p})\hat{k}_1(N\mathfrak{p}^m) &\ll n_K\left(\log\frac{y}{x}\right) \sum_{x^2\leqslant p^h\leqslant y^2, h\geqslant 2} p^{-h}\log p^h\\ &\ll n_K\left(\log\frac{y}{x}\right)(\log y) \sum_{n=p^a, a\geqslant 2, n\geqslant x^2} n^{-1}\\ &\ll n_K\left(\log\frac{y}{x}\right)(\log y)\frac{1}{x\log x},\end{aligned}$$

where the last bound uses the prime number theorem.

(3-b) (See [12, Lemma 3.2]) Let $S(u)$ denote the number of prime power integers $p^h (h \geqslant 2)$ in the interval $[1, u]$. It is easy to see that $S(u) \ll u^{1/2}$ since $S(u) \leqslant u^{1/2} + u^{1/3} + u^{1/5} + \cdots \ll u^{1/2} + u^{1/3}\log u$. Thus,

$$\begin{aligned}\sum_{\mathfrak{p}}^{P} \log(N\mathfrak{p})\hat{k}_2(N\mathfrak{p}^m) &\ll n_K \sum_{p, h\geqslant 2} \log(p^h)\hat{k}_2(p^h)\\ &\ll n_K \int_3^{\infty} (\log u)\,\hat{k}_2(u)\,dS(u)\\ &< -n_K \int_3^{\infty} S(u)d\left(\log(u)\hat{k}_2(u)\right)\end{aligned}$$

$$
\begin{aligned}
&\ll n_K \int_3^\infty u^{1/2}\log(u)(-\hat{k}_2'(u))du \\
&\ll n_K \int_3^\infty u^{1/2}\log(u)\hat{k}_2(u)\frac{\log(u/x)}{2\log x}\frac{x}{u}du \\
&\ll n_K \frac{x}{(\log x)^{3/2}}\int_3^\infty u^{-1/2}(\log u)^2\exp\bigg(-\frac{(\log(u/x))^2}{4\log x}\bigg)du \\
&\ll n_K \frac{x^{3/4}}{(\log x)^{3/2}}\int_3^\infty (\log u)^2\exp\bigg(-\frac{(\log u)^2}{4\log x}\bigg)du \\
&\ll n_K \frac{x^{3/4}}{(\log x)^{3/2}}\int_{-\infty}^\infty t^2\exp\bigg(-\frac{t^2}{4\log x}+t\bigg)dt \\
&\ll n_K x^{3/4}\int_{-\infty}^\infty t^2\exp(-t^2+t\sqrt{\log x})dt \\
&\ll n_K x^{3/4}\int_{-\infty}^\infty \bigg(t+\frac{\sqrt{\log x}}{2}\bigg)^2\exp(-t^2)x^{1/4}dt \\
&\ll n_K x\log(x) < n_K x^{7/4}.
\end{aligned}
$$

For (4),

$$
\begin{aligned}
\sum_{N\mathfrak{p}^m>x^{3+\delta}}\log(N\mathfrak{p})\hat{k}_2(N\mathfrak{p}^m) &\ll n_K\sum_{q>x^{3+\delta}}(\log q)\hat{k}_2(q) \\
&\ll n_K\int_{x^{3+\delta}}^{+\infty}(\log u)\hat{k}_2(u)du \\
&\ll n_K(\log x)^{-\frac12}\int_{(3+\delta)\log x}^{+\infty} t\exp\bigg(-\frac{(t-\log x)^2}{4\log x}\bigg)\mathrm{e}^t dt \\
&\ll n_K x^2(\log x)^{-\frac12}\int_{(3+\delta)\log x}^{+\infty} t\exp\bigg(-\frac{(t-3\log x)^2}{4\log x}\bigg)dt \\
&\ll n_K x^2(\log x)^{-\frac12}\bigg\{\int_{\delta\log x}^{+\infty}3\log x\exp\bigg(-\frac{t^2}{4\log x}\bigg)dt \\
&\quad+(\log x)\int_{\delta\sqrt{\log x}}^{+\infty}t\exp\bigg(-\frac{t^2}{4}\bigg)dt\bigg\} \\
&\ll n_K x^2(\log x)^{-\frac12}\bigg\{3(\log x)^{3/2}\int_{\delta\sqrt{\log x}}^{+\infty}\exp\bigg(-\frac{t^2}{4}\bigg)dt \\
&\quad+(\log x)\exp\bigg(-\frac{\delta^2}{4}\log x\bigg)\bigg\} \\
&\ll n_K x^{2-\frac{\delta^2}{4}}(\log x),
\end{aligned}
$$

where we use the following: $\hat{k}_2(u) \ll \hat{k}_2(x)$ if $|x-u|<1$, and a well known estimate $\int_T^{+\infty}\mathrm{e}^{-t^2/4}dt \ll \mathrm{e}^{-T^2/4}$. □

Lemma 3.2. *Let χ be a global character of C_K.*

(1) *If $N(t)=N_L(t)$ denotes the number of zeros $\rho=\beta+\mathrm{i}\gamma$, of $\zeta_L(s)$ with $0<\beta<1$ and $|\gamma-t|\leqslant 1$, then we have*

$$N(t)\ll \log d_L + n_L\log(|t|+2).$$

(2) *If $n(r;s)=n_L(r;s)$ denotes the number of zeros ρ, of $\zeta_L(s)$ with $|\rho-s|\leqslant r$, then we have*

$$n(r;s)\ll 1+r(\log d_L+n_L\log(|s|+2)).$$

(3) *If $N_\chi(t)$ denotes the number of zeros $\rho=\beta+\mathrm{i}\gamma$, of $L(s,\chi,K)$ with $0<\beta<1$ and $|\gamma-t|\leqslant 1$, then we have*

$$N_\chi(t)\ll \log A(\chi)+n_K\log(|t|+2).$$

(4) *If $n_\chi(r;s)$ denotes the number of zeros ρ of $L(s,\chi,K)$ with $|\rho - s| \leqslant r$, then we have*

$$n_\chi(r;s) \ll 1 + r(\log A(\chi) + n_K \log(|s|+2)).$$

Proof. These are standard results and the proof can be found in a lot of literatures. For examples, see [13, 21, 23] and [12, Lemma 2.2]. □

The following is the Siegel zero free region result for $L(s,\chi)$.

Lemma 3.3. *Let χ be a global character of K. There is a positive, absolute, effectively computable constant c_2 such that*

(1) *$L(s,\chi)$ has no zero $\rho = \beta + \mathrm{i}\gamma$ in the region*

$$\beta \geqslant 1 - {c_2}^{-1}(\log A(\chi) + n_K \log(|\gamma|+2))^{-1},$$
$$\gamma \geqslant (1 + c_2 \log A(\chi))^{-1},$$

where $\gamma \neq 0$.

(2) *$L(s,\chi)$ has at most one zero in the region*

$$\beta \geqslant 1 - (c_2 \log A(\chi))^{-1},$$
$$\gamma \leqslant c_2 \log A(\chi))^{-1}.$$

If such a zero exists, it must be simple and real, and χ must be trivial or quadratic.

Proof. See [13], or [15] and [21]. □

Before finishing this part, we quote the Deuring-Heilbronn phenomenon here, a discussion of which can be found in [12, Section 5].

Lemma 3.4 (Deuring-Heilbronn phenomenon)**.** *There are positive, absolute, effectively computable constants c_7 and c_8 such that if $\zeta_L(s)$ has a real zero β_0, then $\zeta_L(\sigma + \mathrm{i}t) \neq 0$ for*

$$\sigma \geqslant 1 - c_8 \cdot \frac{\log(\frac{c_7}{(1-\beta_0)\log(d_L \tau^{n_L})})}{\log(d_L \tau^{n_L})},$$

where $\tau = |t| + 2$ with the single exception $\sigma + \mathrm{i}t = \beta_0$.

Corollary 3.5. *There is a positive, absolute, effectively computable constant c_{10} such that any real zero β_0 of $\zeta_L(s)$ satisfies $1 - \beta_0 \geqslant {d_L}^{-c_{10}}$.*

Proof. See [12, Corollary 5.2]. □

3.2 Standard model

In this part, we will recall the main model of [12] for our method here. We have included the relevant details for the convenience of the readers.

We need to consider the Artin L-series $L(s,\phi,L/K)$ (see [12, 13, 15, 21]), where ϕ is the character of an irreducible representation of $G = \mathrm{Gal}(L/K)$. We have

$$-\frac{L'}{L}(s,\phi,L/K) = \sum_{\mathfrak{p}} \sum_{m \geqslant 1} \Phi_K(\mathfrak{p}^m) \log(N\mathfrak{p})(N\mathfrak{p})^{-ms},$$

where

$$\Phi_K(\mathfrak{p}^m) = \frac{1}{e_{\mathfrak{p}}(L/K)} \sum_{\alpha \in I_{\mathfrak{p}}(L/K)} \phi(\tau^m \alpha),$$

where $\tau = (\frac{L/K}{\mathfrak{p}})$ is one representative of the Frobenius element corresponding to $\mathfrak{p}$, $I_{\mathfrak{p}} = I_{\mathfrak{p}}(L/K)$ is the inertial subgroup of the decomposition group $G_{\mathfrak{p}} = \mathrm{Gal}(L_{\mathfrak{Q}}/K_{\mathfrak{p}})$ and $e_{\mathfrak{p}}(L/K) = |I_{\mathfrak{p}}|$ is the ramification index of $\mathfrak{Q}$ over $\mathfrak{p}$.

If $\mathfrak{p}$ is unramified in L then $\Phi_K(\mathfrak{p}^m) = \phi(\alpha^m)$. If L/K is abelian, then all irreducible ϕ are characters (and hence by the class field theory, correspond to Hecke characters).

Lemma 3.6. *Let C be a conjugacy class of G and g a representative of C, $H = \langle g \rangle$ and $E = L^H$ the fixed field of g. Then we have*

(1)

$$F_C(s) := -\frac{|C|}{|G|} \sum_{\phi \text{ irreducible}} \bar{\phi}(g) \frac{L'}{L}(s, \phi, L/K) = -\frac{|C|}{|G|} \sum_{\chi \in \hat{G}(L/E)} \bar{\phi}(g) \frac{L'}{L}(s, \chi, E),$$

where $\hat{G}(L/E)$ denotes the group of characters of $G(L/E)$, and

(2)

$$F_C(s) = \sum_{\mathfrak{p}} \sum_{m \geqslant 1} \theta(\mathfrak{p}^m) \log(N\mathfrak{p})(N\mathfrak{p})^{-s},$$

where

$$\theta(\mathfrak{p}^m) = \begin{cases} 1, & \text{if } \left(\frac{L/K}{\mathfrak{p}}\right)^m = C, \\ 0, & \text{if } \left(\frac{L/K}{\mathfrak{p}}\right)^m \neq C, \end{cases}$$

and $|\theta(\mathfrak{p}^m)| \leqslant 1$ if $\mathfrak{p}$ ramifies in L.

Proof. This is an exercise of representations theory. See [13, Section 5]. □

The previous lemma allows us to reduce the density problem to the case of a cyclic extension, for which we can use just the abelian L-series of Hecke.

The following lemma (see [13, 15]) describes a functional equation that $L(s, \chi, E)$ satisfies.

Lemma 3.7. *Let $L(s, \chi) = L(s, \chi, E)$ be the L-series associated to $\chi \in \hat{G}(L/E)$,*

$$A(\chi) = d_E N_{E/\mathbb{Q}}(\mathfrak{f}_0(\chi)),$$

where $\mathfrak{f}_0(\chi)$ denotes the finite conductor of χ,

$$\delta(\chi) = \begin{cases} 1, & \text{if } \chi \text{ is principal}, \\ 0, & \text{otherwise}. \end{cases}$$

There are nonnegative integers $a = a(\chi)$ and $b = b(\chi)$ such that

$$a(\chi) + b(\chi) = n_E.$$

Set

$$\gamma_\chi(s) = \left\{ \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \right\}^b \left\{ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \right\}^a$$

and

$$\Lambda(s, \chi) = (s(s-1))^{\delta(\chi)} A(\chi)^{s/2} \gamma_\chi(s) L(s, \chi).$$

Then $\Lambda(s, \chi)$ satisfies the functional equation

$$\Lambda(s, \chi) = W(\chi) \Lambda(1 - s, \bar{\chi}),$$

where $W(\chi)$ is a certain constant of absolute 1.

Furthermore, $\Lambda(s, \chi)$ is entire of order 1 and does not vanish at $s = 0$.

Let

$$J_j(\chi) := -\frac{1}{2\pi \mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \frac{L'}{L}(s, \chi) k_j(s) ds$$

and

$$I_j := -\frac{1}{2\pi \mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} F_C(s) k_j(s) ds,$$

where $F_C(s)$ is defined in Lemma 3.6.

By Lemma 3.7, we have

$$I_j = \frac{|C|}{|G|} \sum_{\chi \in \hat{G}(L/E)} \bar{\chi}(g) J_j(\chi), \tag{3-1}$$

where g is a representative of C.

We have two ways to express I_j. One way is using the inverse Mellin transform and the other is using the residue theorem.

By the inverse Mellin transform, and we have

$$J_j(\chi) = \sum_{\mathfrak{p}} \sum_{m \geqslant 1} \chi(\mathfrak{p}^m) \log(N\mathfrak{p}) \hat{k}_j(N\mathfrak{p}^m),$$

since

$$-\frac{L'}{L}(s, \chi, E) = \sum_{\mathfrak{p}} \sum_{m \geqslant 1} \chi(\mathfrak{p}^m) \log(N\mathfrak{p})(N\mathfrak{p})^{-ms}.$$

Also, by Lemma 3.6,

$$\begin{aligned} I_j &= \frac{|C|}{|G|} \sum_{\chi \in \hat{G}(L/E)} \bar{\chi}(g) \sum_{\mathfrak{p}} \sum_{m \geqslant 1} \chi(\mathfrak{p}^m) \log(N\mathfrak{p}) \hat{k}_j(N\mathfrak{p}^m) \\ &= \sum_{\mathfrak{p}} \sum_{m \geqslant 1} \theta(\mathfrak{p}^m) \log(N\mathfrak{p}) \hat{k}_j(N\mathfrak{p}^m). \end{aligned}$$

Lemma 3.8. (1) $J_j(\chi) = \delta(\chi)k_j(1) - \sum_\rho k_j(\rho) + O(n_E k_j(0)) + O(k_j(-\frac{1}{2})(\log A(\chi) + n_E))$, *where the sum runs over all the nontrivial zeros of* $L(s, \chi, E)$*, and all the implied constants are absolute and effectively computable.*

(2)

$$\frac{|G|}{|C|} I_j \geqslant k_j(1) - \sum_\rho k_j(\rho) - c_6 \left\{ n_L k_j(0) + k_j\left(-\frac{1}{2}\right) \log d_L \right\},$$

where the sum runs over all the nontrivial zeros of $\zeta_L(s)$ *and* c_6 *is positive, absolute and effectively computable.*

For the proof we need the following proposition.

Proposition 3.9 (The Conductor-Discriminant formula)**.**

$$\prod_{\chi \in \hat{G}(L/E)} A(\chi) = d_L.$$

For a proof, see [13, 22].

Proof of Lemma 3.8. For (1), see [13]. The basic idea is to consider the following integral

$$\begin{aligned} J_j(\chi, T) &\triangleq -\frac{1}{2\pi \mathrm{i}} \int_{\partial B(T)} \frac{L'}{L}(s, \chi, E) k_j(s) ds \\ &= \delta(\chi) k_j(1) - a_\chi k_j(0) - \sum_{|\gamma| < T} k_j(\rho), \end{aligned}$$

where the sum runs over all the zeros $\rho = \beta + \mathrm{i}\gamma$ of $L(s, \chi, E)$ within the rectangle $B(T)$: $[-\frac{1}{2}, 2] \times [-T, T]$. Estimate the integral on each line segment and let T go to the infinity as in [13]. In fact, on the line segment from $-\frac{1}{2} + \mathrm{i}T$ to $-\frac{1}{2} - \mathrm{i}T$,

$$\left| \frac{L'}{L}(s, \chi, E) \right| \ll \log A(\chi) + n_E(\log(|s| + 2))$$

(see [13, Lemma 6.2]). Thus,

$$\left|\frac{1}{2\pi \mathrm{i}}\int_{-\frac{1}{2}-\mathrm{i}T}^{-\frac{1}{2}+\mathrm{i}T}\frac{L'}{L}(s,\chi,E)k_j(s)ds\right| \ll k_j\bigg(-\frac{1}{2}\bigg)\{\log A(\chi)+n_E\}$$

as

$$k_1\bigg(-\frac{1}{2}+\mathrm{i}t\bigg) \ll k_1\bigg(-\frac{1}{2}\bigg)\frac{1}{1+t^2}, \quad \text{if } y \gg x,$$
$$k_2\bigg(-\frac{1}{2}+\mathrm{i}t\bigg) \ll k_1\bigg(-\frac{1}{2}\bigg)\exp(-t^2\log x), \quad \text{if } x \gg 1.$$

To estimate the integral $I_\pm(T)$ on the horizontal line segments from $2 \pm \mathrm{i}T$ to $-\frac{1}{2} \pm \mathrm{i}T$, one uses the method of Landau (see [13, Section 6], [12, Section 3] and [14]), obtaining the estimate

$$I_\pm(T) \ll k_j(\mathrm{i}T)(\log A(\chi) + n_E \log T).$$

Note that $T \to \infty$, $I_\pm(T) \to 0$.

Combining these estimates with Proposition 3.9, we obtain (1).

Now (2) is easy to get from (1) since

$$I_j = \frac{|C|}{|G|}\sum_{\chi\in\hat{G}(L/E)} J_j(\chi)$$

and

$$\zeta_L(s) = \prod_{\chi\in\hat{G}(L/E)} L(s,\chi,E)$$

and we can use Proposition 3.9. □

Now we are ready to explain how we plan to use the standard model for our purposes.

From the rest of this chapter, assume that $y \gg x$ if we apply the first kernel function $k_1(s)$ and $x \gg 1$ if we apply the second one $k_2(s)$. Let $n = n_L/n_K$ which is not less than $|G|/|C|$.

Thus, by Lemma 3.8(2), we have

$$I_j \geqslant \frac{1}{n}\bigg(k_j(1) - \sum_\rho |k_j(\rho)|\bigg) - c_6\bigg\{n_K k_j(0) + k_j\bigg(-\frac{1}{2}\bigg)\bigg(\frac{1}{n}\log d_L\bigg)\bigg\}.$$

Note that

$$k_1(0) = \bigg(\frac{x^{-1}-y^{-1}}{-1}\bigg)^2 \ll x^{-2},$$
$$k_1(-\frac{1}{2}) = \bigg(\frac{x^{-\frac{3}{2}}-y^{-\frac{3}{2}}}{-\frac{3}{2}}\bigg)^2 \ll x^{-3},$$
$$k_2(0) = 1, \quad k_2\bigg(-\frac{1}{2}\bigg) = x^{-\frac{1}{4}}.$$

Thus, the $c_6\{\ \}$ term is bounded by some multiple of

$$T_j = \begin{cases} \dfrac{x^{-2}}{n}\log d_L, & \text{if } j=1,\\ \dfrac{1}{n}\log d_L, & \text{if } j=2. \end{cases}$$

Furthermore, we have

$$I_j = \sum_{\mathfrak{p}}\sum_{m\geqslant 1}\theta(\mathfrak{p}^m)\log(N\mathfrak{p})\hat{k}_j(N\mathfrak{p}^m).$$

Thus,

$$I_1 = S_{1,1} + S_{1,2} + S_{1,3} + \tilde{I}_1,$$
$$I_2 = S_{2,1} + S_{2,2} + S_{2,3} + S_{2,4} + \tilde{I}_2,$$

where the symbols mean the following:

$\tilde{I}_j$ denotes the sum over the primes outside S, unramifying in L, of degree 1 over K and the Artin symbol of $\mathfrak{p}$ under L/K being C such that $N\mathfrak{p} \leqslant y^2$ or $x^{3+\delta}$ when $j = 1$ or 2, respectively.

$S_{1,1}$ denotes the sum over $(\mathfrak{p}, m)$ with $\mathfrak{p}$ ramifying in L. $S_{2,1}$ denotes the sum over $(\mathfrak{p}, m)$ with $\mathfrak{p}$ ramifying in L and $N\mathfrak{p}^m \leqslant x^{10}$.

$S_{1,2}$ denotes the sum over $(\mathfrak{p}, m)$ with $\mathfrak{p}$ in S. $S_{2,2}$ denotes the sum over $(\mathfrak{p}, m)$ with $\mathfrak{p}$ in S and $N\mathfrak{p}^m \leqslant x^{10}$.

$S_{j,3}$ denotes the sum over $(\mathfrak{p}, m)$ with $N\mathfrak{p}^m$ not a rational prime.

$S_{2,4}$ denotes the sum over $(\mathfrak{p}, m)$ with $N\mathfrak{p}^m > x^{3+\delta}$.

Applying Lemma 3.1, we have

$$\begin{aligned}
S_{1,1} &\ll \frac{1}{n}\frac{\log(y/x)}{x^2}\log d_L,\\
S_{2,1} &\ll \frac{1}{n}(\log x)^{\frac{1}{2}}\log d_L,\\
S_{1,2} &\ll \frac{\log(y/x)}{x^2}\log N_S,\\
S_{2,2} &\ll (\log x)^{\frac{1}{2}}\log N_S,\\
S_{1,3} &\ll n_K\frac{(\log(y/x))(\log y)}{x(\log x)},\\
S_{2,3} &\ll n_K x^{7/4},\\
S_{2,4} &\ll n_K x^{2-\frac{\delta^2}{4}}\log x.
\end{aligned}$$

Then the main idea of this model is the following: Pick x, y appropriately. If we assume that for any $\mathfrak{p}$ unramifying in L, of degree 1 over K and the Artin symbol of $\mathfrak{p}$ under L/K being C such that either $N\mathfrak{p} > y^2$ or $x^{3+\delta}$ when $j = 1$ or 2 respectively, or $\mathfrak{p} \in S$ or $\mathfrak{p}$ ramifies in L, then $\tilde{I}_j = 0$ and

$$\frac{1}{n}\left(k_j(1) - \sum_{\rho}|k_j(\rho)|\right) \leqslant c_6' T_j + \sum_{v} S_{j,v}.$$

However, if the left-hand side dominates over $c_6' T_j$ and $S_{j,v}$ by a sufficiently large constant factor, then one gets a contradiction.

So the key component of this model is to find a better lower bound for

$$k_j(1) - \sum_{\rho}|k_j(\rho)|.$$

3.3 Final estimations

In Subsection 3.3, we will prove Theorem 1.1. Let $P_1(C, S)$ be the set of primes of K satisfying (1) to (3) in Theorem 1.1.

From Section 4, we have already seen that the quality of the effective bound depends on the lower bound of $k_j(1) - \sum_{\rho}|k_j(\rho)|$. However, the possible exceptional zero β_0 will cause difficulty. In general, one will be forced to use the Deuring-Heilbronn. Fortunately, there is nothing new here compared with the classical case where $S = \emptyset$.

To simplify our notation, we define β_0 to be the exceptional zero of $\zeta_L(s)$ if it exists, and $\beta_0 = 1 - (c_2 \log d_L)^{-1}$ otherwise, where c_2 is the constant defined in Lemma 3.3, so that $\zeta_L(s)$ has at most one zero in the interval $(1 - (c_2 \log d_L)^{-1}, 1)$.

In either case,

$$k_j(1) - \sum_{\rho} |k_j(\rho)| \geqslant k_j(1) - k_j(\beta_0) - \sum_{\rho \neq \beta_0} |k_j(\rho)|.$$

By using the mean value theorem, we have

$$\begin{aligned} k_1(1) - k_1(\beta_0) &= \left(\log \frac{y}{x}\right)^2 - \left(\frac{y^{\beta_0 - 1} - x^{\beta_0 - 1}}{\beta_0 - 1}\right)^2 \\ &\geqslant \frac{1}{10}\left(\log \frac{y}{x}\right)^2 \min\left\{1, (1-\beta_0)\log\left(\frac{y}{x}\right)\right\}, \\ k_2(1) - k_2(\beta_0) &= x^2 - x^{\beta_0 + {\beta_0}^2} \geqslant \frac{x^2}{10}\min\{1, (1-\beta_0)\log(x)\}. \end{aligned}$$

First suppose

$$1 - \beta_0 \geqslant {c_7}^2 (\log d_L 3^{n_L N_S^n})^{-2},$$

where c_7 is the constant defined in Lemma 3.4. In this case, we use the kernel $k_1(s)$. (Recall that $n = n_L/n_K$.)

The contribution of the zeros ρ of $\zeta_L(s)$ with $|\rho - 1| \geqslant 1$ is bounded by

$$\sum_{|\rho - 1| \geqslant 1} |k_1(\rho)| \leqslant \int_1^{\infty} \frac{2}{t^2} dn(t;1) \ll \log d_L,$$

where $n(t;1)$ is the number of the nontrivial zeros of ζ_L with $|\rho - 1| \geqslant t$ (see Lemma 3.2).

Next, assume that $|\rho - 1| \leqslant 1$ for a nontrivial zero $\rho = \beta + \mathrm{i}\gamma \neq \beta_0$ of ζ_L.

If β_0 as an exceptional zero exists with

$$1 - \beta_0 \leqslant \frac{1}{18} c_2 {c_7}^2 (\log(d_L N_S^n))^{-1},$$

then since $d_L \geqslant 3^{n_L/2}$ for $n_L \geqslant 2$, we have

$$\frac{c_7}{(1-\beta_0)\log(d_L 3^{n^L} N_S^n)} \geqslant \left\{\left(\frac{1}{2}c_2\right)(1-\beta_0)\log(d_L N_S^n)\right\}^{-\frac{1}{2}},$$

and therefore by the Deuring-Heilbronn (see Lemma 3.5, note that when replace d_L by any $Q > d_L$, we still have this statement),

$$\beta \leqslant 1 - c_8 \frac{\log\{\frac{c_7}{(1-\beta_0)\log(d_L 3^{n_L} N_S^n)}\}}{\log(d_L 3^{n_L} N_S^n)} \leqslant 1 - c_{11} \frac{\log\{(\frac{1}{2}c_2)(1-\beta_0)\log(d_L N_S^n)\}^{-1}}{\log(d_L N_S^n)}.$$

On the other hand, if

$$1 - \beta_0 \geqslant \frac{1}{18} c_2 {c_7}^2 (\log(d_L N_S^n))^{-1},$$

then by the zero-free region given by Lemma 3.3,

$$\beta \leqslant 1 - (3c_2 \log(d_L) N_S^n)^{-1}.$$

Hence, we have

$$\beta \leqslant 1 - c_{12} \frac{\log\{(\frac{1}{2}c_2)(1-\beta_0)\log(d_L N_S^n)\}^{-1}}{\log(d_L N_S^n)}, \qquad (*)$$

for some $0 < c_{12} < c_{11}$.

Thus $(*)$ holds for all the cases.

Let

$$B = c_{12} \frac{\log\{(\frac{1}{2}c_2)(1-\beta_0)\log(d_L N_S^S)\}^{-1}}{\log(d_L N_S^n)}.$$

From $(*)$, we have

$$|k_1(\rho)| \ll x^{2(\beta-1)}|\rho-1|^{-2} \ll x^{-2B}|\rho-1|^{-2}.$$

Thus, by Lemma 3.1,

$$\begin{aligned}\sum_{|\rho-1|<1,\rho\neq\beta_0} |k_1(\rho)| &\leqslant x^{-2B}\int_B^1 \frac{1}{t^2}dn(t;1)\\ &\ll x^{-2B}(B^{-2}+B^{-1}\log d_L)\\ &\ll x^{-2B}B^{-1}\log d_L.\end{aligned}$$

As $B \gg (\log(d_L N_S^n))^{-1}$, using the expression of B, we have

$$\sum_{|\rho-1|<1,\rho\neq\beta_0} |k_1(\rho)| \ll (\log(d_L N_S^n))^2\bigg\{\bigg(\frac{1}{2}c_2\bigg)(1-\beta_0)\log(d_L N_S^n)\bigg\}^{2c_{12}\frac{\log x}{\log(d_L N_S^n)}}.$$

Thus we have shown that

$$\begin{aligned}k_1(1)-\sum_\rho |k_1(\rho)| \geqslant{}& \frac{1}{10}\bigg(\log\frac{y}{x}\bigg)^2\min\bigg\{1,(1-\beta_0)\log\frac{y}{x}\bigg\} - c_{13}\log d_L\\ &- c_{14}(\log d_L)^2\bigg\{\bigg(\frac{1}{2}c_2\bigg)(1-\beta_0)\log d_L\bigg\}^{2c_{12}\frac{\log x}{\log d_L}},\end{aligned} \tag{4A-1}$$

for some positive constants c_{13} and c_{14}.

We now complete the proof of Theorem 1.1 in the case

$$1-\beta_0 \geqslant {c_7}^2(\log d_L 3^{n_L})^{-2}.$$

Assume that for any $\mathfrak{p}$ in $P_1(C,S)$, $N\mathfrak{p} > y^2$. Then

$$\begin{aligned}0 = \tilde{I}_1 ={}& \sum_{\mathfrak{p}\in P_1(C,S)} (\log N\mathfrak{p})\hat{k}_1(N\mathfrak{p})\\ \geqslant{}& \frac{1}{10n}\bigg(\log\frac{y}{x}\bigg)^2\min\bigg\{1,(1-\beta_0)\log\frac{y}{x}\bigg\}\\ &- c_{13}\frac{1}{n}\log d_L - c_{14}\frac{1}{n}(\log(d_L N_S^n))^2\bigg\{\bigg(\frac{1}{2}c_2\bigg)(1-\beta_0)\log(d_L N_S^n)\bigg\}^{2c_{12}\frac{\log x}{\log(d_L N_S^n)}}\\ &- c_{15,1}\bigg\{\frac{1}{n}\frac{1}{x^2}\log\bigg(\frac{y}{x}\bigg)\log d_L\bigg\}\\ &- c_{15,2}\bigg\{\frac{1}{x^2}\log\bigg(\frac{y}{x}\bigg)\log N_S\bigg\}\\ &- c_{15,3}\bigg\{n_K\frac{(\log\frac{y}{x})(\log y)}{x\log x}\bigg\} - c_6'\frac{x^{-2}}{n}\log d_L,\end{aligned}$$

where $c_{15,v}\{\cdots\}$ comes from $S_{1,v}$ and $c_6'\{\cdots\}$ comes from T_1.

Fix any positive constant ϵ, and set $y = x^{1+\epsilon}$, $x = (d_L N_S^n)^C$ for sufficiently large C, one gets that the first term dominates over the other terms by a large constant factor. Let us check this.

The $c_{14}\{\cdots\}$ term is bounded by some multiple of

$$\frac{1}{n}(\log(d_L N_S^n))^2 4^{-Cc_{12}} = o\bigg(\frac{(\log x)^2}{n}\bigg)$$

as C goes to ∞, thus it is dominated over by $\frac{1}{n}(\log\frac{y}{x})^2$ by a large constant factor. Also, this term is bounded by some multiple of

$$\frac{1}{n}(\log d_L)^2 4^{1-c_{12}C}\cdot(1-\beta_0)\log d_L,$$

which is $o(\frac{(\log x)^3}{n}(1-\beta_0))$ as C goes to ∞, thus it is dominated over by $\frac{1}{n}(\log\frac{y}{x})^3(1-\beta_0)$ by a large constant factor. From the discussion above one can verify this assertion for the $c_{14}\{\cdots\}$ term.

Since
$$\frac{1}{x^2}\log\frac{y}{x}\log N_S \ll \frac{1}{d_L^{2C}}\log\frac{y}{x} \ll \frac{1}{n}\log\frac{y}{x},$$
thus one can verify this assertion for the $c_{15,2}\{\cdots\}$ term.

Other terms are easy to check. So one draws a contradiction, and we get Theorem 1.1 in this case.

Furthermore, we consider the case
$$1-\beta_0 \leqslant {c_7}^2(\log d_L 3^{n_L})^{-2},$$
where we will use the second kernel function $k_2(s)$. In this case,
$$\log\frac{c_7}{(1-\beta_0)\log(d_L 3^{n_L} N_S^n)} \geqslant \frac{1}{2}\log(1-\beta_0)^{-1}.$$

If $\rho=\beta+\mathrm{i}\gamma$ is a zero of $\zeta_L(s)$ with $|\gamma|\leqslant 1$, and $\rho\neq\beta_0$, then by the Deuring-Heilbronn,
$$\begin{aligned}|k_2(\rho)| &\ll x^{\beta^2+\beta} \ll x^{1+\beta}\\ &= x^2\exp\bigg\{-c_{19}\frac{\log x\log(1-\beta_0)^{-1}}{\log(d_L N_S^n)}\bigg\} x^2(1-\beta_0)^{c_{19}\frac{\log x}{\log(d_L N_S^n)}},\end{aligned}$$
for some positive absolute constant c_{19}. Thus
$$\sum_{|\gamma|\leqslant 1,\rho\neq\beta_0}|k_2(\rho)| \ll x^2(1-\beta_0)^{c_{19}\frac{\log x}{\log(d_L N_S^n)}}\log(d_L N_S^n).$$

If $\rho=\beta+i\gamma$ is a zero of $\zeta_L(s)$ with $|\gamma|\geqslant 1$, and $\rho\neq\beta_0$, we have
$$|k_2(\rho)| \leqslant x^{2-\gamma^2} \ll x.$$
Thus assume $x>2$. Applying Lemma 3.1, we have
$$\begin{aligned}\sum_{|\gamma|>1}|k_2(\rho)| &\ll \sum_{n\geqslant 1} N(|2n|)\,x^{1+4n-4n^2}\\ &\ll x\log d_L\sum_{n\geqslant 1}2^{4n-4n^2} + x\,n_L\sum_{n\geqslant 1}2^{4n-4n^2}\log(2n+1)\\ &\ll x\log d_L,\end{aligned}$$
where $N(T)$ is the number of zeros of $\zeta_L(s)$ in the region $[0,1]\times[T-1,T+1]$.

Thus
$$\begin{aligned}k_2(1)-\sum_\rho k_2(\rho) \geqslant{}& \frac{x^2}{10}\min\{1,(1-\beta_0)\log x\}\\ &- c_{20}x\log d_L - c_{21}x^2(1-\beta_0)^{c_{19}\frac{\log x}{\log(d_L N_S^n)}}\cdot\log(d_L N_S^n),\end{aligned}\tag{4A-2}$$
for some absolute positive constants c_{20} and c_{21}.

We now complete the proof of Theorem 1.1 in the case,
$$1-\beta_0 \leqslant {c_7}^2(\log(d_L 3^{n_L} N_S^n))^2.$$

Assume that for any $\mathfrak{p}$ in $P_1(C,S)$, $N\mathfrak{p}>x^{3+\delta}$. Then
$$0=\tilde{I}_2=\sum_{\mathfrak{p}\in P_1(C,S),N\mathfrak{p}\leqslant x^{3+\delta}}(\log N\mathfrak{p})\hat{k}_2(N\mathfrak{p})$$

$$
\begin{aligned}
&\geqslant \frac{x^2}{10n}\min\{1,(1-\beta_0)\log x\}\\
&\quad - c_{20}\frac{1}{n}\log d_L - c_{21}\frac{1}{n}x^2(1-\beta_0)^{c_{19}\frac{\log x}{\log(d_L N_S^n)}}\cdot\log(d_L N_S^n)\\
&\quad - c_{22,1}\left\{\frac{1}{n}(\log x)^{\frac{1}{2}}\log d_L\right\}\\
&\quad - c_{22,2}\{(\log x)^{\frac{1}{2}}\log N_S\}\\
&\quad - c_{22,3}\{n_K x^{\frac{7}{4}}\} - c_{22,4}\{n_K x^{2-\frac{\delta^2}{4}}(\log x)\} - c_6'\log d_L,
\end{aligned}
$$

where $c_{22,v}\{\cdots\}$ comes from $S_{2,v}$ and $c_6'\{\cdots\}$ comes from T_1.

Fix any positive constant ϵ', and set $x = d_L^C$ for sufficiently large C. One gets that the first term dominates over the other terms by a large constant factor. Let us check this.

First be aware that by the Deuring-Heilbronn (see Corollary 3.5), and the fact that $d_L^{\varepsilon} \gg \log d_L \gg n_L$ for any $\varepsilon > 0$, the first term dominates over $n_K x^{2-\alpha}$ for C sufficiently large for any $\alpha > 0$.

The $c_{21}\{\cdots\}$ term is bounded by some multiple of

$$\frac{1}{n}x^2(1-\beta_0)^{c_{19}C}\log(d_L N_S^n)$$

as C goes to ∞, and thus it is dominated over by the first term.

Since

$$(\log x)^2\log N_S \ll x^{\frac{2}{1+\epsilon'}},$$

thus one can verify this assertion for the $c_{22,2}\{\cdots\}$ term.

Other terms are easy to check now. So one draws a contradiction, and we prove Theorem 1.1 in this case.

The proof of Theorem 1.2 is also similar with slight modification. Just consider $\zeta_K(s)L(s,\chi)$ instead of $\zeta_K(s)$ and we can imitate [12] to prove, and moreover use $d_K N(\chi) N_S$ instead of $d_K N(\chi)$.

4 Landau method

In this section, we will prove Theorem 4.1 (see Theorem 1.3). We are using Landau's idea (see [14]), and the proof also follows [17, 31].

Theorem 4.1. *Let π and π' be two unitary cuspidal automorphic representations of $\mathrm{GL}_d(K)$. Let S be a finite set of finite places of K, and $Q = \max(C(\pi), C(\pi'))$ and assume that the bound for Ramanujuan for π and π' are $< R$.*

Then if $\pi \ncong \pi'$, there exists a place v of K such that $\pi_v \ncong \pi'_v$, $v \notin S$ and

$$N\mathfrak{p}_v \leqslant \begin{cases} CQ^{1+\epsilon}N_S^{\epsilon}, & d = 1,\\ CQ^{2d+\frac{d(d-2)}{dH+1}+\epsilon}N_S^{\frac{d^3(2R+H)}{dH+1}+\epsilon}, & \text{general } d,\end{cases}$$

where C is some effectively computable constant only depending on arbitrarily chosen number $H > 2R$, $\epsilon > 0$, K and d.

Proof of Theorems 4.1 *and* 1.3. Before we start, we pose a condition on π and π': Let $\delta < 1/2$ be any given positive number, and we write the infinite part of $L(s,\tilde\pi\times\pi')$ as: $\prod_{j=1}^{d^2 n_K}\Gamma_{\mathbb{R}}(s + b_j(\tilde\pi\times\pi'))$.

AA-Additional Assumption. The horizontal distance from $\pm H$ and 0 and all $b_j(\tilde\pi\times\pi')$ inside $\mathbb{C}/\mathbb{Z}$ are all greater than δ, namely, for each integer N, $|H - N| > \delta$ and $|\pm H - N - \mathrm{Re} b_j(\tilde\pi\times\pi')| > \delta$.

We want to prove Theorem 4.1 with the assumption (AA) first.

Form

$$S(X,\tilde\pi\times\pi',S) = \sum_{n=1}^{\infty} a_{\tilde\pi\times\pi',S}(n)\,\omega\left(\frac{n}{X}\right),$$

where $a_{\tilde{\pi}\times\pi',S}(n)$ is the n-th coefficient of the incomplete L-function

$$L^S(s,\tilde{\pi}\times\pi')=\prod_{v\notin S}L(s,\tilde{\pi}_v\times\pi'_v)=\sum_{n=1}^{\infty}a_{\tilde{\pi}\times\pi',S}(n)n^{-s}$$

and the weight function $\omega(x)$ defined as a smooth function which may be specified as [31],

$$\omega(X)=\begin{cases}0, & x\leqslant 0 \quad\text{or}\quad x\geqslant 3,\\ \mathrm{e}^{-1/x}, & 0<x\leqslant 1,\\ \mathrm{e}^{-1/(3-x)}, & 2\leqslant x<3,\\ \leqslant 1, & \text{all } x.\end{cases}$$

Consider the Mellin transform

$$W(s)=\int_0^{\infty}\omega(x)x^{s-1}dx,$$

which is an analytic function of s. Fix $\sigma<0$ and let $s=\sigma+\mathrm{i}t$ then

$$W(s)\ll_{A,\sigma}\frac{1}{(1+|t|)^A},$$

for all $A>0$ by repeated partial integration. By Mellin inversion,

$$\omega(x)=\frac{1}{2\pi\mathrm{i}}\int_{(2)}W(s)x^{-s}ds,$$

where the integration is made along the vertical line $\mathrm{Re}s=2$.

Then we have

$$\begin{aligned}S(X,\tilde{\pi}\times\pi',S)&=\frac{1}{2\pi\mathrm{i}}\sum_{n=1}^{\infty}a_{\tilde{\pi}\times\pi',S}(n)\int_{(2)}W(s)\left(\frac{n}{X}\right)^{-s}ds\\&=\frac{1}{2\pi\mathrm{i}}\int_{(2)}X^sW(s)L^S(s,\tilde{\pi}\times\pi'),\end{aligned}$$

where the interchange of the summation and the integral is guaranteed by the absolute convergence along the real line $\sigma=2$. By the standard arguments, plus the fact that all incomplete L-functions of nontrivial characters are entire of order 1, we may shift the integral line to get

$$S(X,\tilde{\pi}\times\pi',S)=\frac{1}{2\pi\mathrm{i}}\int_{-H}x^sW(s)L^S(s,\tilde{\pi}\times\pi')ds,$$

where $H>0$ is to be specified later.

Let

$$L_S(s,\tilde{\pi}\times\pi')=\prod_{v\in S}L(s,\tilde{\pi}_v\times\pi'_v),$$

and

$$L_\infty(s,\tilde{\pi}\times\pi')=L(s,\tilde{\pi}_\infty\times\pi'_\infty)$$

the gamma factor of the Rankin-Selberg product L-function. Then we have the following functional equation

$$L^S(s,\tilde{\pi}\times\pi')=W(\tilde{\pi}\times\pi')N(\tilde{\pi}\times\pi')^{(1/2-s)}G_0(s)G_1(s)L^S(1-s,\pi\times\tilde{\pi}'),$$

where $W(\tilde{\pi}\times\pi')$ is the root number of χ which has absolute value 1,

$$G_0(s)=\frac{L_\infty(1-s,\pi\times\tilde{\pi}')}{L_\infty(s,\tilde{\pi}\times\pi')}$$

and

$$G_1(s)=\frac{L_S(1-s,\pi\times\tilde{\pi}')}{L_S(s,\tilde{\pi}\times\pi')}.$$

We need to estimate $G_0(s)$ and $G_1(s)$ along the vertical line $\sigma=-H$, avoiding to the pole of them. (*To be continued.*)

Lemma 4.2. *Under the assumption* (AA)*, we have*

(1)

$$G_0(-H+\mathrm{i}t) \ll_{H,d,K,\delta} (1+|t|)^{n_K d^2(1/2+H)} \prod_{j=1}^{d^2 n_K} (1+|b_j(\tilde{\pi}\times\pi')|).$$

(2) $G_1(-H+\mathrm{i}t) \ll_{H,d,K,\delta,\epsilon'} N_S^{d^2(2R+H)+\epsilon'}$.

We quote the following results on gamma function for which the proof can be found in a lot of analysis textbooks.

Lemma 4.3 (Stirling formula)**.**

$$|\Gamma(\sigma+\mathrm{i}t)| = \sqrt{2\pi}\mathrm{e}^{-\frac{\pi}{2}|t|}|t|^{\sigma-1/2}\left(1+O_{\sigma,\delta,A}\left(\frac{1}{1+|t|}\right)\right),$$

for all $|t|>A>0$ *and* $s=\sigma+\mathrm{i}t$ *away from any poles by at least distance* δ*.*

Proof of Lemma 4.2. (1) Write $b_j = b_j(\tilde{\pi}\times\pi') = u_j + \mathrm{i}v_j$, and we have $b_j(\pi\times\tilde{\pi}') = \overline{b_j(\tilde{\pi}\times\pi')} = \overline{b_j} = u_j - \mathrm{i}v_j$. Put $s=\sigma+\mathrm{i}t$.

$$\begin{aligned} G_0(s) &= \frac{L_\infty(1-s,\pi\times\tilde{\pi}')}{L_\infty(s,\tilde{\pi}\times\pi')} \\ &= \pi^{-d^2 n_K/2 + d^2 n_K s} \prod_{j=1}^{d^2 n_K} \frac{\Gamma((1-s+\overline{b_j})/2)}{\Gamma((s+b_j)/2)} \\ &\ll_{\sigma,\delta,d,K} \prod_{j=1}^{d^2 n_K} \frac{|t+v_j|^{\frac{1-\sigma+u_j}{2}-\frac{1}{2}}}{|t+v_j|^{\frac{\sigma+u_j}{2}-\frac{1}{2}}} \\ &\ll_{\sigma,\delta,d,K} \prod_{j=1}^{d^2 n_K} |t+v_j|^{1/2-\sigma}. \end{aligned}$$

Hence, under the assumption (AA),

$$|G_0(-H+\mathrm{i}t)| \ll_{\sigma,d,K,H,\delta} (1+|t|)^{d^2 n_K(1/2+H)} \prod_{j=1}^{d^2 n_K} (1+|b_j|)^{1/2+H}.$$

(2) For each $v\in S$, write $L(s,\pi_v) = \prod_{j=1}^d (1-a_{v,j}q_v^{-s})$ and $L(s,\pi'_v) = \prod_{j=1}^d (1-a'_{v,j}q_v^{-s})$. Then

$$L(s,\tilde{\pi}_v\times\pi'_v) = \prod_{j,k=1,\dots,d} (1-\overline{a_{v,j}}a'_{v,k}q_v^{-s})$$

and hence $L(s,\pi_v\times\tilde{\pi}'_v) = \prod_{j,k=1,\dots,d}(1-a_{v,j}\overline{a'_{v,k}}q_v^{-s})$,

$$\begin{aligned} G_1(s) &= \frac{L_S(1-s,\pi\times\tilde{\pi}')}{L_S(s,\tilde{\pi}\times\pi')} \\ &= \prod_{v\in S} \frac{L(1-s,\pi_v\times\tilde{\pi}'_v)}{L(s,\tilde{\pi}_v\times\pi'_v)} \\ &= \prod_{v\in S}\prod_{j,k=1,\dots,d} \frac{1-\overline{a_{v,j}}a'_{v,k}q_v^{-s}}{1-a_{v,j}\overline{a'_{v,j}}q_v^{s-1}}. \end{aligned}$$

Note that by the assumption on the Ramanujan bounds and the results on it (see [1, 18]), we have $|a_{v,j}|, |a'_{v,k}| < q_v^R \ll q_v^{1/2-1/(d^2+1)}$. Hence,

$$|G_1(-H+\mathrm{i}t)| \leqslant \prod_{v\in S}\left(\frac{1+q_v^{2R+H}}{1-q_v^{-(H+2/(d^2+1))}}\right)^{d^2}$$

$$\leqslant \prod_{v\in S}\left(\frac{2q_v^{2R+H}}{1-2^{-(H+2/(d^2+1))}}\right)^{d^2}$$
$$\leqslant 2^{|S|d^2} N_S^{d^2(2R+H)}\left(1-2^{-(H+2/(d^2+1))}\right)^{-d^2|S|}$$
$$\ll_{d,K,\epsilon'} N_S^{d^2(2R+H)+\epsilon'}.$$

Here we use the estimation $a^{|S|} \ll_{K,d,\epsilon'} N_S^{\epsilon'}$.

Proof of Theorem 4.1. The proof is still under the asumption (AA).

Now

$$\begin{aligned}
&S(X,\tilde{\pi}\times\pi',S)\\
&=\frac{1}{2\pi \mathrm{i}}\int_{(-H)} X^s W(s)W(\tilde{\pi}\times\pi')N(\tilde{\pi}\times\pi')^{1/2-s}G_0(s)G_1(s)L^S(1-s,\pi\times\tilde{\pi}')ds\\
&=\frac{1}{2\pi \mathrm{i}}\int_{(-H)} X^s W(s)W(\tilde{\pi}\times\pi')N(\tilde{\pi}\times\pi')^{1/2-s}G_0(s)G_1(s)\sum_{n=1}^{\infty}\frac{a_{\tilde{\pi}\times\pi',S}(n)}{n^{1+H}}ds\\
&=\frac{1}{2\pi \mathrm{i}}\sum_{n=1}^{\infty}\frac{a_{\tilde{\pi}\times\pi',S}(n)}{n^{1+H}}\int_{(-H)} X^s W(s)W(\tilde{\pi}\times\pi')N(\tilde{\pi}\times\pi')^{1/2-s}G_0(s)G_1(s)n^{s+H}ds.
\end{aligned}$$

Here the interchange of the sum and the integral is guaranteed by the absolute convergence of the Dirichlet series, rapid decay of $W(s)$.

Note that when $H > 2R$, we have

$$\sum_{n=1}^{\infty}\left|\frac{a_{\tilde{\pi}\times\pi',S}(n)}{n^{1+H}}\right| \leqslant \zeta_K(1+H-2R)$$

by the arguments in the preliminary subsection on Bound for Ramanujuan.

Applying Lemma 4.2, and the estimation above, we have

$$\begin{aligned}
S(X,\tilde{\pi}\times\pi',S) &\ll_{H,K} \zeta_K(1+H-2R)\int_{(-H)}|X^s W(s)W(\tilde{\pi}\times\pi')N(\tilde{\pi}\times\pi')^{1/2-s}G_0(s)G_1(s)d\,s|\\
&\ll_{H,K,\delta,\epsilon} X^{-H}N(\tilde{\pi}\times\pi')^{1/2+H}\\
&\quad\times\prod_{j=1}^{d^2n_K}(1+|b_j(\tilde{\pi}\times\pi')|)^{1/2+H}N_S^{d^2(2R+H)+\epsilon}\int_{(-H)}W(s)(1+|t|)^{(1/2+H)n_K d^2}d\,t\\
&\ll_{H,K,\delta,\epsilon} X^{-H}C(\tilde{\pi}\times\pi')^{1/2+H}N_S^{d^2(2R+H)+\epsilon}.
\end{aligned}$$

To establish the theorem, we need to bound $S(X,\tilde{\pi},\times\pi',S)$ below. Now assume that $\pi_v \cong \pi'_v$ for all $v \notin S$ such that $N\mathfrak{p}_v \leqslant 3X$. Then $S(X,\tilde{\pi}\times\pi',S) = S(X,\tilde{\pi}\times\pi,S)$.

Now, we have $S(X,\tilde{\pi}\times\pi,S) \gg_{K,d} X^{1/d}/(\log(X)) - |S|$ since by the prime number theory, when X is large, there is a prime $\mathfrak{p}_v$ of K such that $X < q_v^d < 2X$ (prime number theorem and Bertrand), where $q_v = N\mathfrak{p}_v$.

Thus, by Proposition 2.5, $S(X,\tilde{\pi}\times\pi,S)$ is greater than e^{-1} multiples of the primes $\mathfrak{p}_v$ of K of degree 1 outside S such that $X \leqslant \mathfrak{p}_v \leqslant 2X$, when

$$X > 4A|S|^2, \quad S(X,1,S) \gg_{K,d} X^{1/d}/(\log(X)),$$

for large $A > 0$.

Then we have, when $X > 4A|S|^2$,

$$\begin{aligned}
X^{1/d}/(\log(X)) \ll_{K,d,\delta} S(X,\tilde{\pi}\times\pi,S) &= S(X,\tilde{\pi}\times\pi',S)\\
&< C'X^{-H}C(\tilde{\pi}\times\pi')^{1/2+H}N_S^{d^2(2R+H)+\epsilon},
\end{aligned}$$

for some C' depending on H, K, d, δ.

Therefore,

$$X^{H+1/d-\epsilon} \ll_\epsilon X^{H+1/d}\log(X)^{-1} \ll_{H,K,d,\epsilon,\delta} C(\tilde{\pi}\times\pi')^{1/2+H} N_S^{d^2(2R+H)+\epsilon}$$

and thus

$$X \ll_{\epsilon,H,K,d,\delta} C(\tilde{\pi}\times\pi')^{\frac{H+1/2}{H+1/d-\epsilon}} N_S^{\frac{d^2(2R+H)}{H+1/d-\epsilon}}, \tag{4.1}$$

$$X < CQ^{2d+\frac{d(d-2)}{dH+1}+\epsilon} N_S^{\frac{d^3(2R+H)}{dH+1}+\epsilon}, \tag{4.2}$$

where C depends on ϵ and K, d, δ and $H > 2R$. As $X < 4A|S|^2 \ll_{\epsilon,A,K} N_S^\epsilon$, Theorem 4.1 follows under the assumption (AA).

Now for general H, we can choose small δ depending only on K, d and ϵ such that the assumption (AA) holds always for some H' in place of H with $|H'-H| < \delta/2$. Then we still get the theorem for H' in place of H. Since H' here can be taken sufficiently closed to H, we still get the theorem for such H. □

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